Mathematics

Senior 1

Student's Book

Contents

1 SETS

Key unit competence

By the end of the unit, I should be able to use sets, Venn diagrams and relations to represent situations and solve problems related to sets.

Unit outline

- Set concept.
- Relations.
- Inverse and composite relations.

1.1 Introduction to set concept

Activity 1.1

1. Identify and list any ten items in your home that can be grouped together.
2. Name the groups you have identified in step 1 above.
3. Let the group members select one member to present their findings in part 1 and 2 on the chalk board.

From Activity 1.1 above, you may have obtained groups that consist of utensils, domestic animals, furniture and so on. Most things in life have common features. Such features are used to group the things together. A group of items with a common feature is called a set. Examples of sets are: a set of animals, a set of houses, set of cars and so on. An object or item in a set is called a member or an element of the set. This unit will introduce us to the knowledge of sets and a number of terms concerning sets. We will define the terms as we meet them in their respective sections.

1.2 Describing sets

Activity 1.2

1. Find out from the internet, reference books or any other relevant materials how sets are described.
2. By use of examples, show other students how sets are described on the chalkboard.

There are three methods commonly used to describe or represent a set

(a) Statement form method

(b) Roster or tabular form method

(c) Set builder form method

All these methods are used together with a pair of curly brackets within which the description is enclosed.

(a) Statement form method

Consider:

(i) the alphabets a, b, c, d and e. These letters can be described using a statement as follows: {the first five letters of the alphabets}

(ii) the set of the numbers 2, 4, 6, 8, 10. This can be described as: {the first five even numbers} or {the first five natural numbers divisible by 2}

Individually, identify another five sets and describe them in a similar method.

(b) Roster or tabular method

Using this method, we describe a set by listing all the elements in it.

For example, consider the set of all the letters in the word Mathematics. This set can be described by listing all the elements in it. If we let this set be N, then
N = {m, a, t, h, e, i, c, s}

This method of describing sets is known as Roster or Tabular Method.

Note that each element is written only once. For example in the word mathematics, the letters *m, a* and *t,* each appear twice but we list each of them only once.

In roster form, all elements of a set are listed. The elements are being separated by commas and are enclosed with braces {}.

(c) Set builder form method

In this method, we describe a set by using a rule or an algebraic formula which defines **only** the set we want. For example,

A = {x : 1 < x < 10} is read as the set of integers x such that x lies between 1 and 10.

thus A = {2, 3, 4, 5, 6, 7, 8, 9} when a set is described using an algebraic expression, we say we are using **set builder notation.**

The symbol ":" in the statement {x : 1 < x < 10, x ∈N}is used to denote the statement "such that"

1.3 Set Notation

Activity 1.3

Write down the members of the following sets:

1. A set of subjects you study in Senior one class.
2. A set of five utensils used at home.
3. A set of three types of food that make a balanced diet.
4. A set of sexually transmitted diseases.

From Activity 1.3, a set of subjects (S) that you study in Senior one may be: S = {mathematics, physics, chemistry, biology, history, religious education…}. Capital letters are used to represent sets e.g. set S. Members of a set are separated by commas and the set is enclosed with a curly bracket{ }.

For example, a set of odd numbers (O) less than 10 can be written as O = {1, 3, 5, 7, 9}.

A set of the first six letters of an alphabet (A) can be written as A = {a, b, c, d, e, f}.

1.4 Membership of a set

Activity 1.4

1. Write down the set of four legged domestic animals you know.
2. Is a cow a member of that set? If so how can you represent this in set notation?
3. Is a lion a member of the above set? If so how can you represent this in set notation?

We use symbol ∈ to mean "is a member of" and ∉ to mean "is not a member of ". For example, given the set A= { 1, 3, 5, 7, 9, 11 }, 5 ∈ A read as 5 is a member of set A. 8 ∉ A read as 8 is not a member of set A.

1.5 Number of members in a set

Activity 1.5

1. Consider the set of all the members of your class. How many members are in that set?
2. Set A= {1, 3, 5, 7, 9, 11, 13}. How many elements are there in set A?
3. Write down any three sets containing six elements.
4. Suggest the name given to the number of elements in a set.

Consider set T = {0, 1, 2, 3, 4, 5, 6, 7, 8}. The number of elements (n) in set T is 9. This is written as n(T) = 9.

The number of elements in the set is called the cardinality or cardinal number of the set.

Suppose n(A) means the number of elements in set A. If D = {days of the week} then n(D) = 7.

Suppose F = {two legged cows}

L = {green men on the planet earth}

then:

n(F) = 0 and n(L) = 0. This means the two sets F and L have no members. Therefore sets F and L have no cardinal numbers i.e. 0 is the cardinal number in each case.

Example 1.1

List down the set G of vowels in the word "algebra".

Solution

G = {a, e}

Example 1.2

Describe set B using words, given that B = { 1, 3, 5, 7 ...}.

Solution

B is the set of all odd numbers.

Example 1.3

List all the elements of set A given that $A = \{ x: -3 < x < 8, x \text{ is an integer}\}$.

Solution

The elements of set A are $^{-}2, ^{-}1, 0, 1, 2, 3, 4, 5, 6$ and 7.

Example 1.4

Describe set T in words, given that

$T = \{x : x = 2n - 1, \text{ where } n = 1, 2, 3, \dots\}$

Solution

See Table 1.1

n	1	2	3	4	5	6	7
x	1	3	5	7	9	11	13

Table 1.1

Thus, T {the set of all positive odd numbers}

Example 1.5

Sets A and B are subsets of the set of natural numbers, N.

If $A = \{ x : 1 \le x \le 7\}$ and

$B = \{ x : 2 \le x \le 20, x \in \text{set of even numbers}\}$.

List the members of: (i) set A

(ii) set B

Solution

(i) $A = \{ x : 1 \le x \le 7\}$ means that x is a positive integer between 1 and 7 inclusive.

$\therefore \{x: 1 \le x \le 7\} = \{1, 2, 3, 4, 5, 6, 7\}$

(ii) $B = \{x: 2 \le x \le 20, x \in \text{even numbers}\}$ means all even numbers between 2 and 20 inclusive.

$\therefore \{ x : 2 \le x \le 20\} = \{ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.

$\therefore B = \{ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ and $A = \{ 1, 2, 3, 4, 5, 6, 7\}$.

Sets A and B share some members, i.e. 2, 4, 6.

Exercise 1.1

1. List members of the sets below:
   (a) {The set of natural numbers}

(b) {The set of squares of natural numbers}

(c) {x : x is the set of even numbers less than 15}

(d) $\{x: x^2 - 1 = 0\}$

(e) $\{x : \frac{1}{2}x - 3 = 4\}$

2. Copy the following and insert the correct symbol in each case.

(a) –5 ……… {counting numbers}

(b) 3 …………… {even numbers}

(c) 7 ……………… {odd numbers}

3. Describe each of the following sets using a rule.

(a) {2, 4, 6, 8 ...}

(b) {0, 3, 8, 15 ...}

(c) {1, 3, 5, 7 ....}

(d) {2, 5, 10, 17 ...}

4. Name the following sets. Write the number of elements in each set below.

(a) {Kigali, Kampala, Nairobi, Bujumbura, Dar es Salaam}.

(b) {a, e, i, o, u}.

(c) {2, 4, 6, 8, 10}.

(d) {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}.

5. List the members of the following sets. How many members are in each set?

(a) The set of {colours of a rainbow}.

(b) The set of {colours of the Rwandan flag}.

(c) The set of {countries neighbouring Rwanda}.

(d) The set of {letters of the first name of your head teacher}.

6. M is the set of counting numbers between 10 and 20

M= {11, 12, 13, 14, 15, 16, 17, 18, 19}.

Write true or false for each of the following statements:

(a) $12 \in M$ (b) $10 \in M$

(c) $17 \notin M$ (d) $9 \notin M$

(e) $3 \in M$ (f) $22 \notin M$

7. Use set symbols to write the following sets.

(a) April is a member of {the months of the year}.

(b) A is a member of {the set of vowels}.

(c) 3 is a member of {the set of prime numbers}.

(d) 4 is a member of {the set of square numbers}.

(e) K is a member of the set of {alphabets}.

(f) HIV is a member of {sexually transmitted diseases}.

HIV is a sexually transmitted disease. It is a deadly disease and has no cure. Abstain from sex before marriage and stick to one faithful partner.

8. You are given the following sets:

A= {1, 3, 5, 7, 9, 11, 13,15}

B= {2, 4, 6, 8, 10, 12, 14}

C= {2, 3, 5, 7, 11, 13}.

Copy and fill in the missing membership symbol in the boxes in each of the following:

(a) $5 \boxed{?} A$ (b) $5 \boxed{?} B$

(c) $5 \boxed{?} C$ (d) $6 \boxed{?} B$

(e) 13 $\boxed{?}$ B (f) 9 $\boxed{?}$ C

(g) 2 $\boxed{?}$ A (h) 17 $\boxed{?}$ A

9. Write the cardinal numbers in this question.
   (a) n(M), where M = {set of days in February}
   (b) n(S), where S = {set of sides of triangle ABC}
   (c) n(D), where D = {set of days of September}
   (d) n(C), where C = {set of wheels of a car}
   (e) n(B), where B = {set of wheels of a bicycle}

1.6 Subset of a set

Activity 1.6

1. Consider the set A={lion, cow, buffalo, goat, pig, dove, tilapia, weaver bird, crocodile, pigeons, snake}.
2. In pairs, discuss and write down other sets that can be formed from set A above.
3. Name the sets you formed in step 2 above.

From Activity 1.6 above, you learnt that, a set can be obtained by taking some or all the elements of a given set to form another set.

For example, set B = {cow, goat, pig} is a set of some domestic animals. Similarly, we can get set C = {dove, weaver bird, pigeon} which is a set of birds.

Set D = {tilapia} which is a set of fish and so on. A set that is formed by obtaining some elements or all the elements from a given set is called a subset.

The empty set denoted by { } or $\varnothing$ is also a subset of any set. A subset is denoted by symbol $\subset$.

Activity 1.7

Consider the set A= {4, 5, 6}.

In pairs, discuss and write down all the subsets of set A.

How many subsets have you obtained?

Consider another set B.

B= {HIV, Gonorrhea, Syphilis, Trichomoniasis, Genital Herpes}

In pairs, discuss and write down all the subsets of B.

How many subsets have you obtained?

1.7 The number of subsets in a set

The number of subsets of a given set depends on the number of elements. See Table 1.2.

Copy and complete this table:

Set	Subset	Number of Subsets
{ }	{ }	1
{1}		
{1, 2}		
{1, 2, 3}		

Table. 1.2

If the number of subsets of a given set is represented as N_s then the number of subsets is given by;

$N_s = 2^n$

Where n equals the number of elements in the set.

Example 1.6

Set C has 5 elements. How many subsets does it have?

Solution

Since the number of elements is five ($n = 5$) then from $N_s = 2^n$, the number of subsets is

$$N_s = 2^5$$
$$= 32$$

i.e the set has 32 subsets.

Example 1.7

A certain set has 64 subsets. How many elements are there in the set?

Solution

Number of subsets $(N_s) = 64$

$N_S = 2^n$

$64 = 2^n$

$2^6 = 2^n$

$n = 6$

i.e. there are 6 elements.

Example 1.8

Given set $B = \{a, b, c\}$, list all the subsets of set B in your exercise book.

Solution

The subsets of set B are $\{\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and $\{a, b, c\}$.

Note: The number of subsets is obtained by the formulae 2^n where n is the number of elements in a set.

Example 1.9

Given set $A = \{2, 4, 6, 8\}$, find the number of subsets in set A.

Solution

Number of subsets $= 2^n = 2^4 = 16$ *subsets.*

Exercise 1.2

List the subsets of each of the sets in questions 1 and 2.

1. {w, z, y}
2. {a, 3, 1, 2}
3. If there are 128 subsets in a set, how many elements does the set contain?
4. Suppose a set has $4^{(n-3)}$ subsets. How many elements are in this set? (n is the number of elements in a set).

1.8 Subsets of numbers

Activity 1.8

1. Discuss what natural numbers are.
2. Write down the set N of {the first 20 natural numbers}.
3. From set N in 2 above;
   (a) Write a subset E of {even numbers}.
   (b) Write a subset D of {odd numbers}.
   (c) Write a subset P of {prime numbers}.
4. Show the relationship between sets E and N, D and N, P and N in set symbols.

Natural numbers are counting numbers from zero to infinity. The set of natural numbers is represented as set N = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 …}.

Even numbers are numbers divisible by 2. Set E of even numbers is represented as set E = {2, 4, 6, 8 …}.

Therefore the set of even numbers is a

subset of the set of natural numbers. This can be represented as E⊂N.

Odd numbers are numbers which are not divisible by 2. The set of odd numbers is represented by D= {1, 3, 5, 7, 9, 11, 13, 15..........}.

The set of odd numbers is a subset of the set of natural numbers which is written as D⊂N.

A prime number is a number which is divisible by one and itself only. In other words, prime numbers have only two factors.

The set P of prime numbers is represented as set P= {2, 3, 5, 7, 11, 13, 17.......}. The set of prime numbers is a subset of natural numbers represented as P⊂N.

Exercise 1.3

1. (a) Write down the subsets of set D = {Lion, hyena, leopard}.
   (b) How many subsets are they?
2. Set F = {Cassava, potatoes, maize, pumpkin}:
   (a) Write down the subsets of set F.
   (b) How many subsets did you obtain?
3. Set L = {u, v, w, x, y}.
   (a) Write down the subsets of set L.
   (b) How many subsets are they?
4. (a) Given N = {all natural numbers less than 50}. List the members of the following subsets of N:
      (i) Set E = {even numbers}.
      (ii) Set P = {prime numbers}.
      (iii) Set D = {odd numbers}.
   (b) Show the above subsets in set symbols.
   (c) How many subsets are in each of the above set?
5. List the subsets of the given set as described below it:
   (a) Set E = {2, 3, 5, 7, 11, 12, 15}
      (i) Set A = {multiples of 3}
      (ii) Set B = {prime numbers}
      (iii) Set C = {odd numbers}
      (iv) State the numbers of subsets in each case.
   (b) Set E = {1, 5, 10, 15, 20, 25, 30, 35, 40}
      (i) Set A = {multiples of 5}
      (ii) Set B = {multiples of 3}
      (iii) Set C = {multiples of 4}
      (iv) State the number of subsets in each case.

1.9 Venn diagrams

Activity 1.9

Carry out the following activities:

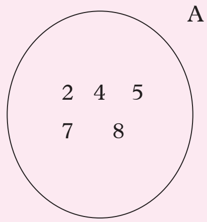
1. Write down the first ten members of the set of even numbers.
2. List and write down any five members of the set of carnivorous animals.
3. Draw a circle. Write the members of the sets in steps 1 and 2 above in separate diagrams.
4. What name would you call such diagrams?

From Activity 1.9, the circular pattern used to represent a set and its elements is called a Venn diagram.

Example 1.10

Given set A= {2, 4, 5, 7, 8}, represent set A on a Venn diagram.

Solution



1.10 Set operations

Sets can be combined in different ways to produce other sets. Such operations may involve finding:

- Intersection of sets
- Union of sets
- Universal sets
- Simple difference of sets
- Symmetric difference of sets
- Complement of sets

(a) Intersection of sets

Activity 1.10

Consider the sets A = {6, 7, 8, 9, 10}, B = {1, 2, 3, 4, 5, 6, 7}, C= {2, 4, 6, 8, 10, 12} and D= {1, 3, 5, 7, 9}.

List the common elements between the following sets:

(a) A and B
(b) B and C
(c) B and D
(d) A and B and C

From Activity 1.10, you noticed that there are common elements that appear in two or more sets. The common elements which appear in two or more sets form the intersection of sets. The symbol used to denote intersection of sets is $\cap$.

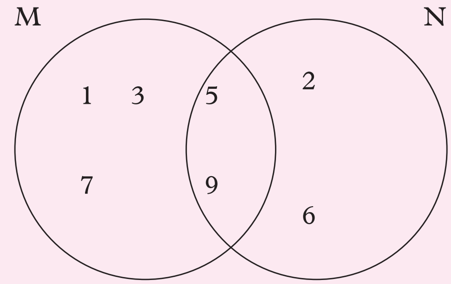
Example 1.11

Given sets $M = \{1, 3, 5, 7, 9\}$ and $N = \{2, 5, 6, 9\}$; find $M \cap N$.

Represent set M and N on a Venn diagram.

Solution

$M \cap N = \{1, 3, 5, 7, 9\} \cap \{2, 5, 6, 9\} = \{5, 9\}$.



The Venn diagram of intersection of sets overlap each other as shown in the example.

Example 1.12

Given that set P = {the first 5 letters of the alphabet} and set Q = {all the vowels};

(a) *List the elements of set P and Q.*

(b) *Find $P \cap Q$.*

(c) *Draw a Venn diagram to represent set $P \cap Q$.*

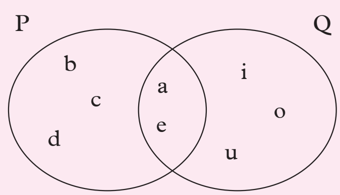
Solution

$P = \{a, b, c, d, e\}$

$Q = \{a, e, i, o, u\}$

$P \cap Q = \{a, b, c, d, e\} \cap \{a, e, i, o, u\} = \{a, e\}$

The Venn diagram is as shown below.



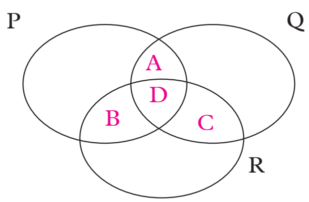
P ∩ Q

Activity 1.11

1. On individual basis, form a set of eight items in your classroom.

2. In pairs, compare and list down common items found in part 1 in your lists.
3. Form groups of three students.
4. Write down the set of common items in each of your lists.
5. How can you represent the common elements in a Venn diagram?

From Activity 1.11, you have noticed that there are three sets with intersection of elements. They can be represented on a Venn diagram as shown below.



Where P, Q and R are the three students in the groups.

Region A represents intersection of sets P and Q only.

Region B represents intersection of sets P and R only.

Region C represents intersection of sets Q and R only.

Region D represents the intersection of all the three sets P, Q and R which is written as $(P \cap Q \cap R)$.
The unlabelled regions represent the elements of each set which is not common with the other sets.

Example 1.13

You are given the following sets;

$A = \{x: x \text{ is a prime number less than } 20\}$

$B = \{x: x \text{ is an odd number less than } 15\}$

$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Draw a Venn diagram to represent the intersection of sets A, B and C.

Solution

First, list the elements of set A, B and C as follows;

$A = \{2, 3, 5, 7, 11, 13, 17, 19\}$

$B = \{1, 3, 5, 7, 9, 11, 13\}$

$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Then list all the common elements between 2 sets

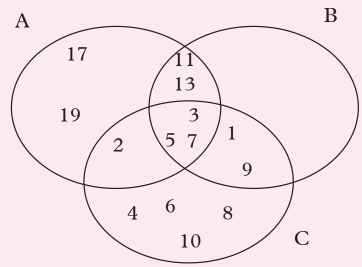
$A \cap B = \{3, 5, 7, 11, 13\}$

$A \cap C = \{2, 3, 5, 7\}$

$B \cap C = \{1, 3, 5, 7, 9\}$

Then list all the common elements between 2 sets

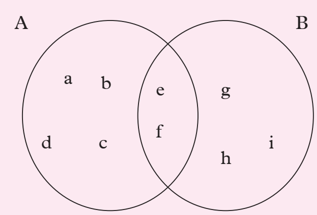
$A \cap B \cap C = \{3, 5, 7\}$. *The Venn diagram is as follows.*



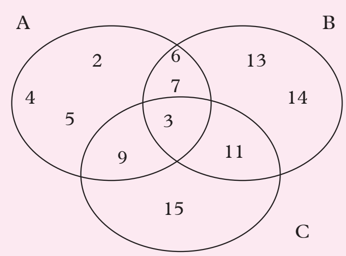
Example 1.14

List elements of the sets in the Venn diagrams below.

(a)



(b)



Solution

(a) *The elements in the intersection are also listed for each individual sets as shown below.*

Set A ={a, b, c, d, e, f}

Set B = {e, f, g, h, i}

(b) *Set A = {2, 3, 4, 5, 6, 7, 9}*

Set B = {3, 6, 7, 11, 13, 14}

Set C = {3, 9,11, 15}

Exercise 1.4

1. Given sets J = {1, 2, 3, 4}, K = {2, 3,5, 7}, L = {1, 2, 5, 8}, M ={3, 4, 5, 8} and N = {9, 10},

   Find:

   (a) $J \cap L$ (b) $J \cap K$

   (c) $K \cap L$ (d) $M \cap N$

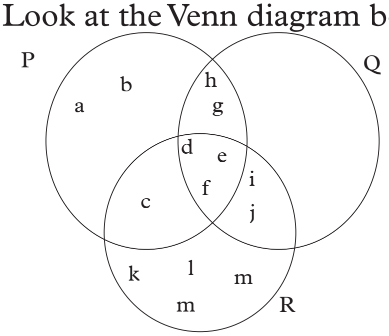
   (e) $J \cap K \cap L$ (f) $K \cap L \cap M$

2. Given sets A = {2, 4, 6, 8, 10, 12}, B = {3, 6, 9, 12, 15} and C = {9, 10, 11, 12, 13, 14, 15, 16, 17}, draw Venn diagrams to represent the following sets:

   (a) $A \cap B$ (b) $A \cap C$

   (c) $B \cap C$ (d) $A \cap B \cap C$

3. Look at the Venn diagram below.

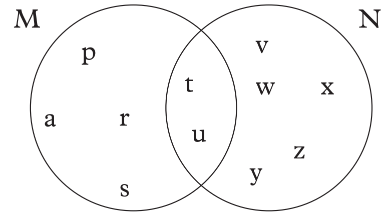


   Write down the elements in:

   (a) $P \cap Q$ (b) $R \cap Q$

4. Set A = {a, b, c, d, e, f, i, j} and set B = {a, e, i}. Draw a Venn diagram to represent $A \cap B$.

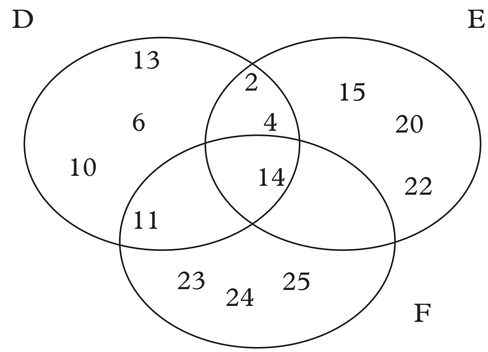
5. Consider the Venn diagram below.



   List the elements of set M and N.

   What are the elements of $M \cap N$.

6. Consider the Venn diagram below.



   (a) List the elements of sets D, E and F.

   (b) List the elements of:

   (i) $D \cap E$ (ii) $D \cap F$

   (iii) $E \cap F$ (iv) $D \cap E \cap F$

(b) Union of sets

Activity 1.12

You are given the universal set = {natural numbers below 12}and sets A = {4, 5, 6, 7, 8}, B = {0, 1,3,4,5}, C = {2, 4, 6, 8, 10} and D = {1, 3, 5, 7, 11}.

In pairs, write down the following sets:

1. The elements of set A which are not in the universal set.
2. The elements of set C which are not in the universal set.

3. The elements of set D which are not in the universal set.
4. The elements of sets A and C which are not in the universal set.

From Activity 1.12 above, you have realized that, elements of two or more sets can be put together to form a set. The set formed is known as the union of sets. The symbol for the union of sets is $\cup$.

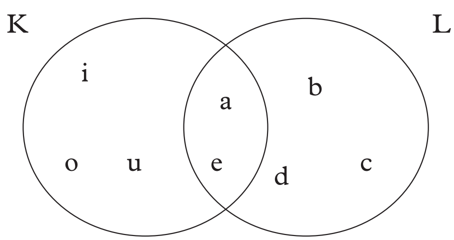
Activity 1.13

You are given sets K= {a, e, i, o, u} and L = {a, b, c, d, e}. Individually, write $K \cup L$.

Represent the union of the sets above on a Venn diagram.

From Activity 1.13 above, there are common elements. In this case they are a, and e. They should be written once.

The Venn diagram should appear as shown below.



The union of set K and L is written as KUL.

Example 1.15

Given the following sets $A = \{a, b, c, d, e, f\}$ *and* $B = \{a, b, c, h, i\}$, *find:*

(i) $n(A)$ *(ii)* $n(B)$

(iii) $n(A \cup B)$

Solution

(i) $n(A) = 6$

(ii) $n(B) = 5$

$$A \cup B = \{a, b, c, d, e, f\} \cup \{a, b, c, h, i\}$$
$$= \{a, b, c, d, e, f, h, i\}$$
$$n(A \cup B) = 8$$

Example 1.16

If $A = \{2, 4, 6, 8\}$ $B = \{1, 2, 3\}$
$C = \{6, 8, 10\}$ $D = \{2, 3, 6\}$, *find:*

(a) $n(A)$ *(b)* $n(B)$ *(c)* $n(C)$

(d) $n(A) + n(B)$

(e) $n(A) + n(C) - n(B)$

(f) $A \cup B \cup C$ *(g)* $n(A \cup B \cup C)$

(h) $A \cup C \cup D$ *(i)* $n(A \cup C \cup D)$

(j) *Draw Venn diagrams to represent* $A \cup B \cup C$ *and* $A \cup C \cup D$.

Solution

(a) $n(A) = 4$ *(b)* $n(B) = 3$

(c) $n(C) = 3$

(d) $n(A) + n(B) = 4 + 3 = 7$

(e) $n(A) + n(C) - n(B) = 4 + 3 - 3 = 4$

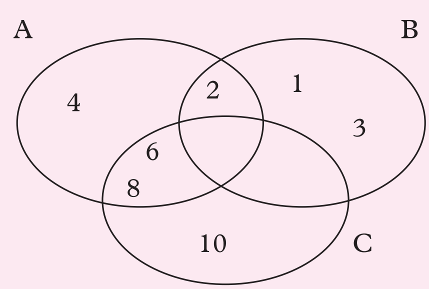
(f) $A \cup B \cup C = \{2, 4, 6, 8\} \cup \{1, 2, 3) \cup \{6, 8, 10\}$
$= \{1, 2, 3, 4, 6, 8, 10\}$

(g) $n(A \cup B \cup C) = 7$

(h) $A \cup C \cup D = \{2, 4, 6, 8\} \cup \{6, 8, 10\} \cup \{2, 3, 6\}$
$= \{2, 3, 4, 6, 8, 10\}$

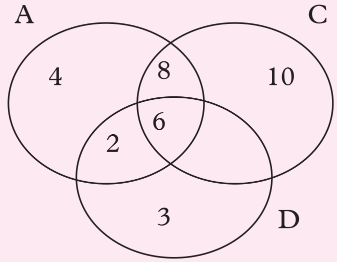
(i) $n(A \cup C \cup D) = 6$

(j) $A \cup B \cup C$



This represents $A \cup B \cup C$

A ∪ C ∪ D



This represents A ∪ C ∪ D

Exercise 1.5

1. Given the sets A = {a, b, c, d}, B = {a, e, i, o}, C = {b, e, d, f} and D = {a, d, e, g}; Find:

   (a) A ∪ B (b) B ∪ C
   (c) A ∪ B ∪ C (d) A ∪ C ∪ D
   (e) C ∪ D ∪ B (f) C ∪ D

2. Find the union of the following sets:

   (a) A = {Even numbers from 0 to 20}
   B = {Integers greater than $^{-}2$ but less than 9}
   C = {Prime numbers between 1 and 7}

   (b) X = {stool, chair, table, bed}
   Y = {chair, beans, lemon}

3. Draw a Venn diagram for the pair of sets given below and work out;
   n (P), n(Q), n(P ∪ Q),
   n(P ∪ Q) – n(P) + n(Q)

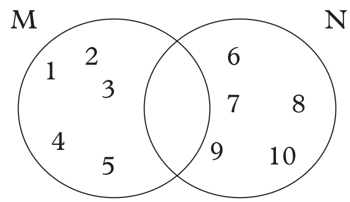
   (a) P = {1, 3, 5, 7, 11, 13}
   Q ={3, 6, 9, 12, 15}

   (b) P = {4, 8, 12, 16}
   Q = {1, 4, 9, 16, 25, 36, 49, 64}

   (c) P = {s, t, u, v, w}
   Q ={q, r, s, t}

   (d) P = {10, 20, 30, 40, 50}
   Q = {5,10,15,20,25}

4. Consider the Venn diagram below.



   List the elements of M ∪ N. Find n (M ∪ N).

(c) Universal sets

Activity 1.14

Consider a school. There are various categories of people in a school. They are pupils, teachers and workers.

1. Use S to represent the set of people in a school.
2. Write all the subsets of S.

We can present the set of all people in the school with sets as follows:

Set S = {pupils, teachers, workers}.

Thus, set S contains all the subsets of the various categories of people in the school. Let us use sets P, T and W to represent the subsets of sets.

Set P = {pupils}

Set T = {teachers}

Set W = {other workers}.

Therefore P⊂S, T⊂S and W⊂S.

Note: A set that contains all the subsets under consideration is known as a universal set. A universal set is denoted by the symbol ε.

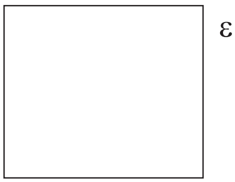
Thus, the set S = {pupils , teachers, workers} is a universal set and is represented as

set ε = { pupils, teachers, workers}.

Other examples of universal sets are:

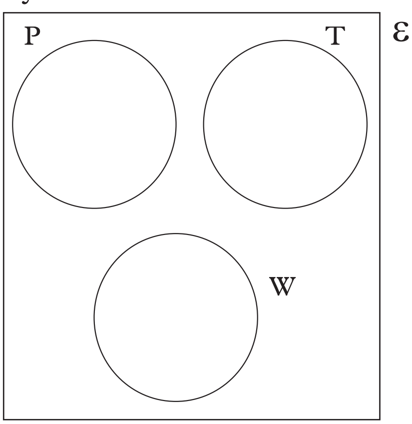
- Set ε { all domestic animals}
- Set ε { whole numbers}.
- Set ε { cereals}.

In a Venn diagram, a universal set is represented by a rectangle shown below.



The following Venn diagram shows the relationship between the universal set ε and the three sets P, T and W.

Note that all the three subsets are disjoint i.e. they have no common members.



Example 1.17

Consider the universal set ε = { all vehicles}. Write down four subsets of set ε.

Solution

Four subsets of set ε are:

Set A = { Cars} Set B = { Lorries}

Set D = { Buses} Set D = {Mini buses}

Example 1.18

A = { 1, 2, 3, 4, 5}

B = { 2, 4, 6, 8, 10}

Choose a universal set, ε, for A and B and represent the sets in a Venn diagram.

Solution

Let ε = {all counting numbers between 1 and 10 inclusive}

$\therefore \quad A \subset \varepsilon$ *and*

$\quad B \subset \varepsilon$

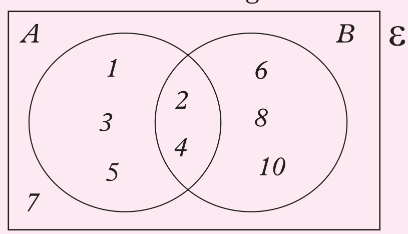
But A and B have members in common

$\therefore$ *Since* $A = \{1, 2, 3, 4, 5\}$ *and*

$B = \{2, 4, 6, 8, 10\}$.

$A \cap B = \{2, 4\}$.

The figure below shows the representation of the three sets in a diagram.



(d) Complement of a set

Activity 1.15

You are given the universal set ε = {natural numbers below 12} and sets A = {4, 5, 6, 7, 8}, B = {0, 1, 3, 4, 5} C = {2, 4, 6, 8, 10} and D = {1, 3, 5, 7, 11}.

In pairs, write down the following sets:

1. The elements of set A which are not in the universal set.
2. The elements of set C which are not in the universal set.
3. The elements of set D which are not in the universal set.
4. The elements of sets A and C which are not in the universal set.

From Activity 1.15, you have observed that some elements in set A are not there in the universal set for instance. Complement of a set is the set of all elements in the universal set that are not members of a given set. The symbol for the universal set is ε.

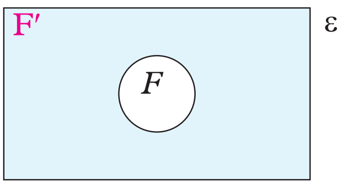
For example ε= {1, 2, 3, 4, 5} and set A = {3, 5}. Then the complement of A = {1, 2, 4} written as A′ = {1, 2, 4}.

Suppose:

ε = { All the teachers in Rwanda}

F = { All the female teachers in Rwanda}

The figure shows set F in relation to the universal set.



The shaded region shows all the teachers in Rwanda who are not female, i.e. all the male teachers. This set is called the **complement** of F denoted as **F′**. Therefore, the members of F′ are all those members of ε who are not members of F.

Example 1.19

Given that:

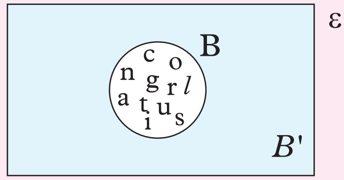
ε = { *a, b, c, d ……z*}

B = { *letters in the word* ***congratulations***}

Describe the set B′ and show set B and B′ in a Venn diagram.

Solution

Draw a rectangle and label it with ε to show the universal set. Within the rectangle draw set B as a subset of ε.



Shade the region within the universal set but outside set B and label it B′. B′ is the set of all the alphabets except those in set B.

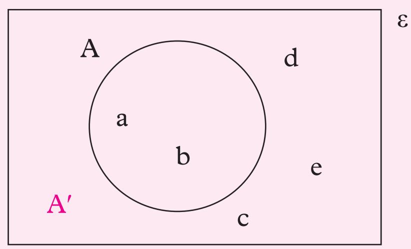
Example 1.20

Given that ε= {a, b, c, d, e} and A= {a, b}, find A′

Solution

ε= {a, b, c, d, e} A= {a, b} thus, A′= {c, d, e}.

This can be represented on a Venn diagram as shown.



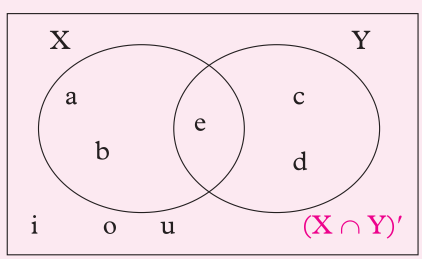
Example 1.21

If ε= {a, e, i, o, u, c, d}, sets X = {a, b, e} and Y= {c, d, e}, find:

(a) $(X\cap Y)'$ *(b)* $(X\cup Y)'$

Solution

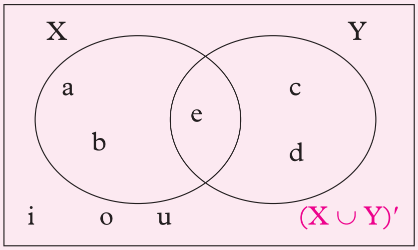
(a) *ε = {a, e, i, o, u, c, b, d}, X= {a, b, e} and Y= {c, d, e} thus, X ∩ Y= {e} and therefore,* $(X\cap Y)'$ *= {a, i, o, u, c, b, d}.*



(b) $\varepsilon = \{a, e, i, o, u, c, d\}$, sets $X=\{a, b, e\}$ and $Y= \{c, d, e\}$.

So, $(X \cup Y) = \{a, b, c, d, e\}$

Then, $(X \cup Y)' = \{i, o, u\}$



Exercise 1.6

1. If $A=\{m, n, k\}$ and $C =\{f, g, h, i, j, k, l, m\}$, find:

   (a) $(A \cap C)'$ (b) $(A \cup C)'$

2. If $\varepsilon = \{2, 3, 4, 5, 6\}$ and $A = \{2, 4, 6\}$. Find A'.

3. If A = {counting numbers from 1 to 15} and B = {even numbers from 2 to 14}, find

   (i) $(A \cup B)'$ (ii) $(A \cap B)'$

4. Given sets $A = \{3, 5\}$, $B = \{7, 9, 11, 13\}$, $C = \{3, 5, 7\}$ and $\varepsilon = \{3, 5, 7, 9, 11, 13\}$. Use Venn diagrams to show the following sets by shading the appropriate regions.

   (i) A' (ii) $(A \cap B)'$

   (iii) B' (iv) $(A \cup B)'$

   (v) $(A \cap C)'$ (vi) $(B \cap C)'$

5. If ε = {integers greater than –4 and less than 5} and $A = \{-3, 0, 2\}$, list the members of :

   (a) A' (b) $A \cap \varepsilon$

   (c) $A \cup \varepsilon$ (d) $A \cup A'$

6. Given that P = {even numbers} list the members in the intersection of Set P and set:

   (a) A = {counting numbers from 1 to 10}

   (b) $B = \{a, b, 3, 4, c, d, 7, 8, F\}$

   (c) $C = \{20 \leq n \leq 30$ and n is a whole number$\}$

   (d) $D = \{5, 7, 9, 12, 25\}$

   (e) $E = \{5, 7, 13, 15, 27\}$

7. If $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{3, 7, 8\}$, list the members of :

   (a) A' (b) $A \cap A'$ (c) $A \cup A'$

8. If ε = {letters of the alphabets} and $V = \{a, e, i, o, u\}$,

   (a) State: (i) $n(\varepsilon)$ (ii) $n(V')$

   (b) Write down all the subsets of V which have two elements only excluding the empty set, { }.

9. Copy the following and insert the correct symbol in each case.

   (a) {4} ……….. {even numbers}

   (b) –5 …… {counting numbers}

   (c) 3 ………… {even numbers}

   (d) { 5, 6 } ………. {odd numbers}

   (e) 7 …………… {odd numbers}

   (f) {6, 7, 8} …… {counting numbers}

(e) Simple and symmetric differences of sets

Activity 1.16

Given set A = {mangoes, pawpaw, oranges}, set B = {apples, pawpaw, strawberry}, set C = {strawberry, apples, passion fruit}.

Identify elements of set:

(a) A which do not belong to set B.

(b) B which do not belong to set C.

(c) B which do not belong to set A.

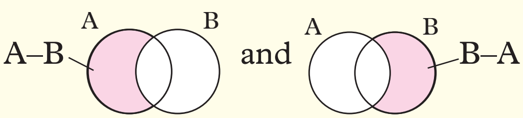
From Activity 1.16, you have realized that some elements in one set are not there in the other set.

Learning points

Simple difference between sets A and B written as A-B or A\B is the set of the elements of set A which are not in set B.

Similary B-A or B/A is difference between sets B and A. This is the set of elements that are in set B and not in set A.

A–B and B–A can be shown using a Venn diagram as follows:



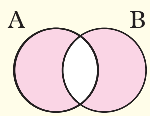
Learning points

The union of sets A-B and B-A is known as the symmetric difference between sets A and B.

It is written in symbols as $A \Delta B$ to mean

$A \Delta B = (A\text{-}B) \cup (B\text{-}A)$

$A \Delta B$ can be shown using a Venn diagram as follows



$A \Delta B$

Example 1.22

Given that $A = \{3, 4, 5, 6, 7, 8\}$ *and* $B = \{2, 4, 8, 12\}$,

find: (a) $A - B$

(b) $B - A$

(c) $A \square B$

Solution

(a) $A - B = \{3, 5, 6, 7\}$

(b) $B - A = \{2, 12\}$

(c) $A \square B = (A\text{-}B) \cup (B\text{-}A$

$= \{2,3,5,6,7,12\}$

1.11 Other special sets

(a) Empty set or Null set

Activity 1.17

Describe ten empty sets.

Learning points

A set which does not contain any element is called an empty set or a null set. Examples of empty sets are;

{A set of natural numbers between $^{-}10$ and 0}.

{A set of girls in a boys only school}, etc.

An empty set is denoted as {} or ϕ.

(b) Disjoint sets

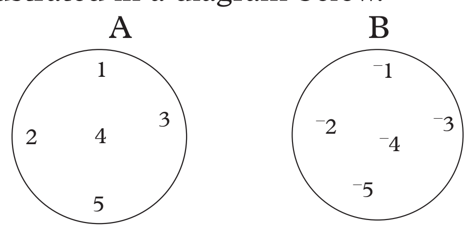
Activity 1.18

In groups of three, discuss what disjoint sets are.

Suppose $A = \{1, 2, 3, 4, 5\}$ and

$B = \{^{-}1, ^{-}2, ^{-}3, ^{-}4, ^{-}5\}$

The members of set A and those of set B are very different. No members of A belongs to set B and vice versa. Set A and Set B are said to be disjoint and can be illustrated in a diagram below.



Learning points

Sets which have no common members are called disjoint sets. For example, the set of alphabets and the set of natural numbers are disjoint sets.

Exercise 1.7

1. Given the sets $P = \{1, 2, 3, 4, 5, 6\}$ and $Q = \{2, 4, 7, 8\}$, find:

   (a) $P - Q$ (b) $Q - P$ (c) $Q \square P$

2. If set A = {tomatoes, onions, green pepper} and B = {onions, potatoes, rice}, find:
   (a) A – B (b) B – A
   (c) A □ B
3. Sets C = {3, 4, 5, 6, 7} and D = {5, 6, 7}. Find:
   (a) C – D (b) D – C
4. Set K = {1, 2, 3, 4, 5}, L = {2, 4, 6, 8, 10} and M = {4, 8, 12, 16}, find:
   (a) K - L (b) L - K (c) L - M
   (d) M - L (e) K – M (f) M – K

1.12 Comparison of sets

(a) Equivalent sets

Activity 1.19

In pairs, consider set A = {1, 2, 3}

B = {Monday, Tuesday, Wednesday, Thursday}

C = {a, b, c}

Tell your class partner, which sets are equivalent?

Suggest a reason for your answer.

Learning points

If two sets P and Q are such that n (P) = n (Q) but the elements are different, the two sets are called **equivalent** sets.

For example,

P = {1, 2, 3, 4, 5}

Q = {a, e, i, o, u},

the elements in P are different from the elements in Q.

But n(P) = n(Q) = 5

Therefore P and Q are equivalent.

Two or more sets are said to be equivalent if they have the same number of elements. For instance set A = {1, 6, 3} is equivalent to set B = {a, b, c} as they both have three elements each.

Example 1.23

Show that G = {7, 3, 2, 1} is equivalent to M = {a, z, w, n}.

Solution

The number of elements in set G is 4 and the number of elements in set M is 4, so the two sets G and M are equivalent.

(b) Equal sets

Activity 1.20

In groups of three,

1. Tell your members what equal sets are.
2. Consider the set:
   C = {Pencil, book, pen}
   D = {Table, chair, desk}

Are the two sets equal?

Discuss the difference between equal and equivalent sets

Learning points

Two or more sets that contain exactly the same elements that are equal in number, they are called **equal sets**.

For example if,

N = {1 ,2 ,3, 4, 6}

P = {1, 2, 3, 4, 6}

N and P contain the same elements and

∴ N = P ⇒ n (P) = n (N)

Two or more sets are equal if they are equivalent and their elements are exactly the same. For instance set N = {7, 8, 9} is equal to set M = {7, 8, 9}.

Exercise 1.8

State whether the pairs of sets in questions 1-6 below are equivalent, equal or neither.

1. $\{3, 2, 1\}$ and $\{1, 2, 6\}$
2. $\{a, b, z\}$ and $\{z, b, a\}$
3. $\{r, s, t\}$ and $\{r, s, m, t\}$
4. {Leah, Carol} and {Carol, Leah}.
5. {cup} and {plate, cup}.
6. {Set of positive integers} and {Set of natural numbers}

DID YOU KNOW?

It is not good to create differences with other people in the society. We should learn how to live happily and share so that we can value our peace!

1.13 General problems on sets using Venn diagram

Example 1.24

In a class, 15 students play cricket, 11play hockey, 6 play both games and everyone plays at least one of the games.

Find the total number of students in the class.

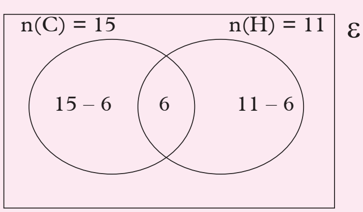
Solution

$\varepsilon = ?$

$n\ (\text{cricket}) = 15$

$n\ (\text{hockey}) = 11$

$n\ (C \cap H) = 6$



$n(\varepsilon) = 15 - 6 + 6 + 11 - 6$

$= 9 + 6 + 5$

$= 20$

Number of pupils in the class is 20.

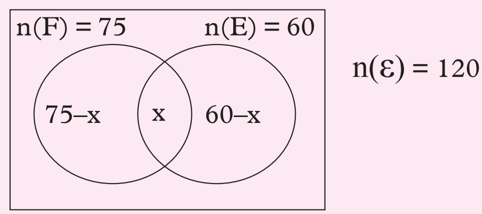
Example 1.25

All the 120 students at Fawe Girls School learn French or English or both. 75 Learn French and 60 Learn English. How many students learn both languages?

Solution

$n(\varepsilon) = 120 \qquad n\ (F) = 75 \qquad n\ (E) = 60$

$n\ (F \cap E) = x$



$$75 - x + x + 60 - x = 120$$
$$135 - x = 120$$
$$-x = -15$$
$$x = 15$$

Those who study both languages is 15

Example 1.26

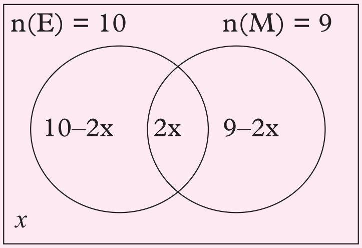
Out of 17 teachers in a school, 10 teach Economics and 9 teach Mathematics. The number of teachers who teach both subjects is twice that of those who teach none of the subjects. With the aid of a Venn diagram, find the number that teach:

(a) *Both subjects*

(b) *None of the subject*

(c) *Only one subject*

Solution



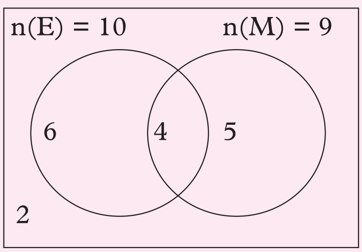
n(ε) = 17

$n(\varepsilon) = 17 \qquad n\ (E) = 10$

$n\ (M) = 9 \qquad None = x$

$n\ (E \cap M) = 2x$

$$
\begin{aligned}
10 - 2x + 2x + 9 - 2x + x &= 17 \\
19 - x &= 17 \\
-x &= 17 - 19 \\
-x &= -2 \\
x &= 2
\end{aligned}
$$



n(ε) = 17

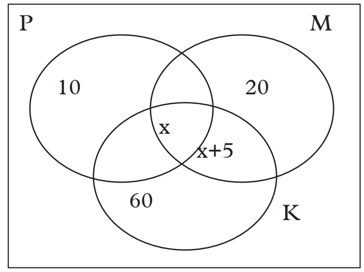
(a) Both subjects = 4 teachers

(b) None of the subjects = 2 teachers

(c) Only one subject = 6 + 5 = 11 teachers

Exercise 1.9

1. In a class of 40 students, 26 play football and 20 play volleyball. 17 students play both games. How many students play none of the games at all?
2. In a group of 20 girls, 16 play football, 12 play hockey and 2 do not play either games. How many play both games?
3. The Venn diagram below shows the number of senior one students in a school who like Mathematics (M), Physics (P) and Kinyarwanda (K). Some like more than one subject in total 55 students like Mathematics.



ε = ?

   (a) How many students like the three subjects?

   (b) Find the total number of senior one students in the school.

   (c) How many students like Physics and Kinyarwanda only?

4. In a survey of course preferences of 110 students in a senior six class, the following facts were discovered. 21 students like engineering only, 63 like engineering, 55 like medicine and 34 like none of the two courses.

   (a) Draw a Venn diagram to represent this information.

   (b) How many students like Engineering or Medicine?

   (c) How many students like Engineering and Medicine?

   (d) How many students like Medicine only?

5. A survey was carried out on 120 people about their breakfast habits. On a particular day, 55 ate eggs for breakfast that day and 40 drank juice while 25 took both eggs and drank juice for breakfast.

   (a) Represent the information on a

Venn diagram.

(b) How many people had neither eggs nor juice that morning?

6. A survey was done on 150 Rwandese about which newspapers they read. 83 read the New Times, 58 read the Monitor, 36 read neither of those two papers. How many people read both newspapers?

7. A survey was carried out on 50 people about the hotels they like for taking lunch from among Hilltop, Serena and Lemigo. It was found out that, 15 people ate at Hilltop, 30 people ate at Serena, 19 people ate at Lemigo, 8 people ate at Hilltop and Serena, 12 people ate at Hilltop and Lemigo, 7 people ate at Serena and Lemigo. 5 people ate at Hilltop, Serena and Lemigo.
   (a) Represent the information on a Venn diagram.
   (b) How many people ate at Hilltop only?
   (c) How many ate at Hilltop and Serena but not at Lemigo?
   (d) How many people did not eat from any of these three hotels?

8. A survey was carried out in a shop to find out how many customers bought bread, milk, both or neither. Out of a total of 79 customers for the day, 52 bought milk, 32 bought bread and 15 bought neither. Draw a Venn diagram to show this information and use it to find out:
   (a) How many bought bread and milk?
   (b) How many bought bread only?
   (c) How many bought milk only?

9. In a cleanup exercise carried out in Kibuye town, a group of students were assigned duties as follows; all of them were to collect waste paper. 15 were to sweep the streets but not plant trees along the streets, 12 were to plant trees along the streets 5 of them were to plant trees and sweep the streets. Draw a Venn diagram to show this information and use it to calculate the number of children in the group.

DID YOU KNOW?

It is important to maintain the environment by keeping our surrounding clean. Planting trees around us helps to keep the environment conducive by providing fresh air.

10. In a class the students are required to take part in at least two sports chosen from football, gymnastics and tennis. 9 students play football and gymnastics; 19 play football and tennis; 6 play all the three sports. If there were 30 students in the class, draw a Venn diagram to show this information. With the help of your diagram, find out how many students did not participate in any of the sports.

1.14 Relations and functions

1.14.1 Introduction to relations

Activity 1.21

1. Write down as many as possible the biological relations you have with your family members.

2. Write down any four relations you have among your classmates.
3. Are there any students in your school that you are biologically related to? If any, write down their names and the relations you have with them.
4. Write down any four mathematical relations you know.

From Activity 1.21, you have realized that you have many relations. You have used statements such as Uwimana is my aunt and so on. In a statement such as 'Mucyo is the uncle of Mutesi'; the phrase "is the uncle of" indicate that there is a biological connection between Mucyo and Mutesi. This tells us that relation is a connection between two or more things.

Examples of relations:

-is a father of, is a cousin of, is a multiple of, is a factor of, is a square of, is taller than, is seated next to and so on.

1.14.2: Properties of relations

As we have seen, a binary relation R links the elements of a set X to the elements of another set Y. This means that through a relation R, a member x contained in the set X ($x \in X$) is related to a member y contained in the set Y ($y \in Y$).

This statement is generally abbreviated as xRy. Note that order in this symbolic representation is important hence $xRy \neq yRx$.

For example let Set X = {boys} and Set Y = {fathers).

This means that a boy (x) in Set X is a son of a father y in set Y.

We can represent this relation R i.e. "*is a son of*" as xRy.

We can represent the inverse relation R' "*is a father of*" as $yR'x$.

We will use this general representation of a relation to help us understand the different types of relations. To do this, let us consider two sets, set X and set Y, and a relation R on set X.

The relation R may have one or more of the following properties:

(a) Reflexive

(b) Symmetrical

(c) Transitive.

(d) Antisymmetric

(a) Reflexive

The relation R is said to be reflexive if for all values x, xRx.

This means all values of x are related to themselves.

Examples of a symmetric relation are:

(a) "equals to" e.g. $x = x; 2 = 2$

(b) "less than or equal to" e.g. $x \leq x; 2 \leq 2$

In a reflexive relation, we have arrows for all values in the domain pointing back to themselves e.g $3 \leq 3$ *implies* $3R3$.

(b) Symmetric

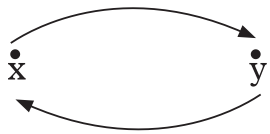
The relation R is said to be symmetric if for all values x and y,

$$xRy \text{ implies } yRx.$$

This means that the relation and its inverse are the same.

An example of a symmetric relation is "is equal to" since we can say x= y and y = x.

In an arrow diagram for a Symmetric relation, we have an opposite arrow for every arrow.



(c) Transitive

The relation R is said to be transitive if for all values x, y and z,

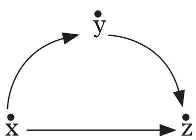
xRy and *yRz implies xRz*

Examples of a transitive relation are:

"is greater than " e.g. if $x > y$ and $y > z$, it implies $x > z$.

"is less than " e.g. if $x < y$ and $y < z$, it implies $x < z$

In an arrow diagram, every arrow between two values x and y, and y and z, has an arrow going straight from x to z.



For example

i) 2 < 3

ii) 3 < 5 then, 2 < 5

(d) Antisymmetric

The relation R is said to be antisymmetric if for all values x and y,

xRy and yRx implies $x = y$

An example of antisymmetric relation is "greater than or equal to"

For example, if x ≥ y, and y ≥ x, then y must be equal to x.

(e) Equivalence relation

An equivalence relation is one that has ALL the following properties:

- symmetric
- transitive
- reflexive

Generally, an equivalence relation is denoted by ~

Example 1.27

Prove that "=" is an equivalence relation.

Solution

Reflexive: it is true that $x = x$ for all values a. Therefore, = is reflexive.

Symmetric: If $x = y$, it is also true that $y = x$. Therefore, = is symmetric

Transitive: If $x = y$ and $y = z$, this means that $x = z$. Thus '=' is transitive.

Therefore '=' is an equivalence relation

1.15 Papygram

Activity 1.22

1. Draw a circle.
2. On it, write the names of ten people that you are related to in your family.
3. Use arrows to show the relationship that exists between you and the list of names and also the relation between the members you named.
4. What would you call such a diagram?

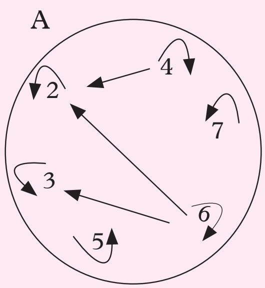
From Activity 1.22, you have drawn a papygram. A papygram is a circular representation of relationships that exist between a given set of things. The relationship is illustrated by the help of arrows.

Example 1.28

Draw a papygram to show the relation "is a multiple of" in the set of numbers

$A = \{2, 3, 4, 5, 6, 7\}$

Solution



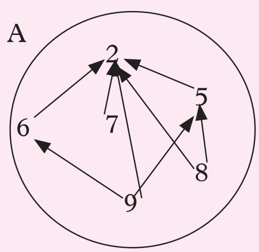
The arrow which goes back to the same number means a number is a multiple of itself.

Examples 1.29

Draw a papygram to illustrate the relation "exceeds by more than 2" for

$A = \{2, 5, 6, 7, 8, 9\}$

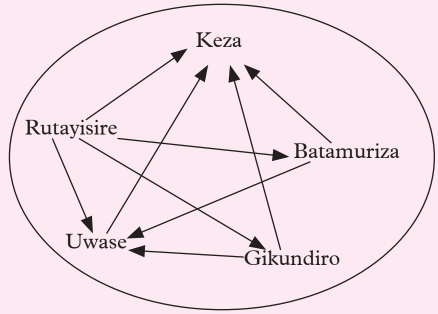
Solution



Example 1.30

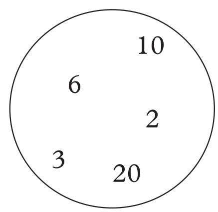
Draw a papygram to show the relation "is older than" for the following relation; Keza is 12 years old, Batamuriza is 14 years old, Rutayisire is 15 years old, Uwase is 13 years and Gikundiro is 14 years old.

Solution



Exercise 1.10

1. Draw a papygram to show the relation 'is a factor of' for $\{2, 3, 4, 6, 7, 8\}$.
2. Complete the arrow diagram to show the relation " is a multiple of".

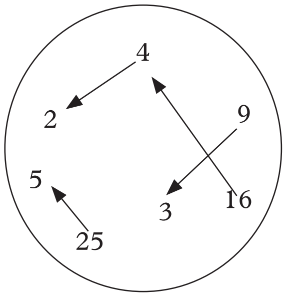


3. The following are weights of some students in a senior one class at Fawe Girls School.

Student	**Weight**
Teta	50 kg
Uwimana	36 kg
Ishimwe	47 kg
Akaliza	40 kg
Bwiza	60 kg

Draw a papygram to show the relation "is heavier than" for the students mentioned above.

4. On the papygram below, state the relation that exists.



5. Use a papygram to illustrate the relations on each of the following sets.
   (a) D = {3, 6, 9,12, 15, 18} ; is a multiple of.
   (b) E= {2, 3, 4, 5, 6, 7}; is a factor of.
   (c) F = {5, 6, 7, 8, 9, 10}; is less than.
6. Determine which of the following relations are transitive, symmetric reflexive, or antisymmetric. Note that a relation may have more than one property.
   (a) a = b
   (b) a > b
   (c) $b^3 = y^3$
   (d) m ≥ n

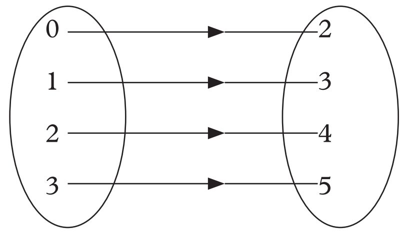
1.16 Terms used to express a relation

(a) Ordered pair

As the name suggests, an ordered pair is an entity comprising of two objects/elements put in a specified order. Suppose x and y are two objects. The ordered pair of x and y is denoted by (x, y). The first element x is known as the first coordinate (or first component), while the second element y is known as the second coordinate (or second component). The order of the elements is significant hence (x, y) ≠ (y, x).

Examples of ordered pairs are (1, 2), (father, son), (child, age), (a, b) and so on.

For example consider the relation "is two more than", shown in the arrow diagram.



We can write down a set of order pairs as{(0, 2)(1, 3)(2, 4)(3, 5)}

(b) Cartesian product

Let X and Y be two sets. The Cartesian product of sets X and Y, denoted by X × Y is the set of the ordered pair (x, y), where x ∈ X and y ∈ Y. This is expressed in symbols as

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

(The | symbol means such that.)

Example 1.31

If Set A = {a, b}, and Set B = {2, 3, 4}, find the Cartesian product A × B.

Solution

A × B is the set of all possible ordered pairs whose first component is a member of set A and the second component is a member of B

A × B = {(a, 2), (a, 3), (a, 4), (b, 2), (b, 3), (b, 4)}

Mathematical representation of relations

As we have seen, a binary relation R links the elements of a set X to the elements of another set Y. Examples of relations are *'is the son of, is a member of, is married to, loves, is heavier than,* etc.

Learning points

The binary relation R between sets X and Y describes/determines a subset of the set formed by the Cartesian product of sets X and Y i.e., a subset of the $X \times Y$, represented as $\{(x, y) \mid E \in X \times Y\}$

The particular, subset described by the relation is specified by giving the condition (predicate P(x, y)) that must be satisfied by the elements in the ordered pairs.

Learning points

As such, a binary relation R between sets X and Y is mathematically represented as follows:

$R = \{(x, y\} \in X \times Y \mid$ Condition or predicate to be met by x and y)$\}$.

Or in symbols,

$$R = \{(x, y) \in X \times Y \mid P(x, y)\}$$

Example 1.32

Set X = {daughters}, Set Y ={mothers}

The relation is ***a daughter of*** *between sets X and Y is represented as*

$R = \{(x,y) \in X \times Y \mid x \text{ is a daughter of } y)\}$

Exercise 1.11

1. Find the cartesian products of the following sets:
   (a) Set A={a, b}, Set B={2, 3}
   (b) Set C={m, n}, Set F={5, 6, 7}
   (c) Set D={p, q, r}, Set E={1, 2, 3}
2. Write in set notation the relation between the following pair of sets
   Set A ={2,3,4}, Set B{4,6,8}

1.17 Mapping

Activity 1.23

On individual basis,

1. Write down set A of the first six whole numbers.
2. Form another set B by squaring each number in set A above.
3. Write the numbers in set A in a circle and their squares in set B in a different circle on the same level side by side.
4. Use arrows to match the numbers in set A to their squares in set B.

What would you call such matching?

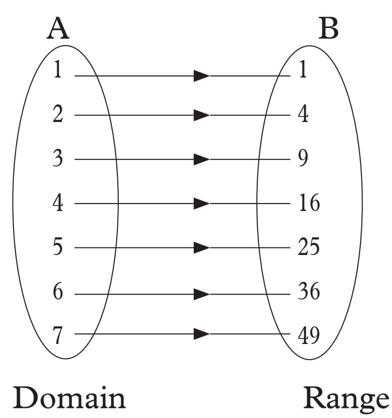
From Activity 1.23, you have squared the elements in set A to give rise to the elements in set B. We say that squaring elements in set A maps onto set B. Your diagram showed the mapping. Set A is called the domain and set B is called the range.

Consider the example below;

A = {1, 2, 3, 4, 5, 6, 7}

B = {The first seven square numbers}

By using the operation of "squaring" each member of set A produces the members of set B. We say that, the operation of "squaring" maps set A onto set B as shown below.



Domain is sometimes referred to as the input and the range as the output.

NB: The difference between relation and mapping is that the element of relations belongs to the same set while in mapping the elements belong to two different sets that is, from the domain to the range.

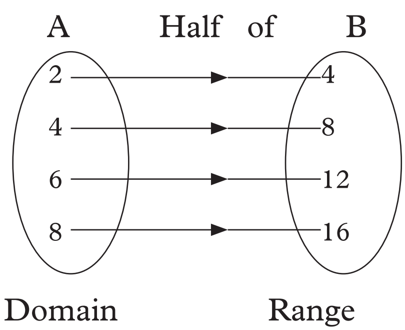
1.17.1 Types of mapping

1. One – to - one mapping

The diagram below shows the relation "is half of" between the following sets:

A= {2, 4, 6, 8}

B= {4, 8,12, 16 }



2. Many – to- one mapping

Activity 1.24

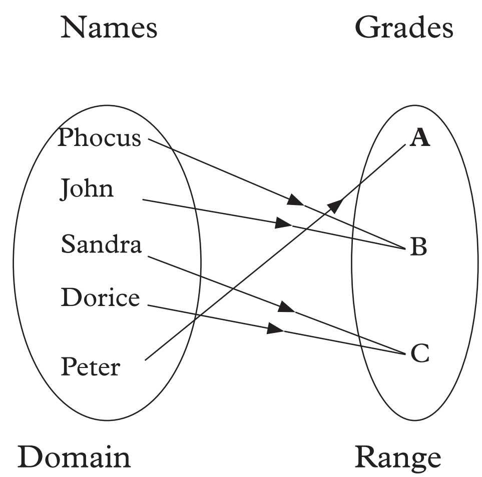
Some S1 students obtained the following grades:

Name	Grades
Phocus	B
John	B
Sandra	C
Dorice	C
Peter	A

Individually,

1. Use the names as the domain and the grades as the range to show the mapping.
2. What name would you call such a mapping?

From Activity 1.24 above, you have observed that several elements in the domain set can be mapped onto one element in the range set. This is called many to one mapping. From the above activity, you should have obtained the following mapping.



3. One- to – many mapping

Activity 1.25

Carry out the following activity.

1. Each student to give three names of his/her siblings. List the names down on a piece of paper.
2. On the domain set S, write the names of each one of you in the group.
3. On the range set T, write the nine names of the siblings listed in step 1 above.
4. Match the name on the domain set to the names of the siblings in the range set with arrows.

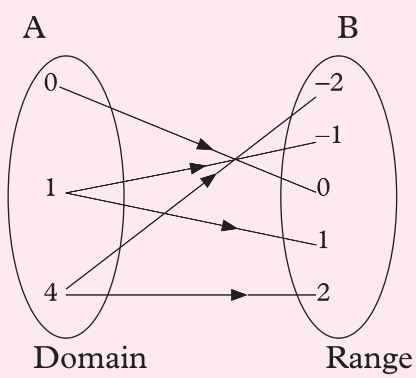
5. What name would you call such a mapping?

From Activity 1.25 above, you have observed that an element in the domain set can be mapped onto more than one element in the range set. Such a mapping is called one to many mapping.

Example 1.33

Show the mapping of the relation "is square of" from A to B where $A = \{0, 1, 4\}$ *and* $B = \{-2 \leq x \leq 2\}$.

Solution



4. Many – to –many mapping

Activity 1.26

Carry out the following activity.

1. You are provided with the following to form the domain set: cat, dog, buffalo, lion, hen, duck, cow.
2. You are also given the following to form the range: carnivorous, herbivorous, domestic, wild, bird, omnivorous.
3. On a sheet of paper, show the above mapping.
4. Are there elements in the domain set that map onto several elements in the range set?
5. Are there more than one element in the domain set mapped onto the same or several elements in the range set?

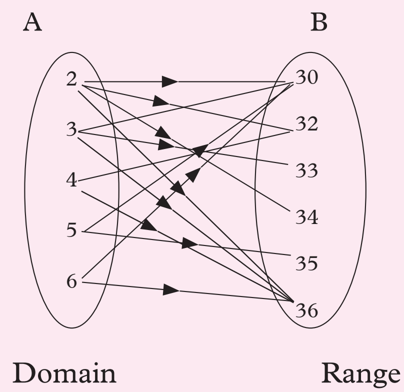
What name would you call such a mapping?

From Activity 1.26 above, you have observed that several elements in the domain set are mapped onto more than one element in the range set. This is called many to many relations.

Example 1.34

Show the mapping of the relation "is a factor of" for the sets $A = \{2, 3, 4, 5, 6\}$ *and* $B = \{30, 32, 33, 34, 35, 36\}$.

Solution



Exercise 1.12

1. Set $M = \{1, 2, 3, 4, 5\}$ is mapped onto set N by the relation "multiply by 4". List the elements of set N and map M onto N.
2. A set A maps onto set B by the operation "multiply by 3 and add 1". The elements of set A are $\{5, 6, 7, 8, 9\}$
   (a) List the element of set B.
   (b) Map set A onto B.
   (c) What type of mapping is this?
3. Optional subjects at Nyagatare

high school are Entrepreneurship, Literature, Kiswahili, Music and Computer science. Murekatete takes Kiswahili and Music, Gatete takes Computer science and Entrepreneurship, Joy takes Literature and Computer science and Hope takes Kiswahili only. Map pupils to the subjects they take. What type of mapping is this?

4. The marks obtained in a test by Carine, Bernice, Delice, Sandrine, John and Sifa were 5, 4, 7, 5, 9 and 7 respectively. Illustrate the mapping of marks to students. What type of mapping is this?

1.18 Graphs of relation

Activity 1.27

Carry out the following activities.

1. Given set A = {1, 2, 3, 4, 5, 6} and the relation "multiply by 2 add 1", Find set B, the range.
2. Using sets A and B, order them in pairs such that an element in set A is paired to an element it is mapped onto in set B.
3. What name would you call such a pairing in step 2 above?
4. Draw a cartesian plane in your graph books and plot the pair on points in step 3 above. Let the domain be in the x – axis and the range be in the y – axis.
5. Use a straight line to join the points you plotted in step 4 above.
6. What name would you call such a graph?

From activity 1.27, you should have observed the following

1. B = {3, 5, 7, 9, 11, 13}
2. Ordered pairs of set A and B are (1, 3) (2, 5) (3, 7) (4, 9) (5, 11) (6, 13)
3. The pairs in steps (2 above are called ordered pairs
4. The graph in step 4 represents a relation between corresponding values of sets A and B. If we denote the elements of A as a and those of set B as b, the relation between them can be denoted as $b = 2a + 1$. The graph is a straight line.

Graphs of relation are drawn by pairing the domain to the range where the domain represent the values of x and the range represent the values of y as an ordered pair (x, y).

Example 1.35

Consider the domain $A = \{0, 1, 3, 4\}$ set A is mapped onto set B by the relation $x \rightarrow 2x$. Find the range, the ordered pair and hence draw the graph of the relation.

Solution

The range of values will be:

$x \rightarrow 2x$

If $x = 0 \rightarrow 2 \times 0 = 0$

$x = 1 \rightarrow 2 \times 1 = 2$

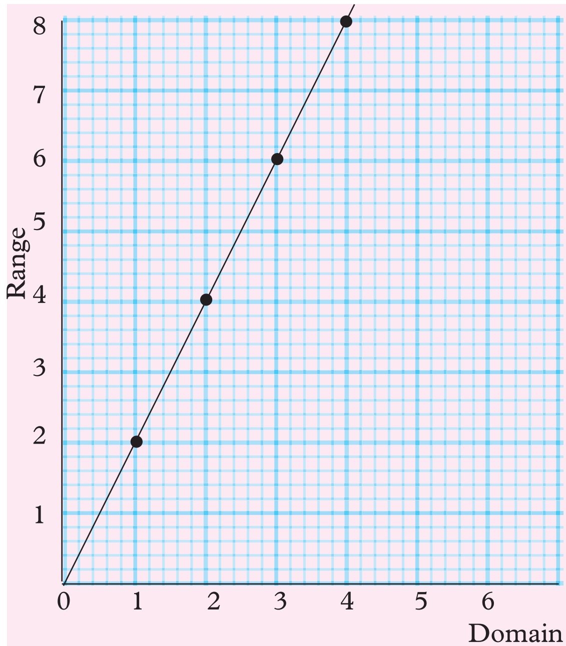
$x = 3 \rightarrow 2 \times 3 = 6$

$x = 4 \rightarrow 2 \times 4 = 8$

Range = {0, 2, 6, 8}

Ordered pairs: (0, 0) (1, 2) (3, 6) (4, 8)

The graph takes the following shape.



Example 1.36

Consider the relation "divide by 2 and subtract 2 from the result" for the set x = {6, 8, 10, 12, 14}. Find the range, the ordered pair and hence draw the graph of the relation.

Solution

This relation can be written as $x \rightarrow \frac{x}{2} - 2$

if $x = 6, \rightarrow \frac{6}{2} - 2 = 1$

$x = 8, \rightarrow \frac{8}{2} - 2 = 2$

$x = 10, \rightarrow \frac{10}{2} - 2 = 3$

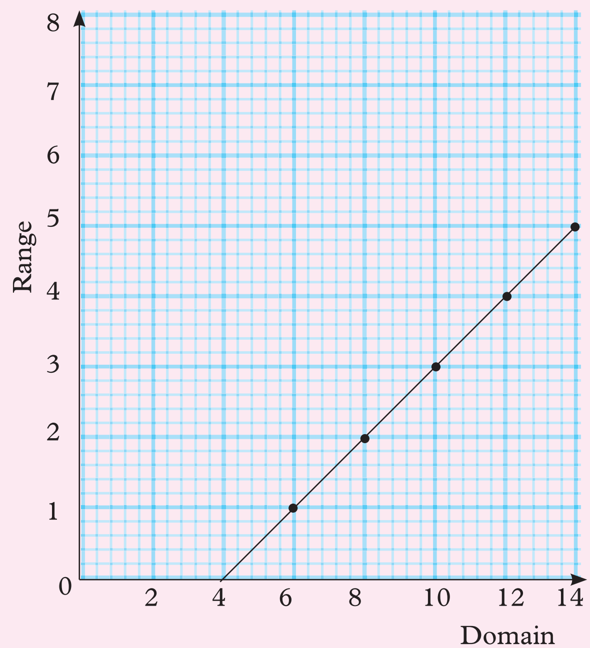
$x = 12, \rightarrow \frac{12}{2} - 2 = 4$

$x = 14, \rightarrow \frac{14}{2} - 2 = 5$

Range = {1, 2, 3 ,4 ,5 }

Ordered pairs: (6, 1), (8, 2), (10, 3), (12, 4), (14, 5)

The graph representing this is as shown.



BE WARE!!

Do you know that one –to- many or many –to- many sexual relationships between members of the opposite sex can spread HIV/ AIDS? You are advised to stay away from sex before marriage.

Exercise 1.13

1. Given A= {0, 1, 2, 3, 4}, use the relation " is multiplied by 2" to list the elements of B. Draw the Cartesian graph to represent this information.
2. Given set P = {2, 4, 6, 8, 10}, use the relation " half the value " to give set Q.
   (a) List the elements of set Q.
   (b) Map set P onto set Q.
   (c) Draw the graph of P against Q.
3. Given the domain={0, 1, 2, 3, 4, 5}, list the elements of the range and draw the graph to represent the given

relation in each case.

(a) Multiply by 3.

(b) Multiply by 2 and add 1.

(c) Multiply by 3 and subtract 2.

4. Draw a graph for the relation $x \to 4x$ for the domain $\{0, 1, 2, 3, 4\}$.

5. Given the domain $\{x : -3 \leq x \leq 3\}$, use the relation "square " to list the element of the domain and the range. Map the relations.

6. If $P= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, write down the ordered pair to illustrate each of the relation Q on set P.

(a) $Q: x \to x^2$

(b) $Q: x \to x + 5$

(c) $Q : x \to x - 3$

1.19 Functions

Activity 1.28

Work out the following:

(a) $x \to 2x^2$, when $x = -3$

(b) $y \to y^2 + 1$, when $y = -1$

(c) $t \to 2t^2$, when $t = 4$

(d) $m \to \frac{m}{2} + 5$, when $m = 4$

Choose one representative from your group to illustrate on the chalkboard their findings.

From Activity 1.28, you have realized that a function is a type of mapping in which every object on the domain has one and only one image in the range. Mappings, which are functions, are

one – to - one mappings and many – to-one mappings.

Consider a function such as $x \to 4x$. Let f be the function $x \to 4x$. One way of writing this is $x \to 4x$. This is read as : f is the function which maps x onto 4x. The most common and important notation is $f(x) = 4x$ read as f of x equals $4x$.

Example 1.37

Given the function $f(x) = 4x + 2$, find $f(2)$.

Solution

$f(x) = 4x + 2$

$f(2) = 4 \times 2 + 2 = 8 + 2$

$= 10$

$f(2) = 10$

Example 1.38

The function $g(y) = 3y^2 + 2y$. Find $g(2)$.

Solution

$g(y) = 3y^2 + 2y$

$g(2) = 3(2)^2 + 2(2)$

$= 4 \times 3 + 4 = 16$

Hence, $g(2) = 16$

Example 1.39

The function $h(x) = ax^2 - 2$ and $h(2) = 18$.

Find: (a) *the value of a.*

(b) $h(-1)$

(c) $h(3)$

Solution

(a) $h(x) = ax^2 - 2$

$h(2) = a(2)^2 - 2 = 4a - 2$

$4a - 2 = 18$ *since* $h(2) = 18$,

Then, $4a - 2 = 18$

implying $4a = 20$

Therefore, $a = 5$

(b) *Substitute the value of a*

$h(x) = 5x^2 - 2$

So, $h(-1) = 5(-1)^2 - 2$

$\Rightarrow h(-1) = 3$

(c) $h(x) = 5x^2 - 2$

So, $h(3) = 5(3)^2 - 2 = 5 \times 9 - 2 = 43$

$\Rightarrow h(3) = 43$

Example 1.40

The function $f(x) = \frac{x^2 - 1}{2x - 1}$. *Find* $f(4)$.

Solution

$$f(x) = \frac{x^2 - 1}{2x - 1}$$

$$\text{So } f(4) = \frac{4^2 - 1}{2 \times 4 - 1} = \frac{16 - 1}{8 - 1} = \frac{15}{7}$$

$$\text{Hence, } f(4) = \frac{15}{7}$$

Exercise 1.14

1. Given that $f(x) = 3x^2 + 2$, find $f(2)$.
2. The function $g(x) = 2x^2 + 3x + 1$. Find $g(-3)$.
3. If $f(x) = \frac{x^2 + 1}{x - 3}$, find $f(4)$.
4. If $f(y) = 2y - 3$, find:
   (a) $f(-4)$ (b) $f(0)$ (c) $f(2)$
5. If $g(x) = \frac{x^2 + 1}{x - 3}$, find:
   (a) $g(4)$ (b) $g(0)$ (c) $g(-4)$
6. The function $f(x) = 2x^2 + b$ and $f(-2) = 10$. Find the value of b.
7. Given that the function $g(y) = cy + 9$ and $g(5) = 49$, find the value of c and $g(4)$.
8. It is given that $f(x) = \frac{ax}{x^2 - 2}$ and $f(2) = 12$. Find:
   (a) The value of a. (b) $f(1)$
9. The function $f(x) = 3x$ has domain values as {1, 2, 3, 4, 5}. Find the values of the range.
10. If $f(x) = 2 - \frac{1}{2}x$ has the domain {-2, 0, 2, 4, 6}, find the range.
11. $f(x) = 3x^2 + bx - 3$. Find b when $f(2) = 15$.
12. $h(x) = ax^2 + 5x - 3$ and $h(1) = 0$. Find:
    (a) The value of a (b) $h(2)$
    (c) $h(-2)$ (d) $h(0)$
    (e) $h(5)$

1.20 Inverse of a function

Activity 1.29

1. Work out the following and show the working clearly.
   $y + 3 = 0$ $x - 2 = 0$
   $2x + 4 = 0$ $\frac{x}{2} + 6 = 0$
   $x^2 - 16 = 0$
2. Explain your method step by step?
3. Volunteer to show your working on the board.

From Activity 1.29, you should have noticed that for a simple equation of the form $x + a = b$ where a and b are constants,

(i) We obtain x by subtracting a from both sides and simplifying the right hand side of the equation.

(ii) If $x - a = b$, we obtain x by adding a to both sides and simplifying the right hand side (RHS).

(iii) For ax = b, we divide both sides by a and simplify the RHS to obtain x.

(iv) If $\frac{x}{a}$ = b, we multiply both sides of the equation by a and simplify the RHS.

Example 1.41

Consider the function $f(x) = 2x$. If the domain is $\{0, 1, 2, 3, 4\}$, find the range values. Hence, draw the arrow diagram.

Solution

The range values will be:

$f(0) = 2 \times 0 = 0$

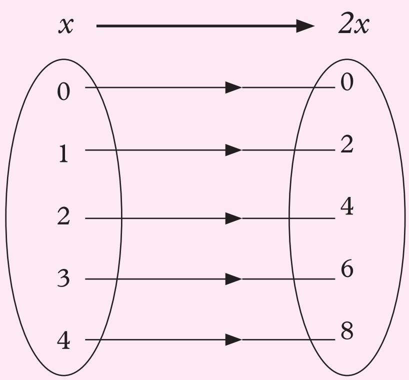
$f(1) = 2 \times 1 = 2$

$f(2) = 2 \times 2 = 4$

$f(3) = 2 \times 3 = 6$

$f(4) = 2 \times 4 = 8$

The arrow diagram for $f(x) = 2x$ will be as follows.

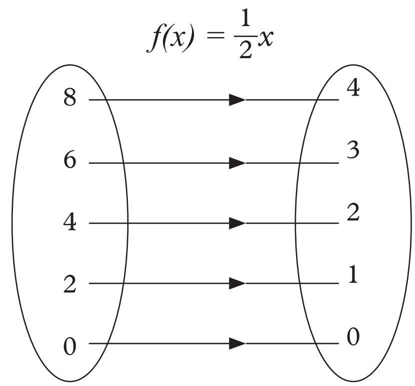


Activity 1.30

Carry out the following activities:

1. Consider the function in Example 1.41 above.
2. Use the domain in the Example and the function f(x) = $\frac{1}{2}$x.
3. Obtain the range.
4. Compare the range you obtained in step 3 above to the range in Example 1.27.
5. What do you notice?

From Activity 1.30, you should notice that, when you change the function to $f(x) = \frac{1}{2}x$, it takes the reverse effect of $f(x) = 2x$ on the same domain. This is called the inverse of the function denoted as $f^{-1}(x)$. From the activity above, you should obtain the following arrow diagram.



Example 1.42

Find the inverse of the following functions ($f^{-1}(x)$):

(a) $f(x) = x + 4$

(b) $g(x) = 2x - 3$

(c) $f(x) = x^2 + 1$

(d) $f(x) = 3x^2 + 2$

Solution

(a) $f(x) = x + 4$

Let $y = f(x)$

$y = x + 4$ *(Interchange the y and x)*

$x = y + 4$ *(Solve for y)*

$x - 4 = y$

$y = x - 4$

$f^{-1}(x) = x - 4$

(b) $g(x) = 2x - 3$

Let $y = g(x)$

$y = 2x - 3$ (Interchange y and x)

$x = 2y - 3$ (Solve for y)

$x + 3 = 2y$

$2y = x + 3$

$y = \frac{x+3}{2}$

$g^{-1}(x) = \frac{1}{2}(x+3)$

(c) $f(x) = x^2 + 1$

Let $y = x^2 + 1$ (Interchange y and x)

$x = y^2 + 1$ (Solve for y)

$x - 1 = y^2$

$\sqrt{y^2} = \sqrt{x-1}$

$y = \sqrt{x-1}$

$f^{-1}(x) = \sqrt{x-1}$

(d) $f(x) = 3x^2 + 2$

Let $y = 3x^2 + 2$ (Interchange y and x)

$x = 3y^2 + 2$ (Solve for y)

$x - 2 = 3y^2$

$\frac{3}{3}y^2 = \frac{x-2}{3}$

$\sqrt{y^2} = \sqrt{\frac{x-2}{3}}$

So, $y = \sqrt{\frac{x-2}{3}}$

Hence $f^{-1}(x) = \sqrt{(\frac{1}{3}(x-2))}$

Exercise 1.15

Find the inverse of the following functions

1. $f(x) = x - 6$
2. $f(x) = x + 2$
3. $g(x) = 2x + 3$
4. $g(x) = 3x - 1$
5. $f(x) = x^2 + 2$
6. $f(x) = 3x^2 - 1$
7. $h(x) = 2x$
8. $h(x) = 4 - 9x^2$
9. $h(x) = \frac{1}{2}x$
10. $f(x) = \frac{1}{3}x^2 - 1$
11. $g(x) = x$
12. $g(x) = \frac{1}{x}$

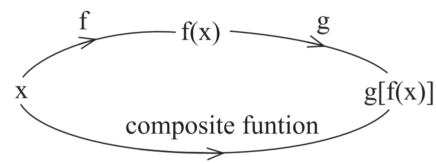
1.21 Composite functions

Activity 1.31

Carry out the following activities:

1. You are given the functions $f(x) = 3x$ and $g(x) = x + 2$.
2. Use a number such as 4 and find $f(4)$ then apply the funtion g to the results i.e. $g[f(4)]$ i.e start with function *f* followed by function *g*.
3. What value did you obtain?
4. What name would you call a function composed of f and g?

From Activity 1.31, you should learn that when you combine two or more functions together to form one, the resultant function is called composite function.



From the activity, you should triple number 4 first and then add 2. This can be illustrated as below:

$[4 \rightarrow 12 \rightarrow 14]$

These two stages $f(4) = 12$ and $g(12) = 14$ are combined as $gf(4) = 14$. This means functions *f* followed by function *g*.

Example 1.43

Given that $f(x) = x + 3$ and $g(x) = 2x$, find:

(a) $fg(x)$ (b) $gf(x)$

Solution

(a) $fg(x) = f[gx] = f(2x)$

Replace x by 2x to get $f(2x) = 2x + 3$

Hence $fg(x) = 2x + 3$

(b) $gf(x) = g[fx] = g(x + 3)$

Replace x by x + 3 to get

$g(x + 3) = 2(x + 3)$

Hence $gf(x) = 2x + 6$.

Example 1.44

The function $f(x) = 2x - 1$ *and* $g(x) = x^2 + 2$

Find :

(a) $fg(x)$ *(b)* $gf(x)$ *(c)* $gf(3)$

Solution

(a) $fg(x) = f[g(x)] = f(x^2 + 2)$

Replace x by $x^2 + 2$ *to get*

$f(x^2 + 2) = 2(x^2 + 2) - 1$

This gives $fg(x) = 2x^2 + 4 - 1$

$= 2x^2 + 3$

Hence $fg(x) = 2x^2 + 3$

(b) $gf(x) = g[f(x)] = g(2x - 1)$

Replace x in $x^2 + 2$ *by* $(2x - 1)$ *to get*

$$g(2x - 1) = (2x - 1)^2 + 2$$
$$= (4x^2 - 4x + 1) + 2$$
$$= 4x^2 - 4x + 3$$

(c) $gf(3) = 4 \times 3^2 - 4 \times 3 + 3$

$$= 36 - 12 + 3$$
$$= 24 + 3$$

Hence $gf(3) = 27$

Exercise 1.16

1. Given the function $f(x) = 4x$ and $g(x) = x - 2$, find:
   (a) $gf(x)$ (b) $fg(x)$
2. Given that $f(x) = 3x - 1$ and $g(x) = 2x + 5$, find:
   (a) $fg(x)$ (b) $gf(x)$
3. If $g(x) = xz$ and $f(x) = 3x$, find: (a) $gf(x)$ (b) $fg(x)$
4. Given the following functions, find $fg(x)$:
   (a) $f(x) = 2x$, $g(x) = x + 3$
   (b) $f(x) = 2x + 1$, $g(x) = x - 3$
   (c) $f(x) = x - 1$, $g(x) = 2x^2 - 3$
   (d) $f(x) = x^2 - 1$, $g(x) = x + 1$
5. Given that $f(x) = 3x + 4$ and $g(x) = x - 1$, find:
   (a) $fg(x)$ (b) $gf(x)$
   (c) $gf(2)$
6. If $f(x) = x^2 + 1$ and $g(x) = 2x$, find:
   (a) $fg(x)$ (b) $gf(x)$
   (c) $gf(2)$ (d) $fg(2)$
7. If $f(x) = 3x$ and $g(x) = x^2 + 3$, find the value of *x* for which, $gf(x) = fg(x)$.
8. If $f(x) = 2x + 3$ and $g(x) = 3x$, find $fg(x)$.
9. The function $f(x) = 2x - 1$ and $g(x) = x + 5$, find $fg(x)$.
10. Given that $f(x) = 3x + 1$, $g(x) = 2x - 5$ and $h(x) = x^2 - 4$, find:
    (a) $fgh(x)$ (b) $hgf(x)$
11. If $f(x) = 3x + 1$, find $f^2(x)$.
12. The function $f(x) = 2x - 5$, find $f^2(x)$.

DID YOU KNOW?

Functions are applied in financial mathematics and other areas of economics and entrepreneurship? Such as demand and supply, calculating simple and compound interests and many more.

Summary

1. Set – It is a group of items with a common feature.
2. Member of a set – It is an object or item in a set.

3. Subset – It is a set which is formed by obtaining some elements in all the elements from a given set.
4. Venn diagram – It is a circular or rectangular pattern used to represent sets and its elements.
5. Intersection of sets – It is the set formed by common elements which appear in two or more sets.
6. Union of sets – It is the set formed by putting together elements of two or more sets.
7. Complement of a set – It is a set of all elements in the universal set that are not members of a given set.
8. Difference between sets A and B (A-B) – It is a set formed by the elements appearing in set A but not in set B.
9. Relation – It is a connection between two or more numbers or things.
10. Papygram – It is a circular representation of the relationship that exists between a given set of numbers or objects.
11. Domain – It is the input set in mapping.
12. Range – It is the output set in mapping.
13. Function – It is a type of mapping in which every object on the domain has one and only one image in the range.
14. Composite function – It is a function formed by combining two or more functions.

Unit Test 1

1. Use set symbols to write the following sets.
   (a) Monday is a member of the days of the week.
   (b) {2, 4, 6, 8} is a subset of even numbers.
   (c) {Lion, cheetah, cat} is a subset of the cat family.
   (d) {–2, –8, –0} is not a subset of natural numbers.
   (e) {Cow, dog, hyena} is not a subset of domestic animals.
2. Given that set P = {natural numbers less than 20}, list the elements of the following subsets.
   (a) Set A = {even numbers}
   (b) Set B = {odd numbers}
   (c) Set C = {prime numbers}
   (d) Set D = {multiples of 4}
   (e) Set E = {multiples of 3}
3. Consider the sets A{1, 2, 3, 5, 6, 8}, B {2, 4, 6, 8} and C{1, 3, 5, 7}. Find and draw the Venn diagrams for:
   (a) $B \cap C$ (b) $A \cap C$
   (c) $A \cap B$ (d) $B \cup C$
   (e) $A \cup B$.
4. Given ε = {letters of the word elephatiasis}, Set A = {all vowels}, Set B = {first five letters of the English alphabet} find:
   (a) A (b) B
   (c) $A \cup B$ (d) $A \cap B$
   (e) $B - A$ (f) $A - B$

5. At a department in a university, 100 students were enrolled. 38 took Mathematics, 20 took Economics and 3 took both Mathematics and Economics. Draw a Venn diagram to represent this information.
   (a) How many students took none of the subjects?
   (b) How many students at least took Maths or Economics?
   (c) How many students took exactly one subject?
6. In a family of 48, 24 members like cassava and 36 members like potatoes. 12 members like both cassava and potatoes.
   (a) How many members like cassava only?
   (b) How many members like neither cassava nor potatoes?
7. Draw a papygram to illustrate the relation mentioned below for the given sets:
   (a) A = {4, 6, 7, 9, 12, 13}; the relations is: "exceeds by more than 3"
   (b) B= {- 2, -1,0 ,1 ,2 , 3, 4, 5, 6, 7, 8, 9 }; Relation: "is the square root of"
   (c) C = {6, 7, 8 , 9 }; Relation is "is less than or equal to"
8. Set A maps onto set B by the operation "divide by 2 add 3". If set A = {4, 8, 12, 16, 20, 24};
   (a) list the elements of set B.
   (b) Find the ordered pairs.
   (c) Draw the graph of the relation.
9. The function $f(x) = 2x^2 - 1$. Find:
   (a) f(4) (b) f(2) (c) f(10)
10. Find the inverse of the following functions:
    (a) $f(x) = x + 3$
    (b) $f(x) = 3x^2 + 2x – 1$
    (c) $f(x) = 2x^2 + 1$
    (d) $f(x) = 6x – 5$
11. Given the function $f(x) = 5x$ and $g(x) = 2x – 2$, find
    (a) f g(x) (b) g f(x)

2 SETS OF NUMBERS

Key Unit Competence

By the end of this unit, I should be able to use operations to explore properties of sets of numbers and their relationships.

Unit outline

- Sets of numbers, its subsets and notation.
- Rational numbers.
- Irrational numbers.
- Relationship between sets of numbers.

2.1 Sets of numbers, its subsets and notation

Activity 2.1

Carry out the following activities:

1. How many sets of numbers do you know? List them down.
2. Using a mathematical dictionary or the internet, define the sets of numbers you listed in step 1 above.
3. Give an example of each set of numbers you listed.
4. State the relationship between the set of numbers that you listed.
5. One of you to present your work to the rest of the class.

From activity 2.1, you learnt that the universal set of numbers contains other sets (subsets) such as the following:

1. **Natural numbers:** are counting numbers and include zero. A set of natural numbers is denoted by N i.e N = {0, 1, 2, 3, 4, 5 …….}.
2. **Integers:** Are whole numbers, negative whole numbers and zero. A set of integers is denoted by z i.e. z = { …. -3, -2, -1, 0, 1, 2, 3, 4 ….}
3. **Decimal numbers:** Are numbers whose fractional part is written starting with a decimal point followed by digits e.g. 0.5, 3.14, 0.143 etc. The number to the left of the decimal point is a whole number. The first number to the right of the decimal point represents tenths $(\frac{1}{10})$, the next one hundredths $(\frac{1}{100})$ and so on.
4. **Rational numbers:** are numbers that can be expressed as a fraction where the denominator and the numerator are integers. The set of rational numbers is denoted by $Q = \{\frac{3}{2}, \frac{-1}{20}, 0.143, 16, -18, -0.21\}$
5. **Irrational numbers:** are numbers that cannot be expressed as a quotient of two integers such that the denominator is not zero i.e. cannot be expressed as a ratio. An irrational number is denoted by I e.g. $I = \{3.14159 \ldots, \sqrt{2}, 2.71828, \ldots \sqrt{99}\}$
6. **Real numbers:** are numbers that can be found on a numberline. They include both rational and irrational numbers. A set of real numbers is denoted by R e.g. $R = \{\sqrt{2}, 1.5, -3.4, \frac{1}{2}\}$

Sets of numbers are related to each other in the sense that one set of numbers is a subset of another set of numbers. This is shown by the following Venn diagram (Fig. 2.1).

In this unit, we will learn about the set of real numbers and its subsets e.g. rational numbers and irrational numbers.

Venn diagram showing the sets of numbers.

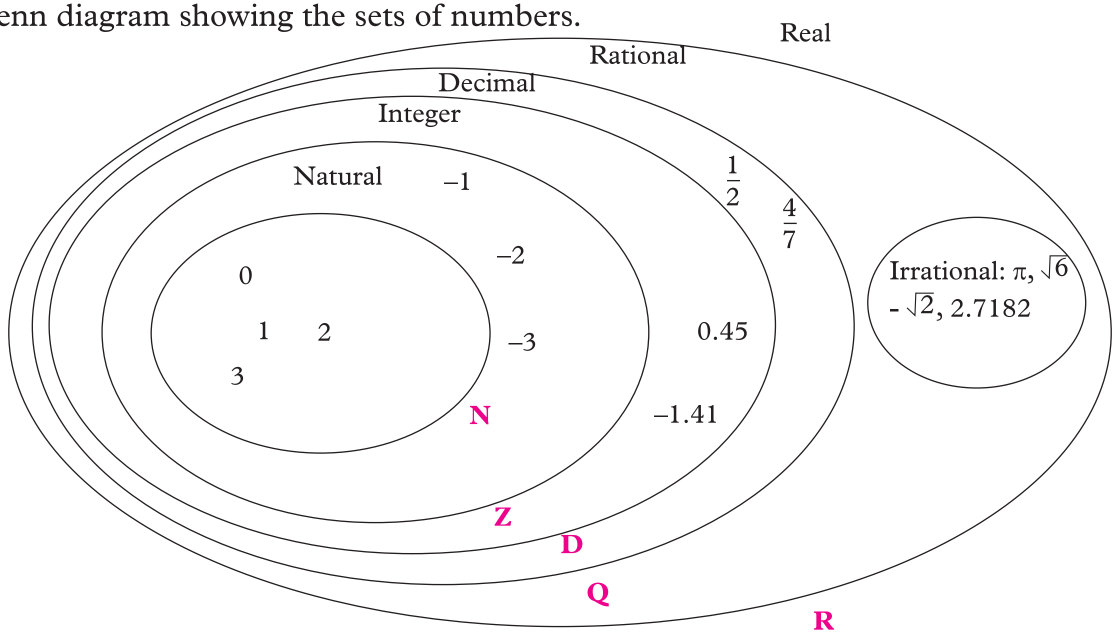


Fig. 2.1

2.2 Set of natural numbers (N)

Activity 2.2

Carry out the following activities:

1. Count the following objects in your class.
   (a) desks.
   (b) Mathematics textbooks in your class.
   (c) pencils in your class.
   (d) students in your class.
2. Write the following numbers in words:
   (a) 12 679 (b) 789 000
   (c) 12 345 798 (d) 363 879 987
3. What name do you call such numbers?
4. What is the place value of each digit in step 2 above?

From Activity 2.2 above, you realize that when counting, we usually begin by one, followed by two, then three and so on. The numbers we use in counting are called Natural numbers.

The set of natural numbers is denoted by
N= {0, 1, 2, 3, 4, ...}

On a number line, natural numbers are represented as follows:

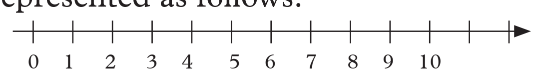


Fig. 2.2

The dotted lines and the arrow head indicate that the numbers can go on forever in the direction shown. We therefore say, they are infinite (have no end).

The number 1 264 583 for instance is read as: One million two hundred sixty four thousand five hundred eighty three. The number 12 345 798 is read as twelve million three hundred forty five thousand seven hundred ninety eight in words.

The place value of a number shows us the position of a specific digit in a number.

For example, the place value of 60 984 is as shown in tabe 2.1.

Place value	Ten thousands (Tth)	Thousands (T)	Hundreds (H)	Tens (T)	Ones (O)
Number	6	0	9	8	4

Table 2.1

Example 2.1

In Rwanda, the Genocide against the Tutsi happened in 1994. Write the place value of each digit in the year 1994.

Solution

The year 1994 has four digits. The place value of 1 is thousands, the first 9 is hundreds, the second 9 is tens and 4 is ones.

1 9 9 4

Ones

Tens

Hundreds

Thousands

Let us always love one another as brothers and sisters. Peace, love and unity are important aspects in our day to day living.

Example 2.2

Write the following numbers in words:

(a) *917 405* *(b)* *80 165*

(c) *439 357 890*

Solutions

(a) *917 405* *Nine hundred seventeen thousands four hundred and five.*

(b) *80 165* *Eighty thousand one hundred sixty five.*

(c) *439 357 890* *Four hundred thirty nine million three hundred fifty seven thousand eight hundred ninety.*

2.2.1 Subsets of Natural Numbers

Activity 2.3

Carry out the following activities.

(a) You are given the set of natural numbers between 0 and 20.

(b) Use a mathematical dictionary or internet to define the terms: even, odd and prime numbers.

(c) From the set described in part (a) above:

(i) Draw the subset of odd numbers.

(ii) Draw the subset of even numbers.

(iii) Draw the subset of prime numbers.

(iv) How many even numbers are prime numbers?

(v) How many odd numbers are prime numbers?

(d) Represent the above subsets of numbers in a Venn diagram.

From Activity 2.3, you should have found several subsets of natural numbers as follows:

(a) Even numbers

Even numbers are numbers which are divisible by 2 or numbers which are multiples of 2.

From Activity 2.3, even numbers from 0 to 20 are 2, 4, 6, 8, 10,12, 14, 16, 18 and 20.

The set of even numbers is E= {2, 4, 6, 8, ..}.

(b) Odd numbers

These are numbers which leave a remainder of 1 when divided by 2.

From Activity 2.3, odd numbers between 0 and 20 are 1,3,5,7,9,11,13,15,17 and 19.

The set of odd numbers is D= {1, 3, 5, 7, .}.

(c) Prime numbers

These are numbers which are divisible by 1 and themselves only.

From the Activity 2.3, prime numbers between 0 and 20 are 2, 3, 5, 7, 11, 13, 17 and 19.

The set of prime numbers is P = {2, 3, 5, 7, 11, 13 ....}.

Example 2.3

Given the following numbers 25, 44, 4, 51, 256, 7, 8, 21, 2 draw a Venn diagram showing the following subsets.

(a) Even numbers *(b) Odd numbers*

(c) Prime numbers

Solution

Let E = set of even numbers, D be the set of odd numbers and P be the set of prime numbers.

The Venn diagram will be as shown below.

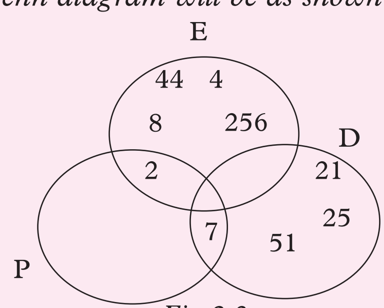


Fig. 2.3

Example 2.4

Given ε = {Natural numbers less than 20}, show on a Venn diagram the subsets E, O and P.

P={Prime numbers}

E= {Even numbers}

O= {Odd numbers}

Solution

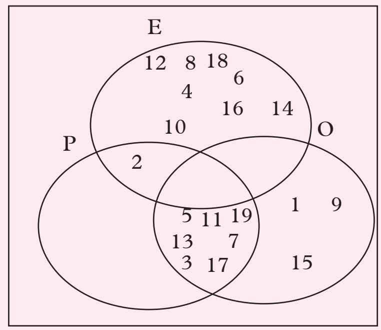


Fig. 2.4

Exercise 2.1

1. Write the following numbers in words:
   (a) 234 (b) 68 901
   (c) 41 343 209 (d) 8 631
   (e) 3 095 542 120 (f) 2 031 401
2. Write the following numbers in numerals:
   (a) Eight hundred and twenty one.
   (b) Sixty five thousand five hundred and thirty two.
   (c) Ninety two thousand eight hundred and fifty four.
   (d) Five million and fifty three thousand.
   (e) One billion thirty seven million four hundred and two.
   (f) Two hundred forty thousand and four.
3. Set ε ={1, 2, 3, 4, 5, 6, 7, 8,.., 50}. Write down the subsets of set A which are;

(a) Even numbers
(b) Odd numbers
(c) Prime numbers

Represent the above in a Venn diagram.

4. Given ε = {1, 4, 8, 11, 16, 25, 49, 53, 75}, list the elements of the following subsets
(a) Even numbers
(b) Odd numbers
(c) Prime numbers

Represent the above on a Venn diagram

2.2.2 Operations on Natural Numbers

Activity 2.4

Add, subtract, multiply and divide the following pairs of natural numbers

(a) 20 and 30 (b) 14 and 17
(c) 12 and 48 (d) 8 and 7
(e) 1000 and 1999

There are four operations we shall look at and their properties:

(a) Addition

The symbol for addition is (+). Given natural numbers **a, b**, and **c**. **a + b = c, c** is the sum of numbers **a** and **b.**

Properties of addition

Activity 2.5

Work out the following:

(a) 24 + 46 and 46 + 24. What is the difference between the two sums? What is the name given to such a property?
(b) (32 + 52) + 47 and 32 +(52 + 47). Is there any difference between the two sums? What name would you call such a property?
(c) 0 + 67 and 67 + 0. Comment on this sum. What name would you call such a property?

From Activity 2.5, you have realized the following:

1. Given numbers a and b the sum a and b is equal to the sum b and a. From the Activity 2.5 part (a) 24 + 46 = 46 + 24 = 70. This is called the commutative property of addition. When you interchange the terms, the sum remains the same.

   For example, 5 + 3 = 3 + 5 = 8
2. Given the natural numbers a, b, and c, from Activity 2.5 part (b) you realize that (32 + 52) + 47 = 32 + (52 + 47). This is called the associative property of addition.

   The associative property allows regrouping of the terms in any order and the sum remains the same at the end of the operation.
3. Given the natural number a, a + 0 = 0 + a. From Activity 2.5, 0 + 67 = 67 +0. 0 is called the identity element of addition.

(b) Subtraction of natural numbers

Activity 2.6

Work out the following:

(a) 64 - 46 = . What do you call the answer?

(b) 54 – 24 = and 24 – 54 =. What do you obtain? Are the results the same?

(c) (45 – 23)- 21 = and 45 – (23 – 21) =. What do you obtain? Is the difference the same? Comment on the results.

From Activity 2.6 above, the symbol for subtraction is (-). You have realized that given that $x - y = t$, t represents the difference of the two terms. From Activity 2.6 the difference in (a) is 18.

Given the two natural numbers, from the activity 2.6, you noticed that $54 - 24$ is not equal to $24 - 54$. We say that subtraction is not commutative.

Similarly, from the activity 2.6 part (c), $(45 - 23) - 21 \neq 45 - (23 - 21)$. This means that subtraction is not associative.

(c) Multiplication and division of natural numbers

Activity 2.7

Work out the following:

(a) $28 \times 22 = ?$. What names do we give to 28, 22 and the answer in a multiplication problem?

(b) $22 \times 28 = ?$ Is there any difference with the answer you obtained in (a) above?

(c) Is $11 \times (12 \times 13) = (11 \times 12) \times 13$?

(d) Comment on the results of 64×1. What name would you give to the number 1?

The symbol for multiplication is ×.

If the natural numbers are, $a \times b = c$ then a is the multiplier, b the multiplicand and c is the product of the two terms.

From Activity 2.7, you have realized the following properties of multiplication:

1. If you are given two natural numbers a and b, then $a \times b = b \times a$. From Activity 2.7, $28 \times 22 = 22 \times 28$. This is called the commutative property of multiplication.
2. For natural numbers a, b and c, $a \times (b \times c) = (a \times b) \times c$. From Activity 2.7, $11 \times (12 \times 13) = (11 \times 12) \times 13$. This is called the associative property of multiplication.
3. Any natural number multiplied by one gives that natural number. For example $64 \times 1 = 64$. 1 is called the identity element of multiplication.

Note that $a(b + c) = ab + ac$. This is called the distributive property of multiplication over addition.

Example 2.5

Work out:

(a) 1 234+ 110 +72 +4

(b) 5 421 631 + 9 214 + 81

Solution

Arrange the numbers in vertical form aligning the numbers according to their place value:

(a)
$$\begin{array}{r} 1\,2\,3\,4 \\ 1\,1\,0 \\ 7\,2 \\ +\quad 4 \\ \hline 1\,4\,2\,0 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 5\,421\,631 \\ 9\,214 \\ +\quad 81 \\ \hline 5\,430\,926 \\ \hline \end{array}$$

Example 2.6

Work out:

(a) 502 341 – 51 021

(b) 6 502 431 – 471 399

Solution

Arrange the numbers vertically and work them out.

(a)
$$\begin{array}{r} 5\;0\;2\;3\;4\;1 \\ -\;5\;1\;0\;2\;1 \\ \hline 4\;5\;1\;3\;2\;0 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 6\;5\;0\;2\;4\;3\;1 \\ -\;4\;7\;1\;3\;9\;9 \\ \hline 6\;0\;3\;1\;0\;3\;2 \\ \hline \end{array}$$

Example 2.7

Work out: $863 \times 56 \times 12$

Solution

First multiply 863 by 56, then multiply the resulting product by 12.

$$\begin{array}{r} 863 \\ \times\ 56 \\ \hline 5\,178 \\ +\ 43\,150 \\ \hline 48\,328 \end{array} \quad \text{then} \longrightarrow \quad \begin{array}{r} 48\,328 \\ \times\ 12 \\ \hline 96\,656 \\ +\ 483\,280 \\ \hline 579\,936 \end{array}$$

Example 2.8

Work out $612 \div 23$

Solution

$$\begin{array}{rl} 26 & \\ 23\ \overline{)612} & \\ -\ 46 & \ldots\ldots\ldots\ldots\ 23 \times 2 = 46 \\ \hline 152 & \\ -\ 138 & \ldots\ldots\ldots\ldots\ 23 \times 6 = 138 \\ \hline 14 & \ldots\ldots\ldots\ldots\ \text{remainder} \end{array}$$

Thus, $612 \div 23 = 26$ rem 14

Exercise 2.2

1. Work out the following:
   (a) $333 + 667$
   (b) $43 + 65 + 50$
   (c) $45 + (50 + 30)$
   (d) $18100 + 940 + 2693$
   (e) $(8400 + 2200) + 3340$
2. Work out the following:
   (a) $456 - 225$
   (b) $100\,000 - 39\,400$
   (c) $4\,824 - 1\,793$
   (d) $6\,453 - 2\,141 - 1\,340$
3. Copy and complete the subtractions and replace them by the missing digit to make the calculations correct.

   (a) $\begin{array}{r} 5\,8\,6 \\ -\ 2\,\square\,4 \\ \hline 3\,5\,2 \\ \hline \end{array}$ (b) $\begin{array}{r} 1\,\square\,5\,\square \\ -\ \ 2\,8\,9 \\ \hline 1\,2\,6\,5 \\ \hline \end{array}$

   (c) $\begin{array}{r} 4\,\square\,\square\,6 \\ -\ 3\,5\,8\,\square \\ \hline 8\,6\,9 \\ \hline \end{array}$ (d) $\begin{array}{r} 8\,\square\,4\,\square\,7 \\ -\ 6\,3\,8\,\square \\ \hline 7\,9\,0\,7\,5 \\ \hline \end{array}$
4. Multiply the following:
   (a) $20 \times 30 \times 40$ (b) $15 \times (35 \times 20)$
   (c) $(32 \times 40) \times 20$ (d) $943 \times 16 \times 11$
5. Expand the following where y = 3 and x = 2:
   (a) $7(20 + y)$ (b) $13(xy + y)$
   (c) $xy(3x + 2y + 3)$
   (d) $2y^2x\,(3x^2 + 4y)$
6. Evaluate:
   (a) $412 \div 6$ (b) $991 \div 31$
   (c) $56\,560 \div 40$
   (d) $3\,219\,312 \div 47$
7. In Ruhengeri, a farmer harvested 34 500 kg of potatoes in the first season and 24 750 kg of potatoes in the second season. Find the total harvest.
8. In a city, there were 45 600 girls in secondary schools and 39 540 boys. Find the total number of students in the city.
9. A farmer requires 37 posts placed 5 m apart to fence one side of his farm. How many posts would he require if the posts were 4 m apart?
10. An order of lumber contains 30 boards 12 m long each, 25 boards 10 m long each, 36 boards 11 m long

each and 15 boards 16 m long each. How many boards are contained in the order? How many linear metres of lumber are contained in the order?

2.3 The Set of integers (Z)

Activity 2.8

Carry out the following activities

1. Using a Maths dictionary, define what an integer is.
2. What integer would you give to each of the following situation?
   (a) A fish which is 50 m below the water level.
   (b) Temperature of the room which is 42°C.
   (c) A boy who is 2 m below the ground level in a hole.
   (d) A bird which is 3 m high on a tree.

From the above activity, you have learnt that, **integers** are whole numbers which are either **negative or a positive** sign and includes zero.

The set of integers is represented by **Z.**

For example, when measuring temperature, the value of the temperatures of the body or surrounding can be negative or positive.

The normal body temperature is about +37°C and the temperature of the freezing mercury is about -39°C.

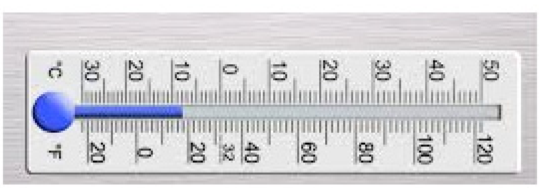


Fig. 2.5

The set of positive integers is {+1, +2, +3, +4, ...} and the set of negative integers is {.......... -5, -4, -3, -1}

Integers can be represented on a number line as shown below.

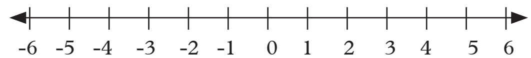


Fig. 2.6

NB: When you move towards the left of the number line, numbers become smaller and smaller. This means -5 is smaller than -2 and -100 is less than -1. We can represent this as -5<-2 and -100<-1.

2.3.1 Subsets of integers

Activity 2.9

Discuss the following:

1. Given that set of integers between -12 and +20, form sets from the set of integers which contain; odd numbers, even numbers, square numbers, factors of 6 and multiples of 3.
2. Draw Venn diagrams showing the following subsets from the set of integers.
   (a) Set of prime numbers.
   (b) Set of negative odd numbers.
   (c) Set of natural numbers.
3. Are there more subsets that you can form from the given set of integers?

From Activity 2.9, you have realized that integers have several subsets such as, set of even numbers, set of odd numbers, set of prime numbers, set of negative numbers and so on.

Example 2.9

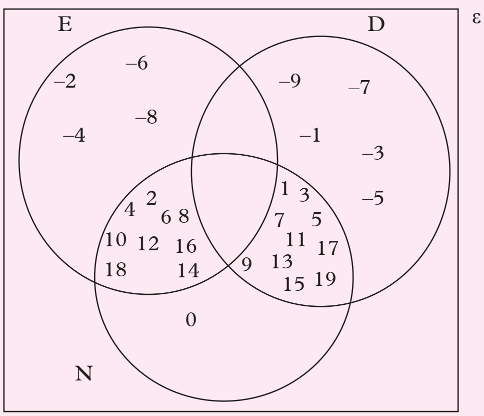
Given ε = {integers between -10 and 20} draw a Venn diagram to show the following subsets of ε:

(a) Set of E = set of positive and negative even numbers, set D = set of positive and negative odd integers, set N = set of natural numbers.

(b) Set D = set of positive and negative odd numbers, set E = positive and negative even integers and set P = set of prime numbers.

Solution

(a)



(b)

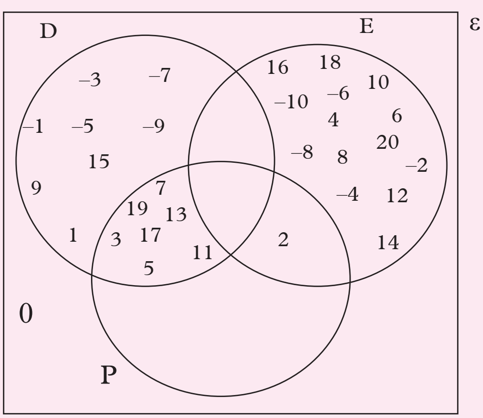


Fig. 2.7

2.3.2 Relationship between integers and natural numbers

Activity 2.10

Discuss and work out the following:

1. You are given ε = {integers between -20 and +20}. Find the subset N from set the universal set.
2. Draw a Venn diagram to show this relationship.
3. Obtain the sets S = square numbers, set F = factors of 16 and set M = multiples of 5. Draw a Venn diagram to show this.
4. What relationship is there between integers and natural numbers.

From activity 2.10, you have realized that set N = set of natural numbers is a subset of integers (Z).

Example 2.10

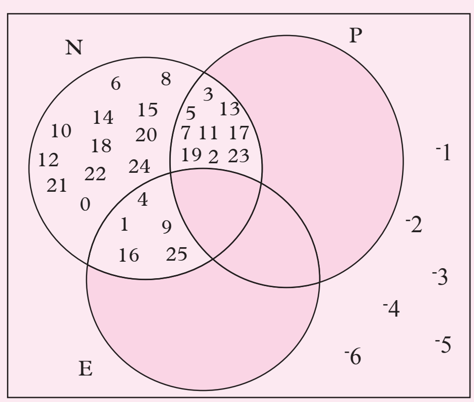
Given set Z = {-6 to +25}

(a) Draw a Venn diagram showing the subsets N ={natural numbers}, subset P={prime numbers} and subset E={ square numbers}.

(b) Draw a Venn diagram showing subsets N, E and D ={multiple of 6}.

Solution

(a)



(b)

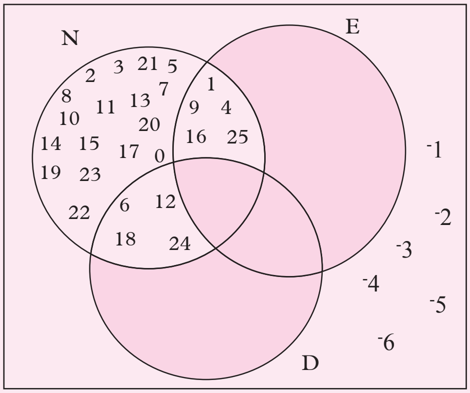


Fig 2.8

Exercise 2.3

1. Given that ε = {-4 to +22}, show on a Venn diagram the subsets P, E and D.
   Given that P = {prime numbers}
   E = {even numbers}
   D = {odd numbers}
2. Given ε = {-6 to +28} and
   Sets E = {even numbers},
   D = {odd numbers} and
   N = {natural numbers
   P = {prime numbers}
   show in a Venn diagram
   (a) subsets E, D and N
   (b) subsets P, D and N.
3. Use Fig 2.9 to answer the following questions. Let P = {prime numbers}
   E = {even numbers}
   N = {natural numbers}

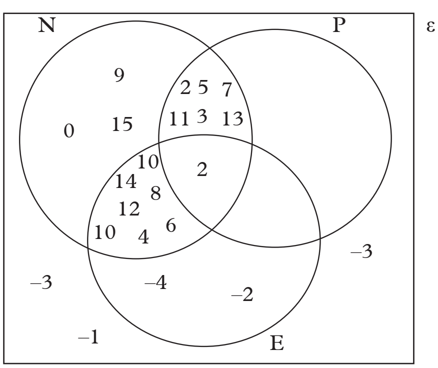


Fig 2.9

(a) Write the elements of:
   (i) Set N (ii) Set E
   (iii) Set P
(b) Write the elements of the universal set.
(c) What relationship is shown by the Venn diagram?

2.2.3 Operation on integers

(a) Addition of integers

Activity 2.11

Work out the following on a number line:

(a) (+3) + (+2)
(b) –(5) + - (3)
(c) (+4) + (-3)
(d) Which side of the number line did you move when adding a negative number?
(e) Which side of the number line did you start with when the given numbers were all negative?

From Activity 2.11, you realize that, addition of integers is best understood by illustrating the movements on the number line.

Example 2.11

Work out (-3) + (+2)

Solution

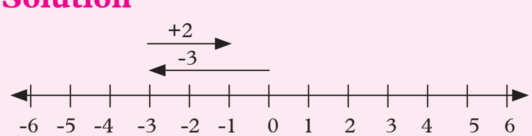


Fig 2.10

(-3) + (+2) = -1

Example 2.12

Work out (-3) + (-2)

Solution

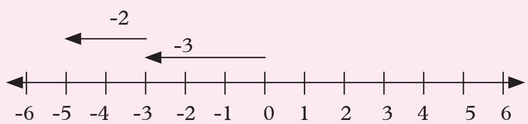


Fig 2.11

(-3)+ (-2) = -5

Example 2.13

Work out (+5) + (-3)

Solution

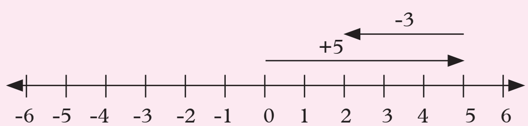


Fig 2.12

(+5) + (-3) = (+2)

Example 2.14

Work out (-3) + (+3)

Solution

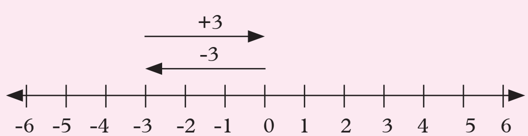


Fig 2.13

(-3) + (+3) = (0)

NOTE: +3 is the additive inverse of -3. Therefore, when we add a number and its additive inverse, we get zero (0) as the result.

Exercise 2.4

Work out the following and show your working on a number line.

1. (-5)+(-4)
2. (-2)+(+6)
3. (-5)+(+5)
4. (-1)+(+4)
5. (+3)+(-2)
6. (+7)+(+3)
7. (-6)+(-2)+(-1)
8. (+3)+(+4)+(-2)
9. (+2)+(-8)+(-3)

(b) Subtraction of integers

Activity 2.12

Work out the following and show your solutions on a number line.

(a) (-4)-(+3)

(b) (+5) – (+3)

(c) (–6) – (-6)

(d) On your number line, which direction do you move when subtracting two negative numbers?

(e) Incase you have two positive numbers that you are finding the difference, which side of the number line would you start with?

From Activity 2.12, you realize that, movements will start from the zero point for subtraction.

You also realize that, in a case such as (-6)-(-6) = 0, the answer is zero which means that, when any number is taken away from itself, it leaves you with nothing which is zero.

Example 2.15

Work out (+4)-(-3)

Solution

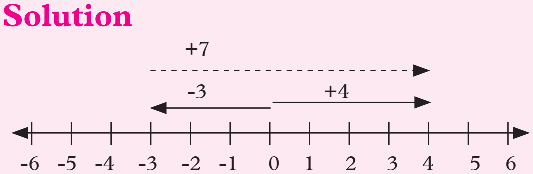


Fig 2.14

(+4)- (-3) =+7

Example 2.16

Work out (-5)-(+3)

Solution

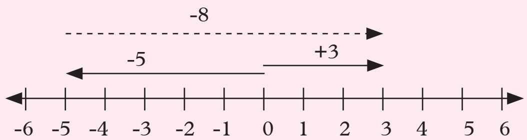


Fig 2.15

(-5)-(+3)= -8

Example 2.17

Work out (+6)-(+4)

Solution

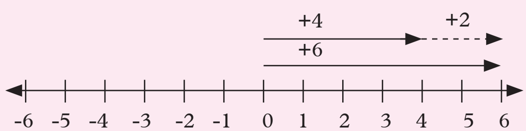


Fig 2.16

(+6) – (+4) = +2

Exercise 2.5

1. Work out the following integers on a number line.

   (a) (–5) – (–4) (b) (–5) – (+5)
   (c) (–7) – (–9) (d) (+3) – (+2)
   (e) (+8) – (–3) (f) (–1) – (–3)
   (g) (+2) – (–2) (h) (–3) – (–7)
   (i) (+4) – (–4) (j) (–10) – (8)
   (k) (–6) – (+7)

2. What is the difference between the following temperatures?

   (a) 15°C and –6°C
   (b) –8°C and –3°C
   (c) –10°C and +10°C
   (d) +5°C and +9°C
   (e) +20°C and –1°C

(c) Multiplication of integers

Activity 2.13

Work out the following:

(a) (+5) × (-6) (b) (+5) × (+6)
(c) (-5) × (+6) (d) (-5) × (-6)
(e) Did you obtain the same results in all the four tasks?

From Activity 2.13, you should have realized the following:

(a) Negative × Negative = Positive
(b) Positive × Negative = Negative
(c) Negative × Positive = Negative
(d) Positive × Positive = Positive

These are the laws of multiplication.

Example 2.18

Work out (-4) × (-7)

Solution

From the laws of multiplication, the result should be positive.

(-4) × (-7) = +28

Exercise 2.6

Work out each of the following integers:

1. (+5) × (–2)
2. (–10) × (–10)
3. (–12) × (+5)
4. (+6) × (+5)
5. (–6) × (+6)
6. (–2) × (+5) × (–3)
7. $(-4)^3$
8. (+5) × (–6) × (–1)
9. $(-3)^2 \times (-2)^2$
10. (–5) × (+20)

(d) Division of integers

Activity 2.14

Work out the following and show your solutions on a number line.

(a) (-4)÷(+4)

(b) (+4) ÷ (+4)

(c) (–4) ÷ (-4)

(d) (+4) ÷ (-4)

(e) Did you obtain the same results in all the four questions above?

From Activity 2.14, you should have realized the following:

(a) Negative ÷ Negative = Positive

(b) Positive ÷ Positive = Positive

(c) Negative ÷ Positive = Negative

(d) Positive ÷ Negative = Negative

These are the laws of division.

Example 2.19

Work out (-5)÷(+5)

Solution

$(-5)\div(+5) = {}^{-}1$

Example 2.20

Work out (+10)÷(⁻2)

Solution

$(+10)\div(-2) = {}^{-}5$

Example 2.21

Work out (-30)÷(-5)

Solution

$(-30)\div(-5) = 6$

Exercise 2.7

1. Work out the following:

   (a) (-28) ÷ (-7)

   (b) (+55) ÷ (-11)

   (c) (+19) ÷ (-3)

   (d) (-48) ÷ (-6)

   (e) (-36) ÷ (+9)

   (f) (-100) ÷ (-10)

2. Work out the following:

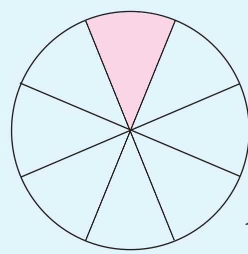
   (a) $\dfrac{(-10) \times (-5) \times (-6)}{(-3) \times (-2)}$

   (b) $\dfrac{(-30) \times (+2) \times (-10)}{(-5) \times (+2)}$

2.4 Fractional Numbers (rational numbers)

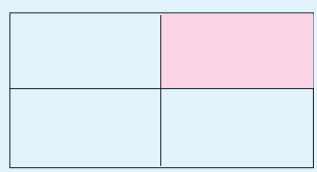
Activity 2.15

Study the following figures and answer the questions that follow.



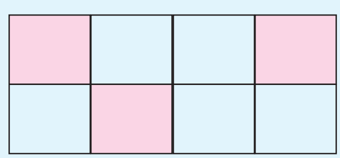
What fraction does the shaded part represent from the whose circle?

Fig 2.17



What is the fraction of the unshaded part?

Fig 2.18



What fraction is represented by the shaded part?

Fig 2.19

What other name would you give to fractions?

How many categories of fractions do you know? Name and describe each of them.

From Activity 2.15, you realize that fractions are **rational numbers**.

Fractions are expressed in the form $\frac{P}{Q}$ where P and Q are integers and the value Q should not be equal to zero i.e. $Q \neq 0$.

If $\frac{2}{3}$ is a fraction, then 2 is called the **numerator** and 3 is called the **denominator.**

2.4.1 Types of fractions

Fractions are grouped into two categories: **proper fractions** and **improper fractions.**

Proper fraction is where the numerator is less than the denominator. Example is $\frac{11}{20}$.

Improper fraction is where the numerator is greater than the denominator. Example is $\frac{17}{5}$. This fraction can be written as a mixed fraction as $3\frac{2}{5}$.

2.4.2 Operations on fractional numbers

(a) Addition and subtraction of fractions

Activity 2.16

Work out the following fractions:

(a) $\frac{1}{2} + \frac{1}{3}$ (b) $3\frac{5}{6} + 2\frac{3}{7}$

(c) $1\frac{2}{3} - \frac{1}{2}$ (d) $\frac{6}{10} - \frac{4}{9}$

What method did you use when working out?

One of you to present your work to the rest of the students.

From Activity 2.16, you realize that, fractions can be added and subtracted easily by first looking for the least common multiple (LCM) of the denominator.

Incase of a mixed fraction, first change the fraction to a proper fraction.

Example 2.22

Work out $3\frac{2}{3} + 1\frac{3}{4}$

Solution

We can first convert the mixed fractions to single fractions.

We get $3\frac{2}{3} + 1\frac{3}{4} = \frac{11}{3} + \frac{7}{4}$

The LCM of 3 and 4 is 12.

Thus, $\frac{11}{3} + \frac{7}{4} = \frac{44+21}{12} = \frac{65}{12} = 5\frac{5}{12}$

Example 2.23

Work out $4\frac{3}{4} - 3\frac{1}{2}$

Solution

We will first convert the mixed fractions to single fractions.

So, $4\frac{3}{4} - 3\frac{1}{2} = \frac{19}{4} - \frac{7}{2}$ *(LCM of 4 and 2 is 4)*

$\frac{19-14}{4} = \frac{5}{4} = 1\frac{1}{4}$

(b) Multiplication and division of fractions

Activity 2.17

Work out the following using any methods that you know:

(a) $5\frac{2}{3} \times 2\frac{9}{5}$ (b) $\frac{2}{5} \times \frac{1}{9}$

(c) $\frac{5}{8} \div \frac{2}{9}$ (d) $5\frac{7}{8} \div 1\frac{6}{5}$

List down the procedure for dividing fractions.

Appoint one of you to show the rest of the students one of the fractions you did.

From Activity 2.17, you have realized that, when the fractions do not have any common factors, the numerator will multiply with the numerator and the denominator will multiply with the denominator.

The rule working is as follows $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$
When there is a common factor existing between the numerator and the denominator, the common factor cancels out by division.

For example, $\frac{a}{b} \times \frac{b}{d} = \frac{a}{d}$ because b is a common factor and it divides by b to give 1.
When dividing fractions, the division sign changes to the multiplication sign and the divider changes to its reciprocal.

The rule, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ applies.

Example 2.24

Work out $\frac{2}{5} \times \frac{3}{4}$

Solution

$$\frac{2}{5} \times \frac{3}{4} = \frac{2 \times 3}{5 \times 4} = \frac{6}{20} = \frac{3}{10}$$

Example 2.25

Work out $4\frac{1}{2} \times 3\frac{2}{9}$

Solution

$$4\frac{1}{2} \times 3\frac{2}{9} = \frac{9}{2} \times \frac{29}{9} = \frac{\overset{1}{\cancel{9}}}{2} \times \frac{29}{\underset{1}{\cancel{9}}} = \frac{29}{2}$$

$$= 14\frac{1}{2}$$

Example 2.26

Work out $\frac{2}{5} \div \frac{1}{2}$

Solution

$$\frac{2}{5} \div \frac{1}{2} = \frac{2}{5} \times \frac{2}{1} = \frac{2 \times 2}{5 \times 1} = \frac{4}{5}$$

Example 2.27

Work out $8\frac{3}{4} \div 4\frac{3}{3}$

Solution

$$8\frac{3}{4} \div 4\frac{3}{3} = \frac{35}{4} \div \frac{15}{3} = \frac{35}{4} \times \frac{\overset{1}{\cancel{3}}}{\underset{5}{\cancel{15}}}$$

$$= \frac{35 \times 1}{4 \times 5} = \frac{7}{4}$$

$$= 1\frac{3}{4}$$

(c) Mixed operations on fractions

Activity 2.18

Work out the following:

$$2\frac{3}{4} \div \left(\frac{6}{5} + \frac{5}{7}\right) \times \frac{1}{4} - \frac{3}{9}$$

Work out the fraction as it appears ignoring the brackets i.e. divide, add, multiply then subtract.

What did you obtain?

Work on the same sum but this time follow the following:

(a) Work out what is in the brackets first.

(b) Next, work out the division part.

(c) Next, work out the multiplication.

(d) Finally, work out the subtraction.

What did you obtain? Are the two answers the same? What do you think may be the cause of the differences?

From Activity 2.18, you have realized that, to work out fractions which involve more than one operation, we use a rule.

This rule is called **BODMAS**. i.e.

B = Brackets

O = Of

D = Division

M = Multiplication

A = Addition

S = Subtraction

The operations are worked out as they appear in the BODMAS.

Example 2.28

Work out $\frac{3}{5} + (2\frac{1}{2} - \frac{2}{3}) \div \frac{5}{6}$

Solution

Step 1: Work out the brackets

$(2\frac{1}{2} - \frac{2}{3}) = \frac{5}{2} - \frac{2}{3} = \frac{15-4}{6} = \frac{11}{6}$

Step 2: Work out on the operation of division

$\frac{11}{6} \div \frac{5}{6} = \frac{11}{\cancel{6}_1} \times \frac{\cancel{6}^1}{5} = \frac{11}{5}$

Step 3: Lastly, work out on addition operator.

We get $\frac{3}{5} + \frac{11}{5} = \frac{3+11}{5} = \frac{14}{5} = 2\frac{4}{5}$

So, $\frac{3}{5} + (2\frac{1}{2} - \frac{2}{3}) \div \frac{5}{6} = 2\frac{4}{5}$

Exercise 2.8

1. Work out the following fractions:

   (a) $\frac{1}{4} + \frac{1}{3}$ (b) $\frac{2}{5} + \frac{1}{5}$

   (c) $\frac{3}{5} + \frac{2}{3} + \frac{4}{9}$ (d) $1\frac{1}{3} + 3\frac{1}{2}$

   (e) $2\frac{4}{5} + 1\frac{6}{7}$

2. Work out the following fractions.

   (a) $\frac{5}{6} - \frac{3}{4}$ (b) $\frac{3}{4} - \frac{1}{5}$

   (c) $\frac{5}{6} - \frac{1}{8}$ (d) $2\frac{3}{4} - 1\frac{4}{5}$

   (e) $7\frac{5}{6} - 4\frac{7}{8}$

3. Multiply the following fractions.

   (a) $\frac{3}{4} \times \frac{1}{2}$ (b) $\frac{8}{33} \times \frac{11}{12}$

   (c) $\frac{15}{6} \times \frac{4}{5}$ (d) $2\frac{1}{2} \times \frac{1}{4}$

   (e) $3\frac{1}{3} \times \frac{3}{8}$ (f) $6\frac{7}{8} \times 2\frac{6}{11}$

4. Work out of the following fractions.

   (a) $\frac{3}{4} \div \frac{1}{4}$

   (b) $2\frac{1}{2} \div \frac{1}{2} \div \frac{9}{70} \div \frac{4}{15}$

   (c) $6 \div \frac{1}{4}$

   (d) $5\frac{2}{3} \div 4\frac{1}{4}$

   (e) $3\frac{1}{2} \div 1\frac{1}{2}$

5. Work out the following fractions.

   (a) $(2\frac{1}{2} \div 1\frac{1}{2}) + \frac{2}{3}$

   (b) $2\frac{1}{2} + (\frac{3}{4} \times 1\frac{1}{4}) - 1\frac{1}{8}$

   (c) $(2\frac{1}{2} \div 7\frac{1}{2}) + \frac{1}{4})$

   (d) $2\frac{1}{2} \div \frac{4\frac{1}{3} - 2\frac{1}{2}}{4\frac{1}{6}}$

   (e) $\frac{3\frac{1}{2} - 1\frac{5}{6} \times \frac{3}{11}}{1\frac{3}{4} + 7\frac{2}{3} \div 3\frac{5}{6}}$

2.5 Irrational Numbers

Activity 2.19

Discuss and carry out the following;

1. Find:

   (a) $\sqrt{8}$ (b) $\sqrt{\frac{2}{5}}$

2. Express 0.24156735.... as a fraction.

   What do you notice?

> Is it possible to get the ratio in step 1 and 2 above?
>
> What do you call such numbers?

You already learnt in previous lessons that, numbers which can be expressed as $\frac{p}{q}$ and recurring decimals such as 0.333333... $= \frac{1}{3}$ fall under rational numbers.

From Activity 2.19, you notice that, there are some decimals which do not recur. Their values keep changing and they go on without an end. For example 3.14159265358.

Similarly, there are some numbers which do not have exact roots neither can they be expressed as fractions, for example $\sqrt{8}$, $\sqrt[3]{10}$ etc.

These numbers are called **irrational numbers**.

If you are asked to find $\sqrt{12}$, 12 is estimated between 9 and 16. Therefore $\sqrt{12}$ is between 3 and 4.

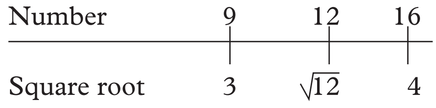


Fig. 2.22

We can see that 3 is less than $\sqrt{12}$ and $\sqrt{12}$ is less than 4. We can represent it as $3 < \sqrt{12} < 4$.

To be more exact, consider the decimals between 3 and 4.

The number 12 is less than half way between 9 and 16, the decimals should be less than half-way between 3 and 4.

Let us try 3.4 and 3.5.

$3.4^2 = 11.56$ and $3.5^2 = 12.25$.

More accurate is $3.45^2 = 11.9025$ and $3.47^2 = 12.409$

So $3.45 < \sqrt{12} < 3.47$.

In two significant figures, $\sqrt{12} = 3.4$

Irrational numbers can be represented on a number line as follows

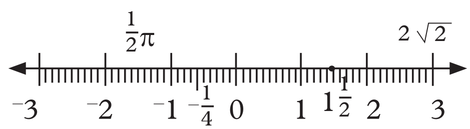


Fig. 2.22

2.6 Decimals

Activity 2.20

Discuss and answer the following questions.

Consider the following numbers in which the digits have been written expressing their total values.

(i) $543 = 5 \times 100 + 4 \times 10 + 3 \times 1$ ... (i)

$\times \frac{1}{10}$ $\qquad \times \frac{1}{10}$

(ii) $543\frac{7}{10} = 5 \times 100 + 4 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$ ...(ii)

$\times \frac{1}{10}$ $\qquad \times \frac{1}{10}$ $\qquad \times \frac{1}{10}$

(iii) $543\frac{1}{4} = 543\frac{25}{100} \longrightarrow (\frac{1}{4} = \frac{25}{100} = \frac{20+5}{100} = \frac{20}{100} + \frac{5}{100} = \frac{2}{10} + \frac{5}{100})$

$$= 5 \times 100 + 4 \times 10 + 3 \times 1 + 2 \times \frac{1}{10} + 5 \times \frac{1}{100} \quad \ldots\text{(iii)}$$

$$\times\frac{1}{10} \qquad \times\frac{1}{10} \qquad \times\frac{1}{10} \qquad \times\frac{1}{10}$$

2. What happens when you move from the left to the right of a decimal number?
3. What is the name given to the last parts in step 1 (ii) and (iii) above?
4. Write the place value of 641015.0014 in a place value table.

From activity 2.20, we observe that on moving from left to right in a number, the place values of each digit reduce to one-tenth of the previous one from left. For example, in (iii) above, we have 5 hundreds, 4 tens, 3 ones, 2 tenths and 5 hundredths.

2.6.1 Types of decimals

In (i) and (iii) the fractional parts of the numbers e.g. $\frac{1}{10}$, $\frac{1}{100}$ are known as decimal fractions.

Decimal fractions are those fractions whose denominators are powers of ten. A number in which there are decimal fractions is usually written with a dot between the whole number part and the decimal fraction part. This dot is called a decimal point.

Thus $543\frac{7}{10}$ can be written as 543.7 and $543\frac{1}{4} = 543\frac{25}{100}$ can be written as 543.25. These are examples of mixed decimals, i.e as they contain a whole number and a decimal fraction.

Each digit occupies a specific place value. For example, the table below shows the place values of digits in the decimal number 4 093.251 8

Thousands	Hundreds	Tens	Ones	Decimal point	Tenths	Hundredths	Thousandths	Ten thousandths
4	0	9	3	•	2	5	1	8

Table 2.2:

Note that the place values of the digits in a decimal number decrease by 1 from the decimal point to the right. A decimal number can either be written as a fraction or in decimal form.

The decimal fraction $\frac{3}{1\,000}$ can be written as 0.003 read as three thousandths or zero point zero, zero three. The digit 3 is in the thousandths place value.

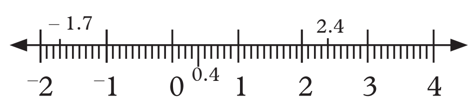


Fig. 2.22

2.6.2 Operations on decimals

Activity 2.21

1. Add, subtract, multiply and divide the following pairs of numbers

(a) 0.245 and 0.8
(b) 0.44 and 0.12
(c) 0.894 and 0.04
(d) 0.94692 and 0.214

2. What rules do you need to observe when carrying out the operations on decimal numbers?
3. Compare your results with those of other members in your class.

(a) Addition and subtraction of decimals

From activity 2.21 you have noticed that when adding or subtracting decimal numbers arrange them vertically, aligning the decimal points. In so doing, digits with the same place value get aligned vertically and are then added or subtracted.

Example 2.29

Work out:

(a) $10.5 + 9.7 + 0.62$

(b) $15.81 - 9.36$

Solution

(a)
$$\begin{array}{r} 10.5 \\ 9.7 \\ +\ 0.62 \\ \hline 20.82 \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 15.81 \\ -\ 9.36 \\ \hline 6.45 \\ \hline \end{array}$$

Exercise 2.9

1. Evaluate the following:
   (a) $5.5 + 3.8$
   (b) $1.8 + 3.6 + 2.9$
   (c) $6.489 + 3.481$
   (d) $203 + 1.626 + 0.813$
   (e) $60.001 + 9.36 + 87.1$
   (f) $0.761\ 12 + 0.983\ 15$
2. Find the value of:
   (a) $10.6 - 4.6$
   (b) $8.2 - 3.9$
   (c) $12.03 - 5.18$
   (d) $6.83 - 7.48$
   (e) $23.001 - 21.062$
   (f) $2.59 - 0.327$
3. Find the value of:
   (a) $5.24 + 0.6 - 4.07$
   (b) $97.32 - 41.1 + 30.6 - 8.7$
   (c) $0.67 + 7.6 - 0.76$
   (d) $69.2 + 331.9 - 39.8$
   (e) $4.832 - 7.48 + 3.903$
   (f) $18.42 - 8.31 - 0.96 + 8.03$
4. The internal diameter of the metal pipe shown in Fig. 2.23 is 3.875 cm. If the thickness of the metal is 0.875 cm, what is the external diameter?

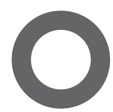


Fig. 2.23

5. Find the sum of the measurements: 124.1 cm, 2.34 cm, 0.86 cm and 21.162 cm correct to 1 decimal place.
6. The length of a football pitch is measured as 97.25 m and its width as 61.82 m. What is the perimeter of the pitch?

(b) Multiplication of decimals

From Activity 2.20, you realized that decimals are multiplied in the same way as whole numbers. Arrange the numbers vertically with the digits right below each and multiply.

The number of decimal places in the product is the sum of the decimal places in the numbers being multiplied (factors).

Example 2.30

Multiply:

0.48×0.006

Solution

0.48	→ has 2 decimal places
$\times\ 0.006$	→ has 3 decimal places
$0.002\ 88$	→ has 5 decimal places

Note: When the number of non-zero digits in the product is less than the required number of decimals, zero(s) are inserted in the product as in Example 2.30 above, so as to give the number of decimal places required.

Exercise 2.10

1. Evaluate:
   (a) 0.7×1.5
   (b) 26.1×8.3
   (c) 57.6×3
   (d) 2.13×0.32
   (e) $0.205\ 6 \times 0.93$
   (f) 0.016×0.005
2. Multiply 37×23 and then use it to write down the answers to:
   (a) 0.23×0.37
   (b) 0.037×2.3
   (c) 3.7×0.23
3. Evaluate:
   (a) 0.03×10
   (b) 0.6×100
   (e) $0.598\ 3 \times 1\ 000$
4. What is the area of a rectangular steel plate 10.5 cm long and 7 cm wide?
5. The electrical resistance of a certain type of wire is 1.256 ohms per cm. Calculate to 2 decimal places the resistance of a piece of wire of length 7.13 cm.

(c) Division of decimals

From Activity 2.21, you notice that, in dividing a decimal by a whole number, we do it the same way as with two whole numbers but we ensure that the decimal point is inserted in the answer vertically aligned with the divided.

Example 2.31

Simplify:

$0.303\ 52 \div 8$

Solution

$$\begin{array}{r|l} & 0.037\ 94 \\ \hline 8 & 0.303\ 52 \\ & -24 \\ & 63 \\ & -56 \\ & 75 \\ & -72 \\ & 32 \\ & 32 \\ & 0 \end{array}$$

In dividing a decimal number with another decimal, multiply the divisor and dividend by a power of 10 to make the divisor a whole number then divide as usual.

Similarly, When dividing a decimal by a power of 10, count the zeros in the divisor, and move the decimal point an equal number of places to the left.

Example 2.32

Evaluate:

(a) $0.376\ 2 \div 0.09$

(b) $82.13 \div 100$

Solution

(a) $\frac{0.376\,2}{0.09} = \frac{0.376\,2 \times 100}{0.09 \times 100}$

$= \frac{37.62}{9} = 4.18$

(b) $82.13 \div 100 = 0.821\,3$

Exercise 2.11

1. Evaluate:

(a) $0.26 \div 2$

(b) $0.714\,42 \div 243$

(c) $0.007\,35 \div 2$

(d) $1.092 \div 0.28$

(e) $0.054\,18 \div 6.02$

(f) $0.182\,9 \div 0.031$

2. Simplify:

(a) $0.681 \div 100$

(b) $0.02 \div 100$

(c) $118.7 \div 100$

3. A length of 3.99 m of shelving is available for storing some copies of a school textbook, which is 4.2 cm thick. How many books can be stored?

4. Woka walks at the rate of 44 paces in 0.5 minutes. If each pace is 8.75×10 cm long; find the time he takes to walk 2.31 km.

5. Twenty five sheets of aluminium of a gauge have a total thickness of 18.75 cm. What is the thickness of each sheet?

(d) Mixed operations on decimals

Activity 2.22

1. Work out the following:

(a) $0.85 + 0.64 \div (0.08 \times 0.14) - 0.06$

(b) $11.574 \times 0.72 \div 0.9 \times 0.76 - 4.84$

(c) $729.084 \div 0.3 \times 0.018 - (69.44 + 20.024)$

2. What operations are you supposed to begin with? Would there be a difference if you carried out the operations as they appear? Show this with an example.

From activity 2.22 you realize that an expression of decimal numbers may include more than one operation. Just like integers and fractions, the order of operation would be as follows: "BODMAS"

(i) Perform the operations within the brackets.

(ii) Perform by division followed by multiplication.

(iii) Perform the addition followed by subtraction.

Example 2.33

Find the value of each of the following:

(a) $5.4 + 3.2 - 8.2$

(b) $\frac{0.3 \times 0.08}{0.05}$

(c) $23.97 + 2\,(11.4 - 6.03) - 7.8$

Solution

(a) $5.4 + 3.2 - 8.2 = 8.6 - 8.2 = 0.4$

(Addition followed by subtraction)

(b) $\frac{0.3 \times 0.08}{0.05} = \frac{0.024 \times 100}{0.05 \times 100}$

$= \frac{2.4}{5} = 0.48$

(c) $23.97 + 2(11.4 - 6.03) - 7.8$

$= 23.97 + 2(5.37) - 7.8$ (brackets first)

$= 23.97 + 10.74 - 7.8$ (Multiplication follows)

$= 34.71 - 7.8$ (finally subtraction)

$= 26.91$

Exercise 2.12

Find the value of:

1. (a) $4.5 + 3(10.7 - 3.75)$

   (b) $\{(5.5 - 2.25) \div 6.5 \times 0.4\}$

   (c) $\frac{86.4 \times 0.03}{0.4}$

   (d) $\frac{0.15 - 1.2 \times 2.8}{0.18 \times 0.8 \times 25}$

   (e) $0.32 \times (2.014 + 0.783)$

   (f) $\frac{46.5 - 3(0.75)}{3.35 + 5.5}$

2. Keza spends $\frac{1}{3}$ of her salary on accommodation and $\frac{2}{5}$ of the remainder on food. What fraction is left for other purposes? (express your answer in decimal form).

> ! It is helpful to always plan and have a budget for your income since it keeps you focused on your money goals.

3. How many whole pieces of string 9.7 cm long can be cut from 1 500 cm of string if there is a wastage of 0.02 cm in every piece?
4. A watering can holds 2.54 litres of water. How many such cans could be filled from a tank that holds 100 litres? How much water would be left over if the tank leaks 0.08 litres of water once filled?
5. A piece of wire is 18.4 cm long. If 7.4 cm is cut from it, what length of wire remains? If the remaining piece is further subdivided into four pieces, what length of wire remains?

2.7 Conversion between decimals and fractions

(a) Changing fractions to decimals

Activity 2.23

Divide the following fractions to obtain a decimal number. Use the long division method to divide.

(i) $\frac{1}{2}$ (ii) $\frac{4}{5}$

(iii) $\frac{12}{1}$ (iv) $\frac{1}{3}$

What do you notice?

Is there a fraction that is not ending when you divide? What do you call such a fraction?

From Activity 2.23, you learn that, fractions are changed to decimals by dividing the numerator by the denominator.

When you divide a fraction such as $\frac{1}{3}$, the same number(s) repeat itself without ending. This number is called a **recurring decimal**.

Example 2.34

Change the following fractions to decimals

(a) $\frac{2}{5}$ *(b)* $\frac{1}{7}$

Solution

(a) $\frac{2}{5}$

$$\begin{array}{r|l} & 0.4 \\ \hline 5 & 2.0 \\ & \underline{2.0} \\ & 00 \end{array} \qquad \frac{2}{5} = 0.4$$

(b) $\frac{1}{7}$

$$
\begin{array}{r}
0.142857 \\
7\,\overline{)\,1.0} \\
-\ 7 \\
\hline
30 \\
-\ 28 \\
\hline
20 \\
-\ 14 \\
\hline
60 \\
-\ 56 \\
\hline
40 \\
-\ 35 \\
\hline
50 \\
-\ 49 \\
\hline
+\ \ 1
\end{array}
\qquad \frac{1}{7} = 0.142857\ldots
$$

(b) Changing decimals to fractions

Activity 2.24

Convert the following decimals into fractions.

(i) 0.4 (ii) 0.12 (iii) 0.025

(ii) How can you tell the number of zeros to include in the fraction as the denominator from a given decimal number?

From Activity 2.24, you realize that, decimals can be converted back to fractions. This can be done by looking at the number of decimal places.

For instance, 0.4 has only one decimal place and we therefore divide by 10 i.e.

$$0.4 = \frac{4}{10} = \frac{2}{5}$$

Similarly, 0.025 has 3 decimal places and we therefore divide by 1000 i.e.

$$0.025 = \frac{25}{1000} = \frac{1}{40}$$

(c) Changing recurring decimals to fractions

Activity 2.25

Convert 0.222 into a fraction.

Follow the following steps

Let y = 0.222 be equation (i).

Multiply equation (i) by 10

You should obtain 10y = 2.22. Let this be equation (ii).

Subtract equation (i) from (ii).

Make y the subject of the formula.

What did you obtain?

Why did we multiply by 10 and not 100 or even 1000?

Try to change the fraction back to the decimal number.

Do you still obtain the same original decimal number?

From Activity 2.25, you have learnt that it is possible to change recurring decimals to fractions.

From the activity, you should have obtained, $0.222\ldots = \frac{2}{9}$

Since only one digit is recurring and it is in the tenth position, we multiply by 100. If we had for instance 0.1233333.. , we would have multiplied by 1 000, since 3 is recurring and it is in the thousandth place value. This helps us to obtain a whole number once we take away the two equations that we form in the process of converting the decimal number to a fraction.

On changing this back to a decimal number, you wil still obtain 0.22222 which is recurring.

A recurring decimal e.g. 0.222 ..., 0.12323 ..., 0.213213.... etc are normally written in short form as $0.\dot{2}$, $0.1\dot{2}\dot{3}$, $0.\dot{2}1\dot{3}$ etc. The dot is put above the recurring digit(s).

Example 2.35

Convert to fraction:

(i) $0.1\dot{6}$ *(ii)* $0.\dot{2}1\dot{3}$

Solution

(i) $0.1\dot{6}$

Let $y = 0.1666.....$ (1)

Multiplying 100 both sides, we get,

$100y = 16.666....$ (2)

Subtracting (1) from (2) we get,

$100\,y = 16.666$

$-y = 0.1666$

$99y = 16.5$

Multiply by 10 to remove the decimal place.

$990y = 165$

Hence $y = \frac{165}{990} = \frac{1}{6}$

(ii) $0.\dot{2}1\dot{3}$

Let $y = 0.213\ 213\ 213........$ (1)

Multiply (1) by 1000

$1000y = 213.213\ 213.........$ (2)

Subtract (1) from (2)

$1000y = 213.213213$

$-y = 0.213213$

$999y = 213$

$y = \frac{213}{999} = \frac{71}{333}$

Therefore, $0.\dot{2}1\dot{3} = \frac{71}{333}$

Exercise 2.13

1. Express the following fractions into decimals.

(a) $\frac{3}{8}$ (b) $\frac{8}{9}$ (c) $\frac{1}{7}$

(d) $\frac{4}{9}$ (e) $\frac{7}{12}$ (f) $\frac{10}{7}$

(g) $\frac{5}{4}$ (h) $\frac{13}{10}$

2. Convert the following decimals to fractions.

(a) 0.2 (b) 0.62 (c) 0.012

(d) 0.001 (e) 1.4 (f) 1.2

3. Convert the following recurring decimals to fractions.

(a) $0.\dot{5}$ (b) $0.7\dot{2}$ (c) $0.1\dot{3}$

(d) $0.\dot{7}1\dot{7}$ (e) $0.1\dot{2}$ (f) $0.48\dot{6}$

(g) $0.303\dot{8}$ (h) $1.1\dot{3}$

2.8 Set of real numbers

Activity 2.26

Use your calculators to find the square roots of:.

(a) $\sqrt{12}$ (b) $\sqrt{36}$

(e) What do notice?

(f) Add and subtract the following:

(i) $5\sqrt{3} + 2\sqrt{3}$

(ii) $2\sqrt{3} - 5\sqrt{7}$

(g) Work out the following:

(i) $2\sqrt{7} \times 5\sqrt{6}$

(ii) $6\sqrt{5} \div 3\sqrt{5}$

Now write down five rational numbers, irrational numbers and real numbers. Deduce the relationship between the two.

From Activity 2.26, you realize that the set of **rational numbers** and the set of **irrational numbers** form the set of **real numbers**.

The set of real numbers is denoted by **R**.

Real numbers are represented on a number line as infinite points or they are set of decimal numbers found on a number line. This is illustrated on the number line below.

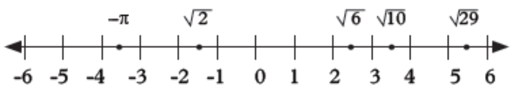


Fig. 2.24

Note: All these sets are subsets of real numbers that is natural numbers, integers, rational numbers, irrational numbers. This can be shown in the Venn diagram below.

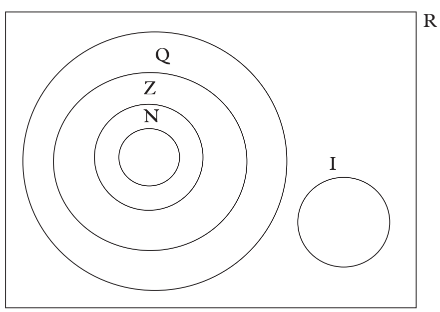


Fig. 2.25

Example 2.36

Given ε = {Set of integers between -10 and +10}, list and show on a Venn diagram the subsets E = {set of even numbers}, D = {set of odd numbers}, set P = {set of prime numbers} and N = { set of Natural numbers}.

Solution

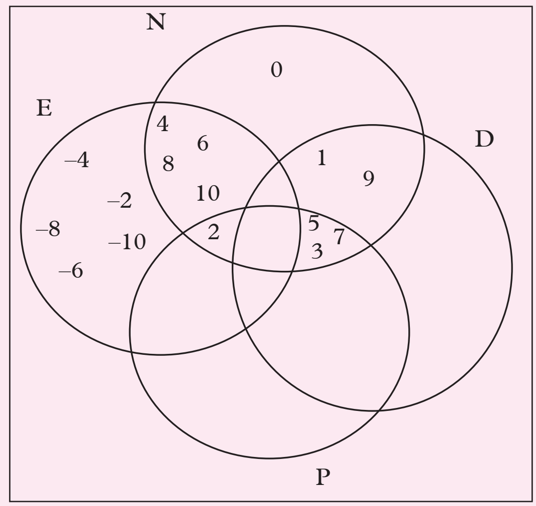


Fig. 2.26

Exercise 2.14

1. Find the square root of each of the following.

   (a) $\sqrt{6}$ (b) $\sqrt{8}$ (c) $\sqrt{3}$

   (d) $\sqrt{10}$ (e) $\sqrt{26}$ (f) $\sqrt{73}$

   (g) $\sqrt{103}$

2. Show the following real numbers on the number line

   (a) $-4, 1.7, \pi, \sqrt{2}, 3.8, -1$

   (b) $-2, 1.1, -\pi, \sqrt{6}, -2.8, 1$

3. Given that $\sqrt{2} = 1.41413$, $\sqrt{3} = 1.732050$ and $\sqrt{6} = 2.443489$, find:

   (a) $\sqrt{6} - \sqrt{2}$ (b $\sqrt{3} + \sqrt{2}$

   (c) $\sqrt{3} - \sqrt{2}$ (d) $\sqrt{6} + \sqrt{2}$

   (e) $\sqrt{6} - \sqrt{3}$

4. Given ε = {-2, 3, 8, 16, 25, 27, 36, 37, $\sqrt{6}$ $\sqrt{11}$, 2.3, 81}, list and show on a Venn diagram the subsets Z, Q and N where:

   Z = {integers}, Q = {rational numbers} and N = { natural numbers}

2.9 Summary of properties of operations on real numbers

(a) Cummutative property

Addition and multiplication of real numbers are commutative.

$a + b = b + a$

$a \times b = b \times a$

For example, if a = 2 and B = 4.5, then

2 + 4.5 = 4.5 + 2

6.5 = 6.5

2 × 4.5 = 4.5 × 2

9 = 9

Subtraction and division of real numbers are NOT commutative.

(b) Associative property

Addition and multiplication of real numbers are associative.

$a + b + c = (a + b) + c = (a + c) + b.$

$a \times b \times c = (a \times b) \times c = (a \times c) \times b$

For example, if a = 5, b = 7 and c = 8, then

5 + 7 + 8 = (5 + 7) + 8

20 = 20

5 × 7 × 8 = (5 × 7) × 8 = (5 × 8) × 7

280 = 280

Subtraction and division of real numbers are not associative.

(c) Distributive property

Multiplication has a distributive property over addition and subtraction of real numbers.

$a(b + c) = ab + ac$

$a(b - c) = ab - ac.$

For example, if a = 2, b = 3 and c = 5, then

2(3 + 5) = (2 × 3) + 2 × 5

16 = 16

2(3 – 5) = (2 × 3) – 2 × 5

$^{-}4 = {}^{-}4$

(d) Identity operators on real numbers

An identity operator on a real number leaves the value of the number unchanged.

Consider the real number *a*

(i) Zero (0) is an identity in *addition* and *subtraction* operator on real numbers.

$a + 0 = a,$

$a - 0 = a$

For example, if a = 4, then

4 + 0 = 4 and

4 – 0 = 4

(ii) One (1) is an identity in *multiplication*, *division* and *index operator* on real numbers.

$a \times 1 = a$

$a \div 1 = a$

$a^1 = a$

For example, if a = 2, then

2 × 1 = 2,

2 ÷ 1 = 2 and

$2^1 = 2$

Unit summary

1. A set (N)of natural numbers, is the set of counting numbers i.e. Set N = {1, 2, 3, 4, ...}.
2. Even numbers, odd numbers, prime numbers, are all subsets of natural numbers.
3. Addition of natural numbers is commutative and associative.

4. Multiplication of natural numbers is commutative and associative.
5. 1 is the identity element of multiplication.
6. Multiplication is distributive over addition.
7. Set of integers Z= {............, -3, -2, -1, 0, 1, 2, 3, ...}.
8. Natural numbers is a subset of integers.
9. Integers are also called direct numbers.

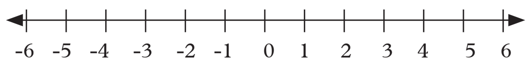


Fig. 2.27

10. Integers increase towards the right on the number line and decrease towards the left of the numberline.
11. Recurring decimals are decimals which do not end.
12. Set of irrational numbers Q= $\{\pi, \sqrt{2}, \frac{1}{\sqrt{2}}, \sqrt{5}, ...\}$
13. The union of rational numbers and irrational numbers give real numbers.
14. Set of real numbers is denoted by R.
15. A set of integers is a subset of rational numbers.
16. A set of rational numbers is a subset of real number.
17. A set of irrational number is a subset of real number.

Unit Test 2

1. Write down a set of all prime numbers less than 30.
2. Use the following number lines to determine the results.

(a)

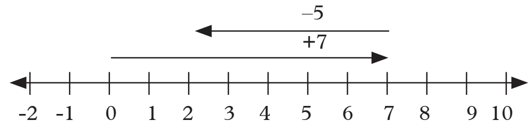


Fig. 2.28

(b)

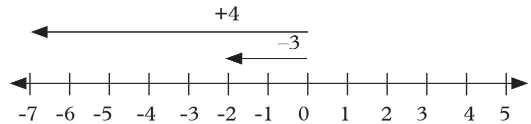


Fig. 2.29

3. Use a number line to find the value of:

   (a) +6 – (+4) (b) -3 – (+5)

   (c) 4 – (-3)

4. Work out the following:

   (a) $\frac{(-a) \times (-b)}{(-c)}$ (b) (-2) × (+3y)

   (c) (+18x) ÷ (-3x)

5. Find the values of;

   (a) {9 + (-2) × (-15)}× (-2 + 7) ÷ 3

   (b) $\frac{-6 + (-5) + 8 \times -2}{-4 + (-2)}$

   (c) -3 × 23 + (-5) × (-1) – 8 × (-4)

   (d) $\frac{9 \times -2}{-4 - (-2)}$

6. Evaluate

   (a) $\frac{1}{2} + \frac{4}{8} \times \frac{16}{3} - \frac{2}{5}$

   (b) $\frac{16}{3} - (2\frac{6}{11} + 6\frac{2}{9}) \div \frac{16}{11} \times \frac{2}{4}$

   (c) 0.45 + (2.45 – 1.009) ÷ 0.2 ÷ 0.008

   (d) 0.547 × 0.12 + 2.1 – 0.46 ÷ 0.007

   (e) 16.57 – (12.432 ÷ 0.04) × (16 × 0.4) + 1.16

3 LINEAR FUNCTIONS, EQUATIONS AND INEQUALITIES

Key Unit Competence

By the end of the unit, I should be able to solve problems related to linear equations, inequalities and represent the solutions graphically.

Unit outline

- Linear functions
- Equations
- Inequalities

3.1 Linear functions

Activity 3.1

Research from the library using mathematics books or the internet the meaning of linear functions.

Discuss and summarize your findings.

Give some examples of linear functions.

Compare your findings with the discussion below.

A linear function, also known as a linear equation, is a relationship between two variables, say x and y, whose graph represents a straight line. This relationship is often written in the form $y = mx + c$ where m and c are constants. Examples of a linear function are $y = x + 1$, $y = 2x - 3$, $y = -3x + 4$ etc. It is also possible to have a constant function in one variable such as $y = 3$, $y = -\frac{5}{2}$, $x = 1\frac{1}{2}$, $x = -2$ etc which can also be written in the form $y = k$ and $x = c$ where k and c are constants. In this unit, we are required to represent linear functions graphically but we will give more details later.

3.1.1 The number line

Let us remind ourselves how we locate rational numbers on a number line. This will be a good starting point to discuss the Cartesian plane and coordinates.

3.1.2 Position of a point on a number line

You are already familiar with the number line.

The number line is a graph or a picture of all the negative and positive numbers.

Activity 3.2

Draw a number line and on it illustrate:

(i) -7, -3, 1, 5, 9, 13, -2.5, 6.5

(ii) All natural numbers between 4 and 15.

(iii) The set of numbers x: $1 \leq x \leq 10$.

(iv) $x : 4\frac{1}{2} \leq x < 7$; $x : 1 \leq x \leq 9$; $\{-2, 0, 5, 7\}$

Fig. 3.1 is an example of a number line. If we mark points on the number line, we can say exactly where they are on the number line.

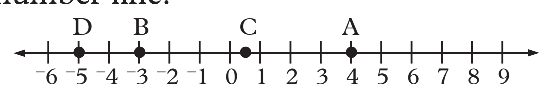


Fig. 3.1

In Fig. 3.1, A is 4 units to the right of zero and B is 3 units to the left of zero. This can be shortened to A (4) and B ($^-3$). Similarly, C is the point C($\frac{1}{2}$) and D is the point D ($^-5$).

A (4), B ($^-3$), C($\frac{1}{2}$) and D(-5) give the **positions** of A, B, C and D on the number line with reference to zero (0).

3.1.3 Position of a point on a plane surface

As an introduction to the work on the Cartesian plane and the coordinates, the discussion below involves the whole class. Participate actively as the teacher leads you through the discussion.

Activity 3.3

Fig. 3.2 shows a plan of the desks in a classroom. Each rectangle represents a desk. Describe the position of the shaded desk.

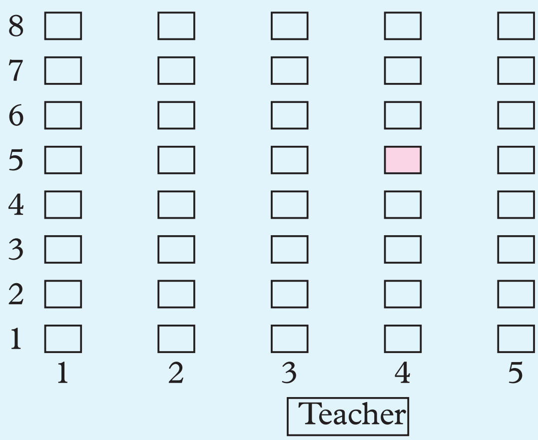


Fig. 3 .2

- You might describe the desk as being in the fourth column and fifth row. If we agreed to write this as (4, 5), it would mean 4 columns across from the left-hand side and then 5 rows up (i.e. from the front towards the back).
- Does (5, 4) mean the same position?
- It is clear that the **order of the numbers inside brackets is important** and so we call pairs like these **ordered pairs**.
- If all the desks in the classroom were removed and then you were asked to put your desk back in the room, exactly where it was before, how would you do it?
- If you knew it was 5 metres from one wall and 6 metres from another, could you do it then? How many places on your classroom floor are 5 m from one wall and 6 m from another? We have to decide which walls we are going to use.

Fig. 3.3 shows one way in which you might fix the position of your desk.

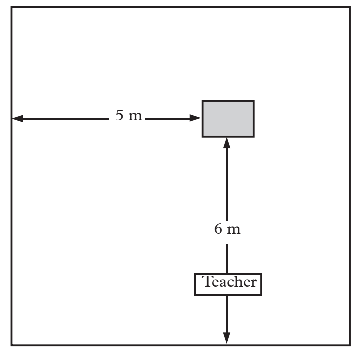


Fig. 3.3

We could say that the desk has an ordered pair (5 , 6), to mean five metres from the left-hand wall and six metres from the front wall.

From the above discussion, you should have concluded the following: to locate a specific point.

i) You need a reference point.

ii) You need the direction.

3.1.4 Drawing and labelling axes

In our desk example, without the desks in place, it would be difficult to locate the exact position of any desk. To locate relative position of any point, we use a more accurate method where we have a reference point as well as a reference pair of lines. The lines are perpendicular, one vertical and the other horizontal, the two lines meet at a point generally denoted by O. For an accurate representation we use a standard grid system (1 cm squares) known as graph paper.

Using a pair of lines on a graph paper and the information in Fig 3.4, we can show the relative position of the shaded desk. We will use a suitable scale on both number lines.

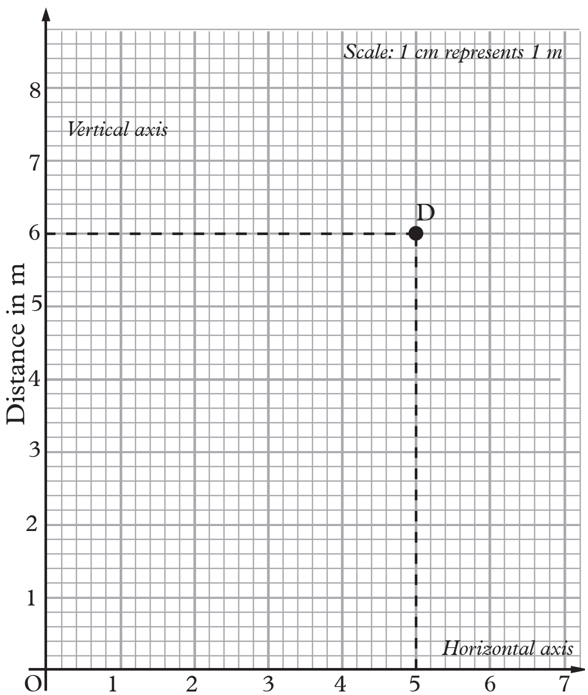


Fig. 3.4

The two reference walls have been replaced by a pair if number lines perpendicular at a point O which replaces the floor corner where the walls meet.

The point marked D represents the relative position of the desk with reference to the two axes and the origin.

3.1.5 Drawing vertical and horizontal lines

The pair of lines discussed above refers only to positive direction of the number line.

Fig. 3.5 shows the number line extending to both positive and negative directions.

We have used a different scale to draw and label vertical and horizontal lines Fig. 3.5.

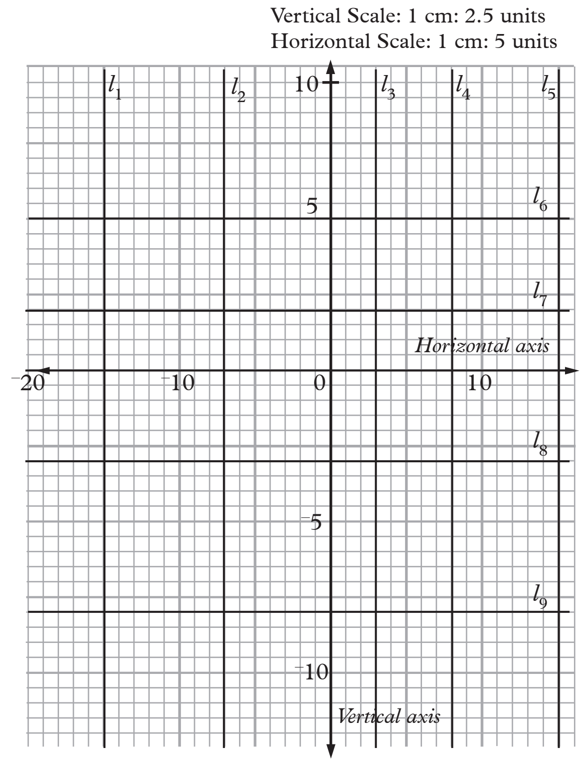


Fig. 3.5

Line l_1 to l_9 in Fig. 3.5 can be described with reference to the two number lines.

For example,

l_1 is a vertical line 15 units from the vertical axis and perpendicular to the horizontal axis. Similarly,

l_6 is a horizontal line, parallel to the horizontal axis and perpendicular to the vertical axis.

Activity 3.4

Describe the remaining lines in a similar way as has been done above.

Exercise 3.1

1. In Fig. 3.6, A(2) gives the position of A and B($^{-}4$) gives the position of B. State the positions of C, D and E in the same way.

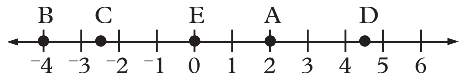


Fig. 3.6

2. Draw a number line from –5 to 5. On it, mark the points A (3), B (–1), C $(-4\frac{1}{2})$ and D $(3\frac{1}{2})$.

3. On a graph paper, draw a number line from $^{-}2$ to 2. On it, mark the points A (1.0), B (0.4), C ($^{-}1.6$), D ($^{-}$ 0.7) and E (1.3).

4. Fig. 3.7 shows points on a centimetre square grid. Starting from the point marked O, and using ordered pairs, the position of R is ($^{-}2$, 3), meaning that R is 2 cm to the **left** of the vertical axis and 3 cm up from the horizontal axis. In a similar way, describe the positions of points P, Q, S and T.

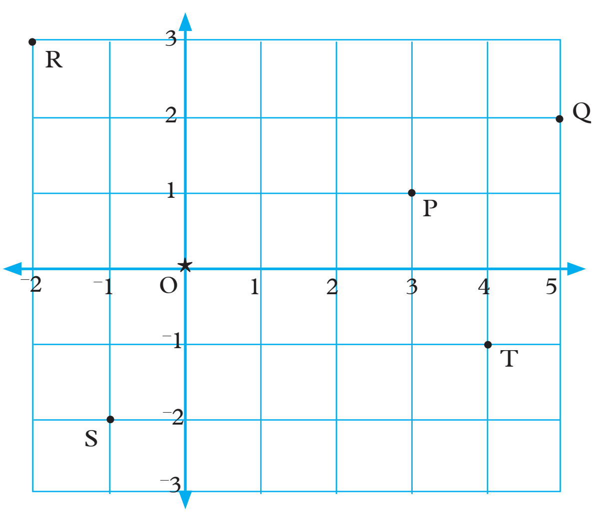


Fig. 3.7

3.1.6 The Cartesian plane

Activity 3.5

Draw a plan of desks in your classroom similar to the one in Fig 3.2.

Identify a pair of suitable walls to act as the vertical and horizontal axes. Using the columns and rows of desks, describe the position of your desk, and those of other members of your group. Compare your answers with the rest of the class.

Use actual measurements, and a suitable scale to locate position of your desk.

The positions of points on a line are found by using a number line, and are written down using a single number e.g. P($^{-}7$).

Learning point

The positions of points on a plane surface are found using two fixed number lines (i.e. two directions), usually at right angles, and are written down using two numbers.

In Fig. 3.8, starting from the zero point, O.

A is 2 units to the **right** and 3 units **up**, written as A (2 , 3).

B is 4 units to the **right** and 0 units **up**, written as B (4 , 0).

C is 0 units to the **right** and 2 units **down**, written as C (0 , $^{-}2$).

D is 2 units to the **left** and 2 units **down**, written as D ($^{-}2$, $^{-}2$).

Activity 3.6

In a similar way, find the positions of points E, F, G, H, I, and J.

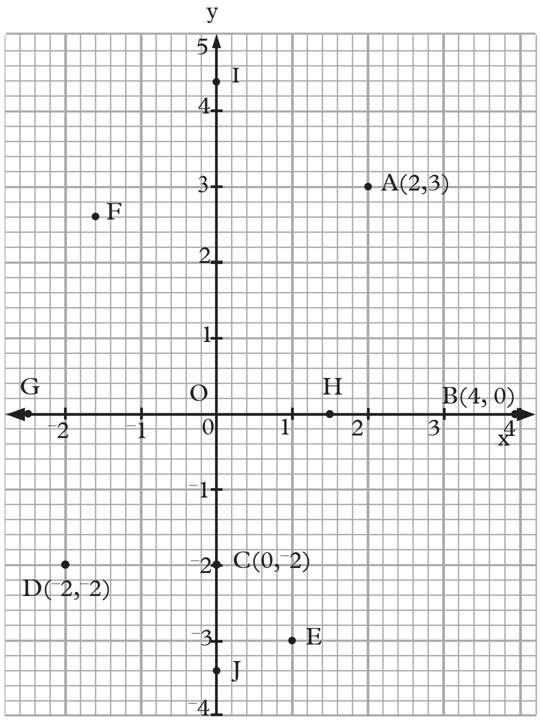


Fig. 3.8

Fig. 3.8 such is a **graph**, of the points A–J. In a graph like this, the number lines are called **axes**. They cross at the zero point of each axis at a point called the **origin**. The axis going across from left to right is called the **x-axis**. It has a positive scale to the right of 0 and a negative scale to the left of 0.

The axis going up the page is called the **y-axis**. It has a positive scale upwards from 0 and a negative scale downwards from 0.

A plane surface with *x* and *y* axes drawn on it, such as Fig. 3.8, is called a **Cartesian plane**. It is on such a plane that we can now plot points and draw required graphs.

> **Point of interest**
>
> Cartesian coordinates is named after the great French philosopher and Mathematician Rene' Descartes (1596-1650) (whose Latin name was Cartesius). Descartes' was the first person ever to use coordinates, and he is therefore considered to be the founder of coordinate geometry.

The ordered pairs that describe the positions of points on a Cartesian plane are called **Cartesian coordinates**. The first number is called the x-coordinate and it gives the distance of the point from the origin in the direction of the x-axis. The second number is called the y-coordinate and gives the distance of the point from the origin in the direction of the y-axis. For example, in (5, 7), 5 is the x-coordinate and 7 is the y-coordinate.

Exercise 3.2

In this exercise work in pairs

1. Name the points in Fig. 3.9 which have the following coordinates.

(a) (0, 1.5) (b) (0, $^-0.5$)
(c) ($^-1$, $^-2$) (d) ($^-2$, 1)
(e) (0.5, 2.5) (f) (1, $^-2.2$)
(g) ($^-2$, $^-1$) (h) ($^-2$, 3)
(i) (2.5, 1) (j) (4, $^-1$)
(k) ($^-1.4$, $^-0.6$) (l) ($^-3$, 0)
(m) (1, 0) (n) (2.5, $^-1$)
(o) ($^-3$, 0)

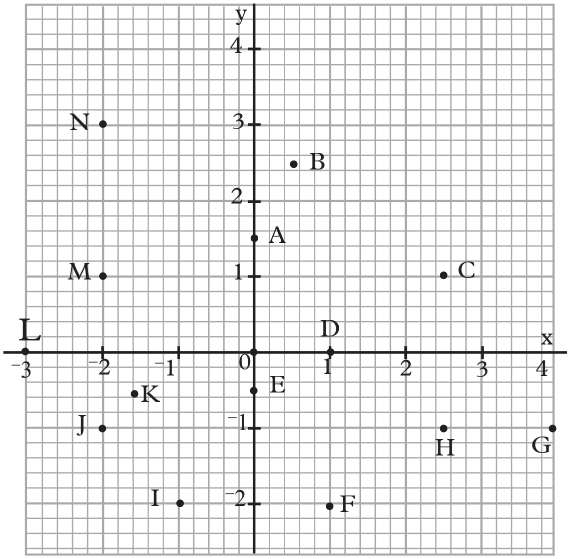


Fig. 3.9

2. What are the coordinates of the points A, B, C, D, E, F, G, H I, J, K in Fig. 3.10?

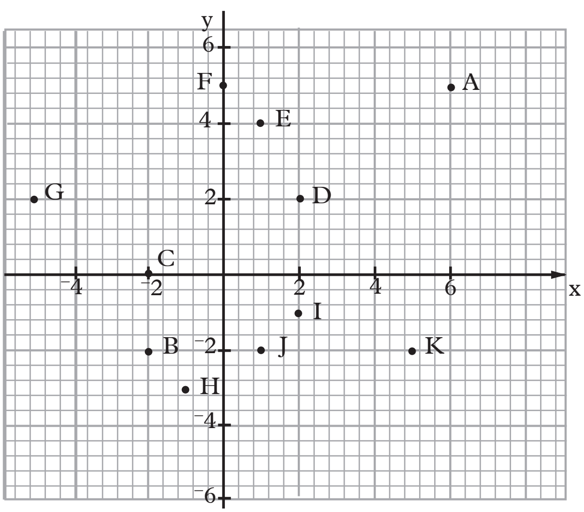


Fig. 3.10

3. In Fig. 3.11 write down the coordinates of the vertices of:
   (a) triangle ABC.
   (b) parallelogram PQRS.

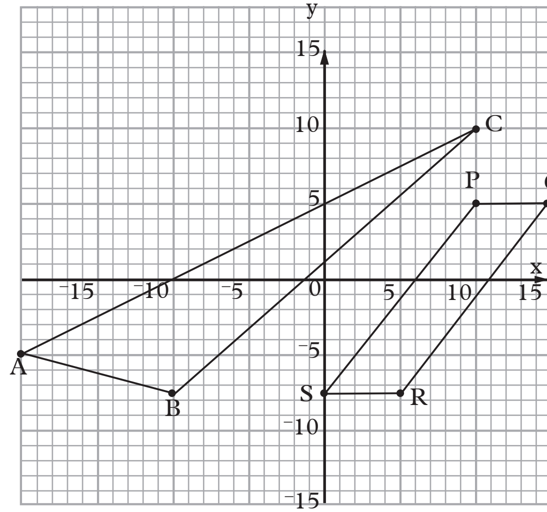


Fig. 3.11

4. State the coordinates of the vertices of the ‘arrow’ in Fig. 3.12.

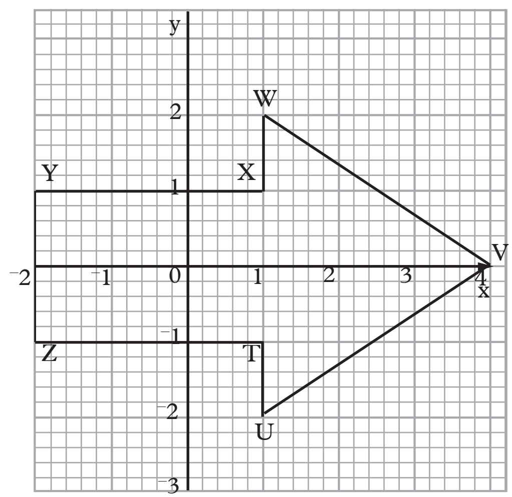


Fig. 3.12

5. Fig. 3.13 shows part of a map drawn on a Cartesian plane.

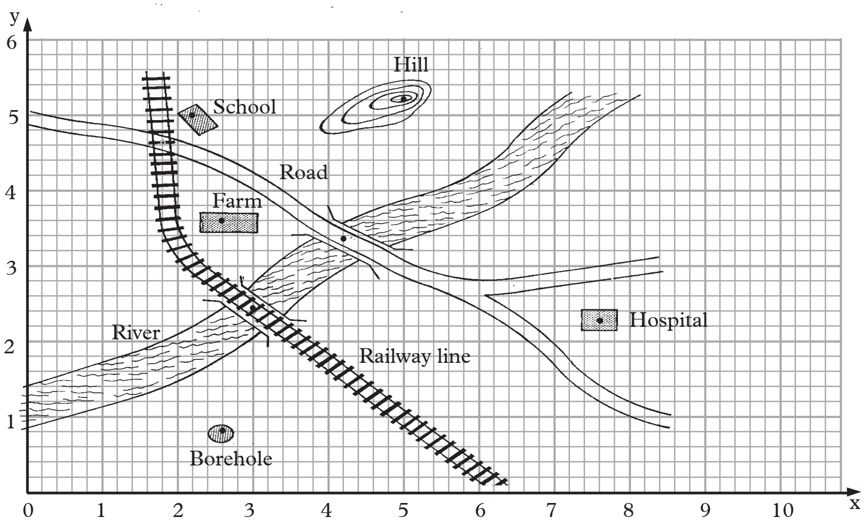


Fig. 3.13

Find the coordinates of the point in:

(a) the school (b) the borehole

(c) the farm (d) the hospital

(e) the top of the hill

(f) the intersection of the road and railway line.

(g) the point where the railway line crosses the river.

(h) the point where the road crosses the river.

3.1.7 Plotting points

To plot a point means to mark its position on a Cartesian plane.

Procedure:

1. Start at the origin.
2. Move along the x-axis the number of steps, and in the direction given by the x-coordinate of the point. Fig. 3.14
3. Move up or down parallel to the y-axis the number of steps, and in the direction given by the y-coordinate of the point.
4. Mark the point with a dot (.) or a cross (×).

To choose the scale of a graph, check what the highest and lowest values of x and y are in the given points. Choose the scale such that the axes include all the numbers. The scale should be as large as possible to make it easy to plot and read coordinates accurately. We normally use 1 cm (square) to represent 1, 2, 5, 10, 20, 50, 100, . . . units. Do not use scales in multiples of 3 or such numbers as 7, 11, etc! They are not easy to subdivide when need arises.

Activity 3.7

Draw a pair of x and y axes and on them mark a scale that extends to both +ve and –ve directions. Mark and label several points on the plane, ensuring that there are at least three points marked in every quarter of the plane.

Example 3.1

Plot the points (2, ⁻3) and (⁻2.4, 1.8) on a Cartesian plane.

Solution

The dotted lines in Fig. 3.14 show the method of plotting.

For (2, ⁻3):

x-coordinate is 2, i.e. 2 steps in the positive direction of x-axis. y-coordinate is ⁻3, i.e. 3 steps in the negative direction of the y-axis.

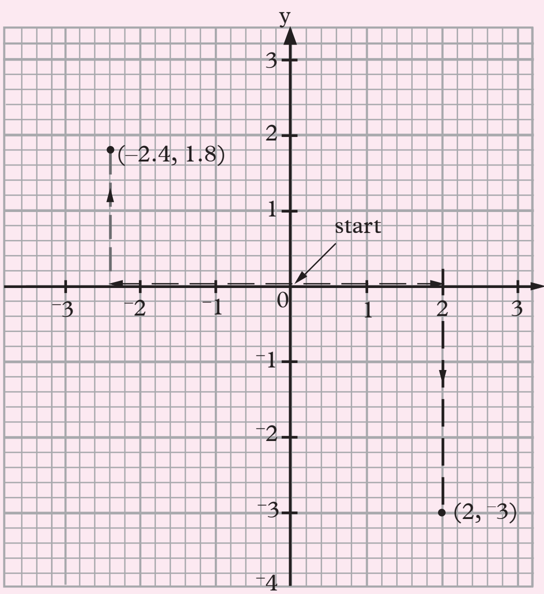


Fig. 3.14

For (⁻2.4, 1.8):

Move 2.4 steps to the left along the x-axis. Move 1.8 steps up parallel to the y-axis.

Note that the dotted arrows in Fig. 3.14 are not normally put on the graph. They are used here only to show the method of plotting the points, and for the purposes of demonstrating.

Example 3.2

The vertices of a quadrilateral are:

A(⁻7.5, ⁻5), B(0, ⁻5), C(7.5, 7.5), D(0, 7.5).

(a) *Using a suitable scale, plot the points A, B, C and D.*

(b) *Join the vertices of quadrilateral ABCD and state what kind of quadrilateral it is.*

(c) *Find the coordinates of the intersection of the diagonals of ABCD.*

Solution

(a) *The highest value of x-coordinate is 7.5 and the lowest is ⁻7.5. The x-axis must include these numbers.*

The highest y-coordinate is 7.5 and the lowest is ⁻5. With a scale of '1 cm represents 1 unit' on either axis, it should be possible to include all the numbers but the figure would be unnecessarily large. A scale of '2 cm represent 5 units' should be appropriate so that the figure is neither too large nor too small. The points are plotted in Fig. 3.15.

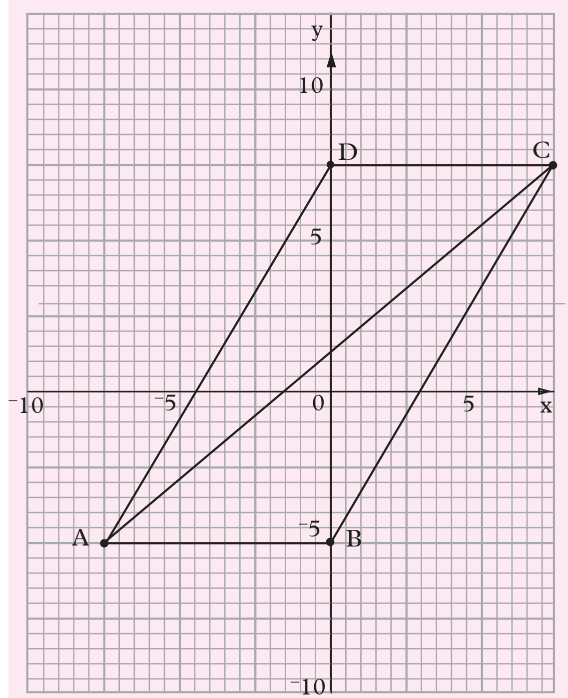


Fig. 3.15

(b) *ABCD is a parallelogram.*

(c) *The point of intersection of the diagonals is (0, 1.25).*

Exercise 3.3

1. On squared or graph paper, plot the following points, taking one square to represent a unit on both axes.

   A ($^-1.5$, $^-0.5$), B (0.5 , $^-2.5$) and C (4 , 1.5).

   Given that ABCD is a parallelogram, find the possible coordinates of D.

2. Using a scale of '1 cm represents 2 units' on both axes, plot the following points.

   A (8, 10), B ($^-8$, $^-10$), C (3, $^-5$), D ($^-6$, 9), E ($^-4$, $^-7$), F (l, 8), G (2, 0), H (0, $^-6$), I ($^-2.4$, 5.2), J ($^-4$, 3.8), K (0, 6.6), L(0.8,$^-7.8$).

3. The vertices of a triangle are X($^-0.5$, $^-0.8$), Y ($^-0.5$, $^-2.4$) and Z (2, 0.8). Find the area of the triangle.

4. Plot the following points, joining each point to the next in the order they are given. A (1, 5), B (3, 5), C (5, 3), D (5, 1), E (3, l), F (l, 3), G (3, 3).

   Draw in BE, GC and GA. What is the name of the solid formed?

5. Look carefully at the following sets of coordinates. Decide, without drawing, what shape they will make when they are joined together.

   (a) (4 , 2), (4 , 4), (4 , 5), (4 , 6)

   (b) (2 , 3), (4 , 3), (5 , 3), (7 , 3)

   (c) (1 , 1), (2 , 2), (3 , 3), (4 , 4)

   Now plot the given coordinates. Join each set of points. Were you right?

3.1.8 Linear graphs

We are now in a position to draw graphs of linear functions. We start with the linear graphs involving only one variable i.e. simple vertical and horizontal lines.

We have learnt how to choose appropriate scale and use it to plot points whose coordinates are given. In this unit, we will expand our graphical work restricting ourselves to linear graphs only.

The ordered pair (x, y) represents coordinates of any point (i.e. a general point) on the Cartesian plane.

Consider the equation $y = 3$. For all values of x, y is always equal to 3. Thus (x, y) may lie anywhere on the line shown in Fig. 3.16.

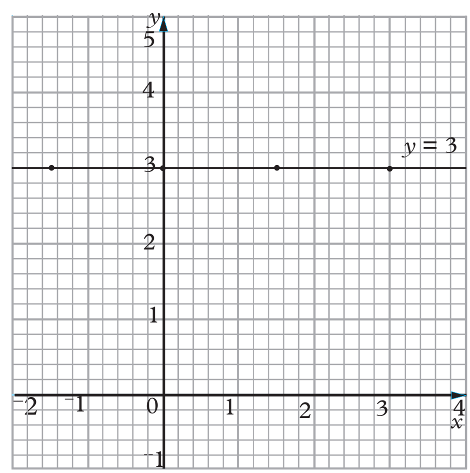


Fig. 3.16

$y = 3$ is called the **equation of the line** and Fig. 3.16 is the graph of the line $y = 3$.

Fig. 3.17 shows lines which are parallel to the axes. The equation of each line is written beside it.

Note that the **equation of the x-axis is** $y = 0$, i.e. the line on which all points have the y-coordinate as 0.

Likewise **$x = 0$ is the equation of the y-axis**.

Activity 3.8

Draw a pair of horizontal and vertical lines and label them x- and y- axes respectively. Using same scale on both axes, draw five different vertical lines and five horizontal lines and state their equations. On a different pair of axes, draw the lines whose equations are x = 4, x = 1, x = -2, x = 3, y = 3, y = -4, y = 5, y = $2\frac{1}{2}$, y = 0. The result of Activity 3.8 should resemble Fig. 3.17 below. These are examples of linear functions in one unknown.

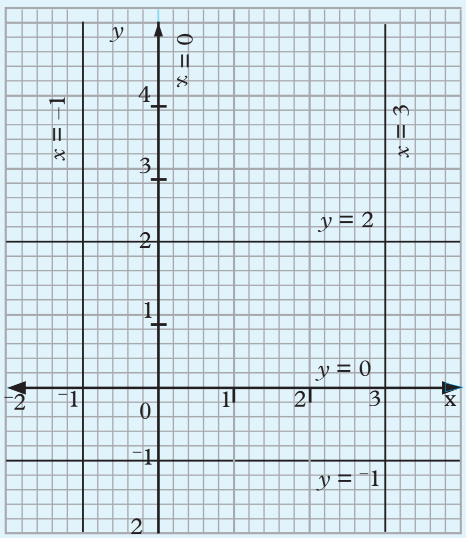


Fig. 3.17

Consider the equation $y = x + 2$ (a linear equation in x and y). For every value of x, there is a corresponding value of y, x and y are called **variables**. The equation gives the connection (or relation) between the variables.

For example, if $x = 0, y = 2$; if $x = 1, y = 3$; if $x = 3, y = 5$, etc. These values can be written as ordered pairs of corresponding x and y values as: (0, 2), (1, 3), (3, 5), etc.

If these pairs are plotted as points on the Cartesian plane, they give the **graph of the equation $y = x + 2$**, as shown in Fig.3.18.

Since the points lie on a straight line, they are joined using a straight edge or ruler.

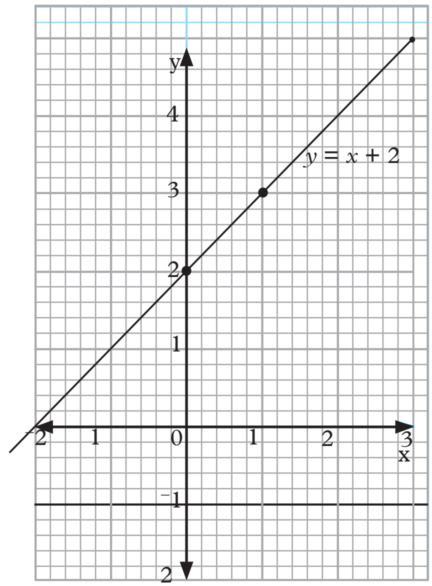


Fig. 3.18

3.1.9 Graphs of a straight-line (Table of values)

When drawing a graph of a linear equation, it is sufficient to plot only two points. In practice, however, it is wise to plot three points. If the three points lie on the same line, the working is probably correct, if not you have a chance to check whether there could be an error in your calculation.

Consider the equation $y = 2x + 3$.

If we assign x any value, we can easily calculate the corresponding value of y.

$y = 2x + 3$,

when $x = 0$, $\quad y = 2 \times 0 + 3 = 3$

when $x = 1$, $\quad y = 2 \times 1 + 3 = 5$

when $x = 2$, $y = 2 \times 2 + 3 = 7$ and so on.

For convenience and ease while reading, the calculations are usually tabulated as shown in Table 3.1

x	0	1	2	3	4
$2x$	0	3	4	6	8
$+3$	3	3	3	3	3
$y = 2x + 3$	3	5	7	9	11

Table. 3.1

From the table the coordinates (x, y) are (0, 3), (1, 5), (2, 7), (3, 9), (4, 11)……

A table such as Table 3.1 is called a **table of values** for $y = 2x + 3$.

Activity 3.9

Using a dictionary or the internet, find the meanings of the terms.

(i) Variable.

(ii) Dependent variable.

(iii) Independent variable.

In general a variable means;

- A quantity which is not constant i.e. a quantity able to assume different numerical values under different conditions.
- In a function such as y = ax + c where a and c are constants, x and y are variables.
- The value of y depends on the value of x i.e. the value of y is determined by the value of x.
- Thus: y is the dependent variable and x the independent variable.

From the calculations, the value of y depends on the value of x. y is therefore called the dependent variable and x the independent variable.

When drawing the graph, the dependent variable is marked on the vertical axis generally known as the **y – axis**. The independent variable is marked on the horizontal axis also known as the **x – axis**.

Fig. 3.19 shows the line whose equation is $y = 2x + 3$.

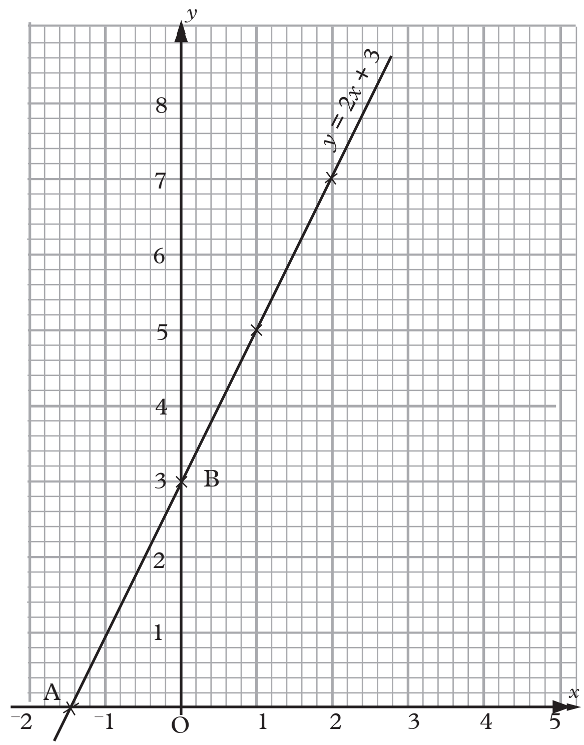


Fig. 3.19

The graph of y = 2x + 3 is neither vertical nor horizontal. It meets the two axes at distinct point A and B. (–1.5, 0) and (0, 3) respectively.

The point B is called the ***y*-intercept** and has cordinates (0, 3).

Exercise 3.4

1. On a Cartesian plane, draw and label the lines whose equations are:

(a) $x = 5$ (b) $y = -2$

(c) $y = 3.5$ (d) $x = -4$

2. For each of the following lines, find the coordinates of any two points on the line. Use the points to draw the lines whose equations are:

(a) $y = x + 1$

(b) $y = 5x - 1$

(c) $y + \frac{1}{4}x = 2$

(d) $2x - y = 3$

Hence, state the coordinates of the point where:

(i) $y = x + 1$ and $y = 5x - 1$ intersect.

(ii) $y + \frac{1}{4}x = 2$ and $2x - y = 3$ intersect.

3. Copy an complete Table 3.2 for the equation $3y + 2x = 21$

x	-6	-3	0	3	6
y	11	-	-	-	3

Table 3.2

(a) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $3y + 2x = 21$

(b) Use your graph to find the value of:

(i) y when $x = -1.5$

(ii) x when $y = 4$

4. Use a suitable scale to draw on the same axes the line:

$y = \frac{1}{3}x$; $y = \frac{1}{3}x + 2$; $y = \frac{1}{3}x - 2$

for values of x from $x = -6$ to $x = 6$

What do you notice about the three lines?

5. Copy and complete Table 3. For $y = 2x - 1$

x	-2	-1	0	1	2
y		-3		1	

Table 3.3

Draw the graph of $y = 2x - 1$.

(b) Find the value of y when $x = 4$.

(c) Find the value of x when $y = -2$.

(d) Write down the coordinates of the points where the line meets the axes.

3.1.10 The *y* and *x* intercept

Activity 3.10

- Consider the linear functions $3x + 2y = 4$ and $x - y = 3$.
- Make separate tables of values for the functions for values of x: $-2 \le x \le 4$.
- Choose an appropriate scale for both axes.
- Plot the points and join them to form two straight lines.
- If your lines intersect, state the coordinates of the common point.
- Does the line $3x + 2y = 4$ meet the axes? If so, state the coordinates, where:
  - (i) It meets the y – axis.
  - (ii) It meets the x – axis.
- Repeat bullet 6 for the line $x - y = 3$.

From activity 3.10, a line which is neither vertical nor horizontal meets the two axis at two points. The two points are

i) The y-intercept and

ii) The x – intercept respectively

From your activity, the line whose equation is $3x + 2y = 4$ meets the y – axis at (0, 2).

Similarly, the line whose equation is $x - y = 3$ meets the y – axis at the point (0, -3).

The points such as (0,2) and (0, -3) are called the y-intercepts.

These intercepts can be obtained without drawing the graphs as follows:

The function $3x + 2y = 4$ □

$2y = -3x + 4$

$y = (-\frac{3}{2})x + 2$

On the y – axis, $x = 0$

$\therefore$ when $x = 0$, $y = (-\frac{3}{2}) \times 0 + 2$

$= 2$

$\therefore$ the y – intercept is the point (0, 2), also written as the y – intercept is 2.

Similarly, we have seen the line $x - y = 3$ meets the y – axis at (0, -3).

The function $x - y = 3 \Rightarrow -y = -x + 3$

$y = x - 3$

The function $y = (-\frac{3}{2})x + 2$ is called the y-intercept form of the equation $3x + 2y = 4$

In general, the general equation of a line can be written in the form $y = mx + c$ where m and c are constants and c is the y-intercepts.

We can use a method similar to the one above to find the x – intercept.

Activity 3.11

Find the x – intercept and y – intercepts.

- Use the functions

  $y - x = 2$, $y - 2x - 4 = 0$,

  $y + x + 3 = 0$, $y = 3x - 2$,

  $y + 2x = 5$, $y - x = 1$

- Draw the graphs of the given functions.

- Copy and complete the table below.

Function	y-intercept	x-intercept
$y - x = 2$		
$3y - 6x - 12 = 0$		
$y + x + 3 = 0$		
$y - 3x = -2$		
$2y + 4x = 10$		
$-2y + x = 2$		

Table 3.4

Exercise 3.5

1. In this question, write each question in the form $y = mx + c$. Hence make a table of values for each question and represent the equations graphically
   (a) $5x + 2y = 0$
   (b) $y - 3x - 1 = 0$
   (c) $2x + y = 3$
   (d) $4x - 2y + 3 = 0$
   (e) $2x + 3y = 3$
   (f) $5 = 5x - 2y$
   (g) $2x + 3y = 6$
   (h) $8 - 7x - 4y = 0$

   Use your graph to find the value of x and y intercept in each case.

2. Find the y-intercept of the following without drawing the graphs.
   (a) $y = 3x + 7$
   (b) $7 - 2x = 4y$
   (c) $4y + x - 8 = 0$
   (d) $2y + \frac{1}{3}x + \frac{1}{6} = 0$
   (e) $\frac{3}{2}y - 15 = \frac{2}{3}x$
   (f) $-10(x + 3) = 0.5y$
   (g) $y = 5x - 4$
   (h) $2y + x = 7$
   (i) $5x + 6y - 4 = 0$
   (j) $2y - 8 = 7x$

3. Find x-intercepts of the lines in question (2) above.

3.1.11 The gradient of a straight line

A bus conductor wishes to move some luggage to the roof rack of his bus. To get onto the roof, he uses an extendable ladder which he places in position AC, as shown in Fig. 3.20.

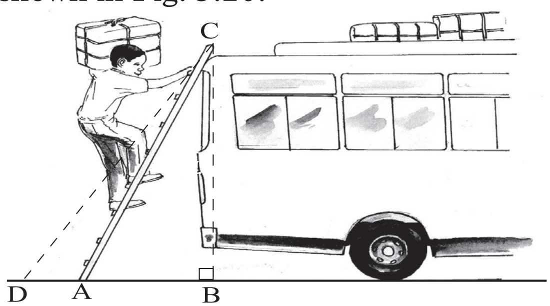


Fig. 3.20

The height BC of the bus is 2.8 m and the distance AB is 2 m. For every one metre that the conductor moves horizontally, what is the corresponding vertical distance?

This may be expressed in ratio form as:

$$\frac{\text{Vertical distance}}{\text{Horizontal distance}} = \frac{2.8}{2} = 1.4$$

Suppose that the conductor extends the ladder further so that it is in position DC (dotted). Will he find the climb to the top of the bus any easier? Why?

If the ladder is in position DC, and given that BD = 2.5 m, what is the corresponding vertical distance for every one metre that the conductor moves horizontally?

You should find that the ratio $\frac{\text{Vertical distance}}{\text{Horizontal distance}}$ is now 1.12.

The ratio, $\frac{\text{Vertical distance}}{\text{Horizontal distance}}$ is a measure of steepness or slope. It is known as gradient. Gradient measures how steep an incline is.

Notice, therefore, that gradient is higher for a steeper incline.

3.1.12 Determining the gradient of a line through known points

The gradient of a straight line is determined in the same way as that of the inclined ladder in Fig. 3.20.

In Fig. 3.21, OC is a straight line through the origin.

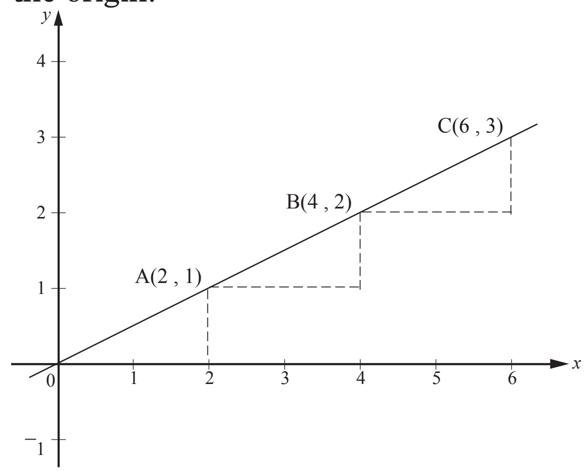


Fig. 3.21

In moving from O to C, we move 1 step vertically for every two steps that we move horizontally.

Thus the gradient of line OC is

$$\frac{\text{Vertical distance}}{\text{Horizontal distance}} = \frac{3}{6} = \frac{1}{2}$$

Activity 3.12

(a) Draw the line whose equation is given as $2y - 3x = 4$

(b) From your graph, find the y-intercept.

(c) Identify any three points A, B, C on the line such that $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

(d) Using points A and B, find the vertical distance $y_2 - y_1$ from A to B and the horizontal distance, $x_2 - x_1$ from A to B.

(e) Find the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ and in their simplest form.

(f) What do you notice about the three quantities in $\frac{y_3 - y_1}{x_3 - x_1}$ and $\frac{y_3 - y_2}{x_3 - x_2}$ (f) above?

(g) Using the bus example method, what is the gradient of the line?

(h) Now, express the function $2y - 3x = 4$ in the form $y = mx + c$. State the values of *m* and *c*.

(i) Comment on the value of m.

From your activity 3.12, you should have observed that :

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

For any line passing through points $P(x_1 , y_1)$ and $Q(x_2 , y_2)$ as shown in Fig. 3.22.

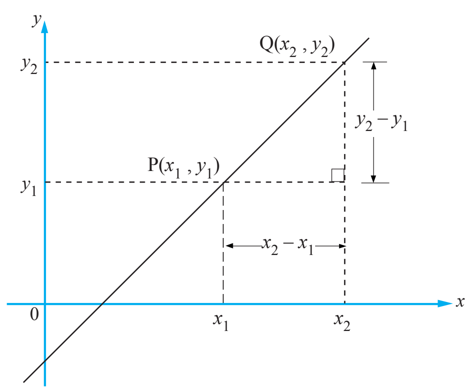


Fig. 3.22

$$\text{Gradient} = \frac{\text{Vertical distance}}{\text{Horizontal distance}}$$

$$= \frac{\text{change in y-coordinate}}{\text{corresponding change in x-coordinate}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

1. If an increase in the x-coordinate causes an increase in the y-coordinate (Fig. 3.23(a)), i.e. the line slopes upwards from left to right, the gradient is positive.

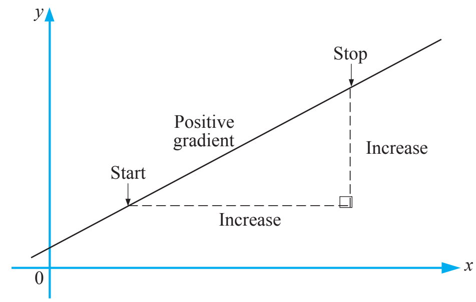


Fig. 3.23(a)

2. If an increase in the *x*-coordinate causes a decrease in the *y*-coordinate (Fig 3.23 (b)), i.e. the line **slopes downwards** from left to right, the gradient is **negative**.

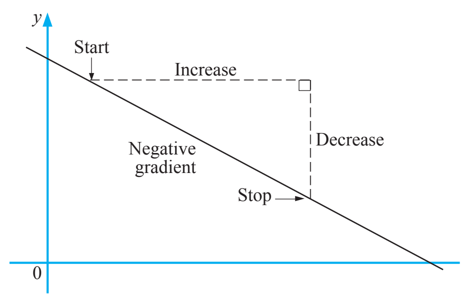


Fig 3.23 (b)

Activity 3.13

On a Cartesian plane plot the points:

(a) (3, 5) , (7, 8) (b) (2, 5), (4, 7)

(c) (0, 4), (2, 0) (d) (-1, 4), (3, -1)

(e) (-1, 2), (1, -1)

Calculate the gradients of the line segments joining the respective pairs of points. In each case describe the inclination of each line segment.

Example 3.3

Determine the gradients of line segments AB and AC in Fig. 3.24.

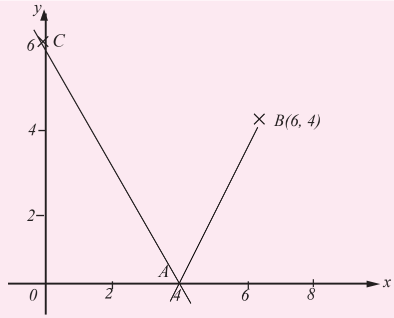


Fig. 3.24

Solution

Point A is (4,0) and point B is (6,4). Using these points, we get

Gradient of line AB $= \dfrac{\textit{change in y-coordinate}}{\textit{change in x-coordinate}}$

$$= \frac{4-0}{6-4} = \frac{4}{2} = 2$$

Point A is (4, 0) and C is (0, 6). In moving from left to right, i.e. from C to A, line AC slopes downwards.

The change in y–coordinate is $6 - 0 = 6$*, and the change in x–coordinate is* $0 - 4 = {}^{-}4$.

$\therefore$ *Gradient of line AC* $= \dfrac{-6}{4} = -1.5$

3.1.13 Gradients of vertical and horizontal lines

Let us now investigate what happens when we find the gradient of vertical or horizontal lines.

Activity 3.14

1. On squared paper, draw the line $y = 3$.
2. Identify two points on the line and use them to find the gradient of the line.
3. Draw another line whose equation is $x = 2$.
4. Using any two points on the line, find the gradient of the line.
5. Comment on your answer to (4) above.

From activity 3.14 you should have established that the gradient of the line $y = 3$ is 0.

i.e. vertical distance = 0

horizontal distance = c where $c \neq 0$

$$\frac{\text{Vertical distance}}{\text{Horizontal distance}} = \frac{0}{c} = 0$$

For the line $x = 2$, the vertical distance = c where $c \neq 0$, but the horizontal distance = 0

$$\therefore \text{ Gradient} = \frac{\text{Vertical distance}}{\text{Horizontal distance}} = \frac{c}{0}$$

The quantity is not defined

$\therefore$ The gradient of the line $x = 2$ is not defined i.e. does not exist since division by zero is not possible.

For a line $x = k$, where k is a constant, gradient is not defined.

For a line $y = c$, where c is a constant, gradient = 0.

Example 3.4

Find the gradients of lines DE and EF in Fig. 3.25.

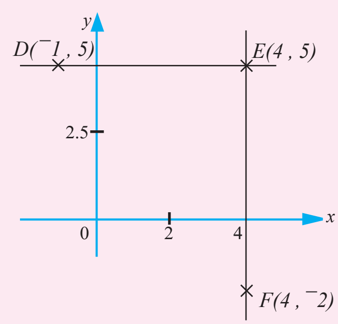


Fig. 3.25

Solution

Moving from D to E, the

change in x-coordinate is $4 - {-1} = 5$,

change in y-coordinate is $5 - 5 = 0$.

$\therefore$ *gradient of line DE is* $\dfrac{0}{c} = 0$

Moving from F to E, the
change in x-coordinate is 4 – 4 = 0,
change in y-coordinate is 5 – $^-$2 = 7

$\therefore$ *gradient of line EF is* $\frac{7}{0}$. *But the value of* $\frac{7}{0}$ *is* ***undefined***. *See note below. Thus the gradient of line EF is undefined.*

Note:

$\frac{6}{3} = 2$ means that 2 (quotient) multiplied by 3 (divisor) equals 6 (dividend).

Similarly, $\frac{0}{7} = 0$ means that 0 (quotient) multiplied by 7 (divisor) equals 0 (dividend), which is true.

But $\frac{7}{0} = 0$ would mean 0 (quotient) multiplied by 0 (divisor) equals 7 (dividend), which is not true.

Likewise, if x is any number, $\frac{x}{0}$ has no meaning.

Thus, $\frac{0}{x} = 0$ while $\frac{x}{0}$ is undefined.

We notice that:

1. If, for an **increase** in the x-coordinate, there is **no change** in the y-coordinate, i.e. the line is **horizontal**, the gradient is **zero**.(all lines parallel to x axis).
2. If there is **no change** in the x-coordinate while there is an **increase** in the y-coordinate, i.e. the line is **vertical**, the gradient is **undefined** or does not exist.(all lines parallel to y axis).

Exercise 3.6

1. For each of the following pairs of points, find the change in the x-coordinate and the corresponding change in the y-coordinate. Hence find the gradients of the lines passing through them. You may plot the points first.
   (a) (0 , 2), (3 , 4)
   (b) (0 , 2), (5 , 0)
   (c) ($^-$2 , $^-$2), (2 , 0)
   (d) ($^-$1 , $^-$2), (1 , 8)
   (e) ($^-$1 , 2), (3 , $^-$2)
   (f) (0 , $^-$2), (0 , 3)
   (g) (2 , $^-$8) ($^-$2 , 8)
   (h) ($^-$2 , 0), (3 , 0)
2. Find the gradient of the line which passes through each of the following pairs of points.
   (a) (3, 5), (9, 8)
   (b) (2, 5), (4, 10)
   (c) (7, 3), (0, 0)
   (d) (1, 5), (7, 2)
   (e) (0, 4), (4, 0)
   (f) ($^-$2, 3), (5, 5)
   (g) ($^-$7, 3), (8 , $^-$2)
   (h) ($^-$3, $^-$4), (3, $^-$4)
   (i) ($^-$1, 4), ($^-$3, $^-$1)
   (j) (3, $^-$1) , (3, 1)
3. Find the gradients of the lines l_1 to l_5 in Fig. 3.26.

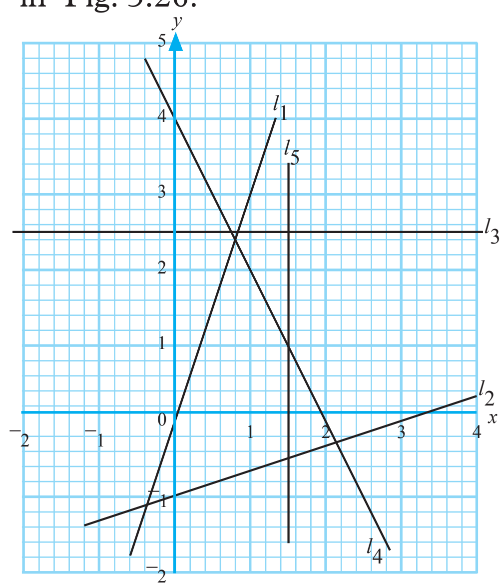


Fig.3.26

4. In each of the following cases, the coordinates of a point and the gradient of a line through the point are given. State the coordinates of two other points on the line.
   (a) (3 , 1), 3 (b) (4 , 5), $\frac{1}{2}$
   (c) ($^-$2 , 3), $^-$1 (d) (5 , 5), $^-\frac{1}{2}$
   (e) ($^-$4 , 3), undefined
   (f) ($^-$4, 3), 0
5. A square has the vertices of A (6, 1) B(6, 6) C(1, 6) and D (1, 1)
   (a) On a squared paper, plot the points A, B, C and D.
   (b) Calculate the gradients of:
      i) AB ii) BC
      iii) DC iv) AD
      v) AC vi) BD
   (c) Comment on your answer.

3.1.14 Gradient and y – intercept

We have just seen that we can find the gradient of a line if we know the coordinates of any two points on the line. It is possible to determine the gradient of a line by rearranging its equation so that it is in the form y = mx + c.

Activity 3.15

Carry out the following activity:
1. Find the coordinates of any two points on the line whose equation is $3x - y = 2$
2. Use the two points to determine the gradient of the line.
3. Express $3x - y = 2$ in the form $y = mx + c$.
4. How does the gradient of the line compare with the coefficient of x in (3) above?
5. Determine the x and y –intercept.

From your activity, you should have found that:

i) The gradient of the line is 3.
ii) The coefficient of x in $y = mx + c$ is equal to the gradient of the line i.e. m = 3.
iii) y – intercept is -2
iv) x – intercept is $\frac{2}{3}$

Note that it is possible to determine the gradient and intercepts of the line $3x - y = 2$ without looking for any points or drawing the graph.

Now consider the equation $3x - y = 2$

$3x - y = 2$

$-y = -3x + 2$ - ← after adding -3x to both sides

$y = 3x - 2$ ← after multiplying each term by -1

⇒ The coefficient of x = 3

Hence its gradient = 3

The y – intercept of the line is – 2 and the constant term in $y = 3x - 2$ is -2

∴ Gradient = coefficient of x = 3

y - intercept = the constant term = -2

The x – intercept can be determined by expressing the equation in the from $x = ky + c$

Thus, x – intercept = c

In general, for the equation of a line in $y = mx + c$

i) The gradient of the line = m
ii) The y-intercept = c

Example 3.5

Use the y-intercept form to determine the gradient and the y-intercept of the line whose equation is

(a) $4x - 3y = 9$ *(b)* $5x + 7y - 14 = 0$

Solution

(i) $4x - 3y = 9$

$4x - 4x - 3y = 9 - 4x$ (subtract $4x$ from both sides)

$-3y = -4x + 9$

$\frac{-3y}{-3} = \frac{-4y}{-3} + \frac{9}{-3}$ (divide both sides by -3)

$y = \frac{4}{3}x - 3$

$y = \frac{4}{3}x - 3$ is in the form $y = mx + c$

$\therefore m = \frac{4}{3}$ and $c = -3$

$\therefore$ gradient of the line is $\frac{4}{3}$

and $y -$ intercept is -3

(ii) $5x + 7y - 14 = 0$

$5x + 7y = 14$ (add 14 to both sides)

$7y = -5x + 14$ (subtract $5x$ from both sides

$y = \frac{-5x + 14}{7}$

$= \frac{-5}{7}x + \frac{14}{7}$

$\therefore$ Gradient $= \frac{-5}{7}$

$y -$ intercept $= 2$

Activity 3.16

Rewrite each of the equations $x - 2y = {}^-6$, $2x + 3y = 6$ and $4x + 2y = 5$ in the form $y = mx + c$ (as we did in Example 2.5). Find any two convenient points on each line and draw the lines on squared paper. Calculate the gradient of each line.

Copy and complete Table 3.5.

Equation of line	The form $y = mx + c$	m	c	Gradient	y-intercept
$x - 2y = {}^-6$	$y = \frac{x}{2} + 3$		3	$\frac{1}{2}$	3
$2x + 3y = 6$					
$4x + 2y = 5$					

Table 3.5

y = mx + c is an equation of a straight line where m represents the gradient and c is the y-intercept of the line. The y-intercept is the value of y at the point where the line crosses the y-axis.

When we write the equation of a line in the gradient-intercept form, we can easily tell the gradient and y-intercept of the line and hence sketch it, if need be.

Example 3.6

Find the gradient and y-intercept of the line whose equation is $4x - 3y - 9 = 0$. Sketch the line.

Solution

$4x - 3y - 9 = 0$ is equivalent to $y = \frac{4}{3}x - 3$.

Comparison with $y = mx + c$ gives

gradient, $m = \frac{4}{3}$,

y-intercept, $c = {}^-3$.

Fig. 3.27 is a sketch of the line.

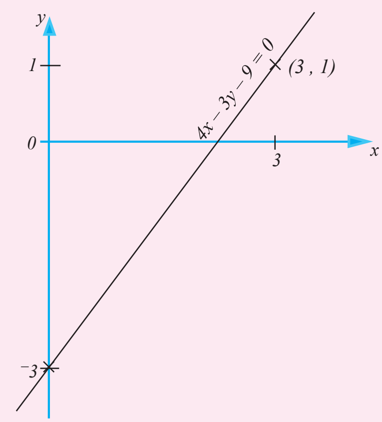


Fig. 3.27

Note that for an increase of 3 units in the x-coordinate, the increase in the y-coordinate is 4 units. Hence, point (3,1) is on the line.

Note: The values of the gradient (m) and y intercept determines the orientation and position of a line in the cartesian plane, as shown in Fig. 3.28 below.

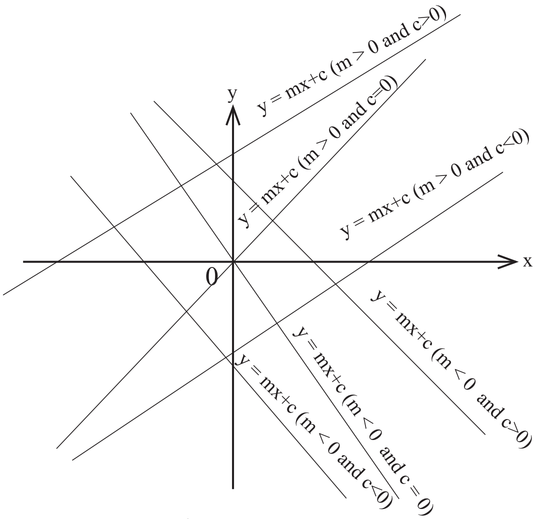


Fig. 3.28

Exercise 3.7

1. In each of the following cases, determine the gradient and y-intercept by writing the equation in the form $y = mx + c$. Sketch the line.
   (a) $5x = 2y$
   (b) $y - 3x - 1 = 0$
   (c) $2x + y = 3$
   (d) $4x - 2y + 3 = 0$
   (e) $2x + 3y = 3$
   (f) $5 = 5x - 2y$
   (g) $2x + 3y = 6$
   (h) $8 - 7x - 4y = 0$
2. Find the y-intercepts of the lines with the given gradients and passing through the given points.
   (a) 3, (2 , 6) (b) ($^-2$, 3), 2
   (c) $^-2$, (7 , 4) (d) (2 , 4), $\frac{1}{2}$
   (e) 0, ($^-3$, $^-2$)
   (f) Undefined, (1, 3)
3. Write down, the equations of lines (a), (b), (c) and (d) in Fig. 3.29 in the gradient – intercept form.

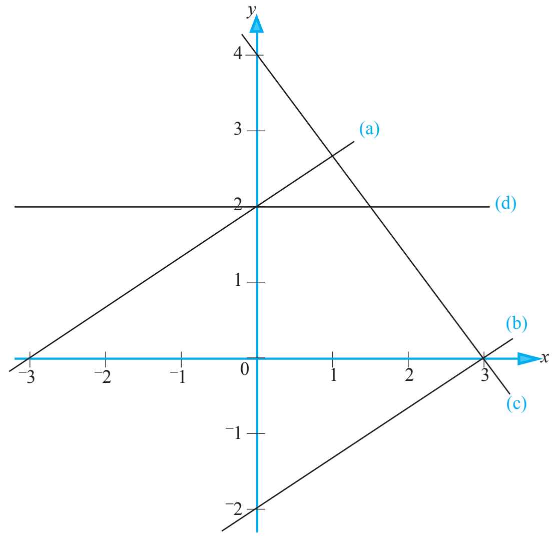


Fig. 3.29

4. Copy and complete table below.

Function	y-intercept	Gradient
2y + 3x = 2		
3x + 4y = 8		
-15 + 9y = 3x		
2x + 2y = 7		
-2x – 4y = 6		
5y – 3x = 2		

Table 3.6

3.2 Linear equations

3.2.1 Meaning of linear equation

A mathematical sentence with the symbol = is called an **equation.** Such a statement expresses the equality of quantities.

An equation may be conditional or an identity.

A **conditional equation** is one that is satisfied by at least one value of the unknown. For example there is only one value of the unknown *x* which satisfies the equation $x + 11 = 15$.

An **identity** is an equation which is true regardless of what values are substituted for the unknown. For Example;

(i) $2x + 2 = 2(x + 1)$

$2(x - 1) + 3 = 2x + 1$

$3x - (2x + 1) = x - 1$

are identities. Whatever value x takes, the statement is true.

(ii) $(x + 7)^2 = 2x^2 + x + 1$ is a conditional equation since it is true only when $x = {}^-3$ or $x = 16$.

A letter such as *x* used above is called an **unknown**.

An equation in which the unknown has power 1 (i.e. it is just *x* or *y*, etc.) is known as a **linear equation**. For example: $x + 3 = 17$, $2m - 5 = 25$, $\frac{n}{4} - 3 = 8$, are linear equations.

3.2.2 Meaning of letters in algebra

In algebra, we use letters and symbols to represent unknown quantities. When used, the letters may have different meanings depending on the type of problem to be solved. Generally, the first letters of alphabet a, b, c.... are used to represent constant values or numbers. The last letters of the alphabet x, y, z are used to represent unknown values to be solved.

Exercise 3.8

1. Which of the following are equations?

(a) $3 - 5 + 1x$ (b) $6 - 5 = 1$
(c) $5 + 2 - 6x$ (d) $4 \times 6x$
(e) $3 - 9x$ (f) $3 \times 8 = 24$
(g) $16 - 9x = 3 + 4x$ (h) $4 + 17 - 3x$
(i) $6 \times 4x = 3 \times 8$ (j) $17 = 13 + 4x$

2. State whether the following are true or false.

(a) $3 + 6 = 7 + 2$ (b) $7 - 3 = 6 - 2$
(c) $6 + 2 = 6$ (d) $5 + 6 + 1 = 11$
(e) $10 + 7 + 4 = 13 + 4$
(f) $5 + 6 - 3 = 9$ (g) $3 + 5 - 2 = 8$

3. State whether the following are true, false or open.

(a) $2 - 5 = 3$ (b) $8 + 13 = 23$
(c) ${}^-2 + 9 = 7$ (d) $7 - x = 0$
(e) $2 = 2x - 3$ (f) $13 - 17 = {}^+7 + {}^-3$
(g) $({}^-24) \div ({}^+3) = ({}^-8)$
(h) $\frac{({}^-6) \times ({}^-5)}{10} = 3$

4. Copy and complete the following to make them true.

(a) $4 + 7 =$ (b) $5 + 1 =$
(c) $4 - 1 =$ (d) $6 - 6 =$

3.2.3 Solving linear equations

Consider the equation $x + 13 = 19$.

This equation may be true or false depending on the value of the unknown. For example, it is only true when $x = 6$, but false when $x = 3$.

The value of the unknown that makes an equation true is called the **solution** of the equation.

To **solve** an equation means to find the value of the unknown which makes the equation true.

Solving linear equations involves changing equations into simple equivalent equations. For example the equation $x + 7 = 5$ is equivalent to $x = -2$.

Thus, $x + 7 = 5$ is true only when $x = -2$.

-2 is called the solution of $x + 7 = 5$

(a) Simple approaches to solving linear equations

Activity 3.17

Copy and complete the table 3.7 below.

Equation	Equivalent equation	Solution of the equation
6 + x = 11	x = 5	[5]
x – 10 = -6		
x + 12 = 17		
10x = 50		
$\frac{1}{2}$ x = 4		
-2x = 10		
x + 5 = 12		
x + a = 8 + a		
X + $\frac{1}{3}$ = 6		
$\frac{4}{5}$ x = 20		

Table 3.7

We say that 6 + x = 11 is equivalent to x = 5 and the solution set of 6 + x = 11 is [5].

From the above activity, you should have identified different approaches to making simple equivalent equations from given equations. These are: -

i) Adding same number to both sides.

ii) Subtracting same number from both sides.

iii) Multiplying both sides by the same number.

iv) Dividing both sides by the same number.

Remember:

1. Adding to both sides a number such as -4, means the same as subtracting 4 from both sides.
2. Multiplying both sides by a number such as $\frac{1}{3}$ means the same as dividing both sides by 3.

What do you think it means to multiply both sides by a number such as 3.7? Discuss.

Example 3.7

Solve the equations

(a) $x + 6 = 10$ *(b)* $x - 4 = 14$

(c) $2x = 14$

Solution

(a) We are required to find a number to which we add 6 to get 10.

The number is 4.

$\therefore$ *The solution is* $x = 4$.

(b) We are required to find a number from which we subtract 4 to get 14.

The number is 18.

$\therefore$ *The solution is* $x = 18$.

(c) We are required to find a number which when multiplied by 2 gives 14.

The number is 7.

$\therefore$ *The solution is* $x = 7$.

The method used in Example 3.6 is known as the **cover-up technique.** Sometimes we use the cover-up technique more than once.

Example 3.8

Solve (a) $2x + 3 = 17$

(b) $^{-}3 + 2x = 5x$

Solution

(a) $2x + 3 = 17$

$\square + 3 = 17$

$14 + 3 = 17$

so, $2x = 14$ *Check:* $2x + 3 = 2 \times 7 + 3$

$2 \times \square = 14$ $\qquad = 14 + 3$

$2 \times 7 = 14$ $\qquad = 17$

$x = 7$

$\therefore$ *Thus, 7 is the solution*

(b) $^{-}3 + 2x = 5x$

$$^{-}3 + 2x = 5x$$
$$\text{so } 5x - 2x = {}^{-}3$$
$$3x = {}^{-}3$$
$$x = \frac{^{-}3}{3}$$
$$x = {}^{-}1$$

Exercise 3.9

1. Copy and complete the following to make them true.
   (a) $18 \div 3 = \square$
   (b) $5 + \square = 7$
   (c) $9 - \square = 7$
   (d) $3 \times \square = 6$
   (e) $6 + \square = 9$
   (f) $24 \div \square = 6$
   (g) $4 \times \square = {}^{-}12$
   (h) $7 + 4 + \square = 13$
   (i) $3 + 4 + \square = 10$
   (j) $8 + 2 + \square = 6$
   (k) $4 + 5 - \square = 2$
   (l) $4 + 6 - \square = 2$
   (m) $6 + \square + 4 = 3$
   (n) $3 + 7 + 1 = 9 - \square$
   (o) $8 + 3 + 2 = 10 + \square$
   (p) $6 + 5 = 11 - \square$
   (q) $6 + 6 - 3 = 12 - \square$
   (r) $3 + 12 - 8 = 15 + \square$

2. State whether the following are true or false.
   (a) $\frac{x}{4} = 5$ when $x = 24$
   (b) $x - 2 = 9$ when $x = 7$
   (c) $20 + x = 28$ when $x = 8$
   (d) $4x = 20$ when $x = {}^{-}5$
   (e) $12 - 3x = 0$ when $x = 4$
   (f) $8 = 9 - x$ when $x = 1$.

3. Solve the following equations.
   (a) $x + 6 = 20$
   (b) $x + 9 = 23$
   (c) $16 + x = 25$
   (d) $17 + x = 25$
   (e) $x - 3 = 12$
   (f) $x - 5 = 1$
   (g) $12 - x = 10$
   (h) $14 - x = 8$
   (i) $4x = 28$
   (j) $7x = 42$
   (k) $3x + 4 = 12$
   (l) $5 + 5x = 5$
   (m) $2x + 16 = 32$
   (n) $7 + x = 63$
   (o) $2x - 8 = 15$
   (p) $18 - 3x = 15$
   (q) $x - 12 = 60$
   (r) $18 - x = 10$
   (s) $30 + 3x = 9x$
   (t) $12x - 60 = 0$

(b) Solving equations by the balancing method

Activity 3.18

Research from the internet the balancing method of solving linear equations. You can also use a mathematics dictionary. Compare your findings with those illustrated in the text book.

Since an equation states the equality of two things, it may be compared to a pair of scales.
If the contents of the two scale-pans balance each other, they will still do so if:

(i) equal amounts are added to both sides,
(ii) equal amounts are taken away from both sides,
(iii) the contents of both sides are doubled, or tripled, or halved, and so on.

Thus, if the two sides balance, they will still do so if **what is done on one side is also done on the other.**

Example 3.9

Solve the equation $8x - 6 = 5x + 9$.

Solution

Imagine a pair of scales with $8x - 6$ *on one side balanced by* $5x + 9$ *on the other (Fig. 3.30 (a)).*

$8x - 6 = 5x + 9$

Add 6 to both sides: $8x - 6 + 6 = 5x + 9 + 6$

$$8x = 5x + 15$$

Subtract 5x from both sides:

$$8x - 5x = 5x + 15 - 5x$$

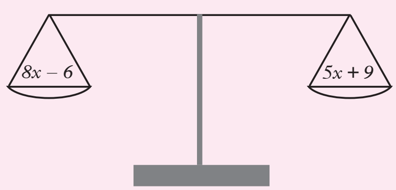
$$3x = 15$$

Divide both sides by 3:

$$\frac{3x}{3} = \frac{15}{3}$$

i.e. $x = 5$ *(Fig. 3.30 (d))*

(a)



(b)

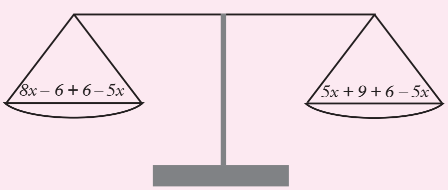
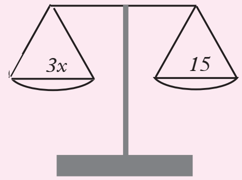


Fig. 3.30 (a) and (b)

(c)



(d)

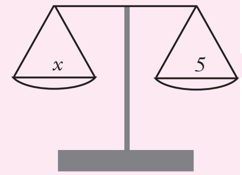


Fig 3.30 (c) and (d)

Check: If $x = 5$, $8x - 6 = 8 \times 5 - 6$

$$= 40 - 6$$

$$= 34$$

and $5x + 9 = 5 \times 5 + 9$

$$= 25 + 9 = 34$$

$\therefore$ $8x - 6 = 5x + 9$ *when* $x = 5$

The reason for adding 6 to both sides and subtracting 5x from both sides is to get rid of the $^{-}6$ *on the left-hand side (LHS) and the 5x on the right-hand side (RHS). A simpler equation is obtained, in which the LHS contains only a term in x, while x does not appear on the RHS.*

The LHS of the equation $3x = 15$ *is divided by 3 to give x. The RHS also has to be divided by 3.*

Example 3.10

Solve the equation

$3x + 34 - 8x = 11 - 9x - 13$

Solution

$3x + 34 - 8x = 11 - 9x - 13$

Simplifying both sides:

$3x - 8x + 34 = 11 - 13 - 9x$

i.e. $^{-}5x + 34 = ^{-}2 - 9x$

Adding 9x to both sides:

$^{-}5x + 34 + 9x = ^{-}2 - 9x + 9x$

$$4x + 34 = ^{-}2$$

Subtracting 34 from both sides:

$4x + 34 - 34 = ^{-}2 - 34$

$$4x = ^{-}36$$

Dividing both sides by 4:

$$\frac{4x}{4} = {}^{-}\frac{36}{4}$$

$$\therefore \quad x = {}^{-}9$$

Example 3.11

Solve $\frac{2}{5}x = 10$

Solution

$$\frac{2}{5}x = 10$$

Multiplying both sides by 5:

$$\frac{2x \times 5}{5} = 10 \times 5$$

$$2x = 50$$

Dividing both sides by 2:

$$\frac{2x}{2} = \frac{50}{2}$$

$$\therefore \quad x = 25$$

Exercise 3.10

Solve the following equations using the balance method and stating the steps as in Examples 3.9 and 3.10.

1. (a) $x + 113 = 153$
   (b) $24 = x + 13$
   (c) $x + 8 = {}^{-}12$
   (d) $x + {}^{-}7 = {}^{-}19$
2. (a) $x - 17 = 37$
   (b) $11 = x - 7$
   (c) $x - 2 = {}^{-}4$
   (d) $x - {}^{-}2 = {}^{-}7$
3. (a) $x - 72 = {}^{-}24$
   (b) ${}^{-}x + 72 = {}^{-}30$
   (c) $4 = x - 5$
   (d) $15 + 2x = {}^{-}3x$
4. (a) $24x = 72$
   (b) $+10 = {}^{-}5x$
   (c) ${}^{-}25x = 200$
   (d) ${}^{-}7 = {}^{-}84x$
5. (a) $6x - 5 = 25$
   (b) $9x + 8 = 35$
   (c) ${}^{-}5x + 5 = {}^{-}15$
   (d) ${}^{-}7x - 6 = {}^{-}20$

3.2.4 Linear equations involving fractions

Earlier in this unit, we defined and described linear equations in one unknown. We also solved simple equations. In this section, we will solve equations involving fractions and brackets.

Expressions such as $\frac{a+b}{c}$ and $\frac{x-4}{x+1}$ are called algebraic fractions.

In order to solve equations involving fractions, we must remember how to find LCM of numbers and algebraic expressions.

Activity 3.19

Find the LCM of the following:

(a) 6, 8, 16	(b) 2, 3, 4
(c) 3y, 4y, 5	(d) x, 5x, 5
(e) 2, x – 3, x + 3	(f) 4ab, 3a, 2b

Decide which one of you will explain to the class the procedure you used to obtain your answers.

Example 3.12

Solve $\frac{2}{5}y + \frac{3}{4} = 10 + \frac{y}{2}$

Solution

$$\frac{2}{5}y + \frac{3}{4} = 10 + \frac{y}{2}$$

LCM of 5,4 and 2 = 20

Multiplying both sides by 20:

$$20\left(\frac{2}{5}y + \frac{3}{4}\right) = 20\left(10 + \frac{y}{2}\right)$$
$$8y + 15 = 200 + 10y$$

Subtracting 10y from both sides:

$$8y + 15 - 10y = 200 + 10y - 10y$$
$$^-2y + 15 = 200$$

Subtracting 15 from both sides:

$$^-2y + 15 - 15 = 200 - 15$$
$$^-2y = 185$$

Dividing both sides by $^-2$

$$\frac{^-2y}{^-2} = \frac{185}{^-2}$$
$$\therefore y = -92\tfrac{1}{2}$$

Note: *Should there be a term like $1\frac{1}{2}x$ in the equation, always write it in the improper fraction form, as $\frac{3}{2}x$, and then proceed as in Example 3.11.*

Exercise 3.11

Solve the following equations using the balance method and stating the steps as in Example 3.11.

1. (a) $\frac{x}{6} - 2 = 10$ (b) $9 - \frac{x}{2} = 5$

(c) $\frac{k}{5} = 0$ (d) $p - 2\frac{1}{2} = 6\frac{1}{2}$

2. (a) $^-3\frac{3}{4} = x + 1\frac{2}{5}$

(b) $4\frac{1}{2} = 5q - \frac{1}{4}$

(c) $2p - 8 = p - 3$

(d) $t + 7 = 17 - 4t$

3. (a) $3\frac{1}{2} + 2\frac{1}{4}f = 17\frac{1}{2} - 1\frac{1}{4}f$

(b) $1\frac{1}{2}x + \frac{1}{4} = \frac{1}{2}x + 3\frac{1}{4}$

(c) $\frac{0.1}{x} + \frac{3.9}{x} = 12$

4. (a) $2\frac{1}{2}b - 4 = 10 - 3\frac{1}{4}b$

(b) $1\frac{1}{2}h - 4\frac{1}{2} = 1\frac{1}{2} + \frac{1}{2}h$

5. (a) $\frac{x}{3} + 3 = 3$ (b) $^-3 - \frac{x}{2} = 4$

(c) $\frac{1}{x} = 2\frac{1}{2}$ (d) $\frac{2}{p-1} = \frac{5}{p}$

3.2.5 Equations involving brackets

Study the identities given below.

(i) $a + (b + c) = a + b + c$
(ii) $a + (b - c) = a + b - c$
(iii) $a - (b + c) = a - b - c$
(iv) $a - (b - c) = a - b + c$

Use these rules to remove brackets before solving equations involving brackets.

Remember that the dividing line, as in $\frac{3m+2}{4}$, is both a division/and a bracket.

Sometimes brackets are used to hold quantities together. For example if we wish to multiply both x and 4 by 3, we write $3(x + 4)$. The multiplication sign is invisible just as it is in 4x which means $4 \times x$. $3(x + 4)$ means "3 times everything in the bracket." So we have $3 \times x$ and 3×4 and we write it as $3(x + 4) = 3x + 12$.

Note: Observations

An expression such as $-a\,(a + b)$ means "multiply $-a$ by each number inside the bracket."

Thus $-a\,(a + b) = -a \times a + -a \times b$

$= -a^2 - ab$

Your skills in arithmetic operations apply in algebra.

Remember

i) A positive number x a positive number = a positive number

ii) A positive number x a negative number = a negative number.
iii) A positive number ÷ a negative number = a negative number.
iv) A positive number ÷ a positive number = a positive number

Activity 3.20

Copy and complete the following:

4 (x – ?) = 4x – 8

4x + ? = -2 (-x – 5) + ?x

2 (1 – x) + ? (x + 3) = x + ?

2 (x - ?) – 6 = 2 (? – 4)

-2 (x + 6) + 5 = __ + __

2 (x – 3) + ? = -2 (-x – 4)

Did you observe the following:

1. That the multiplier outside the bracket multiplies every number inside the brackets?
2. If the multiplier outside the bracket is negative, the numbers inside the brackets change signs to the opposite?

Example 3.13

Solve the equation:

$9x - (4x - 3) = 11 + 2(2x - 1).$

Solution

$9x - (4x - 3) = 11 + 2(2x - 1)$

Removing brackets:

$9x - 4x + 3 = 11 + 4x - 2$

Simplifying both sides:

$5x + 3 = 4x + 9$

Subtracting 4x and 3 from both sides:

$5x - 4x = 9 - 3$

i.e. $x = 6$

Example 3.14

Solve the equation $\frac{5x+2}{4} - \frac{3}{2} = \frac{7x+1}{3}$

Solution

$\frac{5x+2}{4} - \frac{3}{2} = \frac{7x+1}{3}$

Multiply both sides by 12 (LCM of 4, 2 and 3)

i.e $12 \times \left\{\left(\frac{5x+2}{4}\right) - \frac{3}{2}\right\} = 12 \times \left(\frac{7x+1}{3}\right)$

i.e. $3(5x + 2) - 6 \times 3 = 4(7x + 1)$

Removing brackets:

$15x + 6 - 18 = 28x + 4$

Simplifying LHS:

$15x - 12 = 28x + 4$

Subtracting 28x from both sides and adding 12 to both sides:

$15x - 28x = 4 + 12$

i.e. $-13x = 16$

Dividing both sides by $^{-}13$:

$\frac{^{-}13}{-13}x = \frac{16}{-13}$

i.e. $x = ^{-}\frac{16}{13}$

Example 3.15

Solve the following equations.

(a) $\frac{2x+7}{3} - \frac{7x+1}{4} = 0$

(b) $\frac{2a+36}{a} - \frac{4}{5} = 0$

Solution

(a) $\frac{2x+7}{3} - \frac{7x+1}{4} = 0$

$\frac{4(2x+7) - 3(7x+1)}{12} = 0$ *(Use LCM to get same denominator)*

$$4(2x+7)-3(7x+1)=0 \quad \textit{(Multiply both sides by 12)}$$

$$8x+28-21x-3=0 \quad \textit{(Open brackets)}$$

$$-13x+25 = 0$$

$$-13x = -25$$

$$\therefore\ x = \frac{25}{13} = 1\frac{12}{13}$$

(b) $\dfrac{2a+36}{a} - \dfrac{4}{5} = 0$

$$\frac{5(2a+36)-4a}{5a} = 0$$

$$5(2a+36)-4a = 0$$

$$10a+180-4a = 0$$

$$6a+180 = 0$$

$$6a = {}^{-}180$$

$$a = {}^{-}30$$

Exercise 3.12

Solve the following equations.

1. $4d + (5 - d) = 17$
2. $12m + (1 - 7m) = 23$
3. $23 = 7 - (3 - 4t)$
4. $(8w - 7) + (5w + 13) = 0$
5. $24 - (5 + 3x) = 8x + (4 - 5x)$
6. $3(5x - 1) = 4(3x + 2)$
7. $5(3c + 4) - 3(4c + 7) = 0$
8. $4(3k - 1) = 11k - 3(k - 4)$
9. $4(3x - 5) - 7(2x + 3) + 2(5x + 11) = 5$
10. $7(5x - 3) + 10 = 2(3x - 5) + 3(5 - 7x)$
11. $\dfrac{x+1}{2} - \dfrac{x-2}{3} = \dfrac{1}{8}$
12. $\dfrac{1}{8} - \dfrac{14y-3}{2} = \dfrac{y-4}{4}$
13. $\dfrac{5-c}{4} = \dfrac{c-4}{2} + 6$
14. $\dfrac{3x+1}{2} = \dfrac{4x-3}{3} + 3$
15. $3y - \frac{2}{5}(2y - 5) = 12$
16. $\dfrac{p-3}{12} - \dfrac{3(p-1)}{8} = \dfrac{2}{5}$
17. $\dfrac{3(2x+3)}{4} = 2(3x - 2) + 4$
18. $\dfrac{x-1}{7} + 1 = \dfrac{5x+1}{5}$
19. $\frac{2}{3}(4 - 2y) - \frac{3}{5}(2 - 4y) = 2$
20. $\frac{3}{4}(8 - 4z) - \frac{2}{3}(3 - 2z) = 1$
21. $\dfrac{x}{5} + 1\frac{1}{2}x - \dfrac{11}{20} = \dfrac{5x-1}{3}$
22. $\dfrac{7e-5}{5} - \dfrac{2e-1}{10} = \dfrac{4e-3}{15}$
23. $\dfrac{2}{3y} + \dfrac{3}{5} = \dfrac{1}{4y}$

3.2.6 Forming and solving linear equations

To solve a word problem in which a number is to be found:

(i) Introduce a letter to stand for the number to be found (the unknown).

(ii) Form an equation involving this letter by expressing the given information in symbols instead of words.

(iii) Solve the equation to get the required number.

Activity 3.21

Some practical problems can be solved by forming equations.

Solve the following problems by forming an equation in each case.

Explain either in words or on a diagram

what your letter stands for and always end by answering the question asked.

a) The width of a rectangle is x cm. Its length is 4 cm more than its width. The perimeter is 48 cm. What is the width?

b) A cup of ice cream costs x shillings and a cone costs 3 shillings less. One cup and two cones together cost 54 shillings. How much is a cup?

a) Write the length of the rectangle in terms of x.
   - Draw a diagram to represent a rectangle and on it mark the dimensions.
   - Write down an equation for the perimeter of the rectangle.
   - Solve the equation for x.
   - State the width of the rectangle.

b) Write down the cost of a cone in terms of x.
   - Write down the cost of a cone in terms of x.
   - Write down an equation for the total cost of the cup and the 2 cones.
   - Solve the equation for x.
   - State the cost of a cone of ice cream.

At the end of your activity, compare your results with the working shown below.

a) The width is x cm

the length is $(x + 4)$cm.

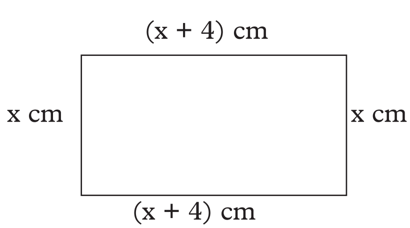


Fig 3.31

$x + (x + 4) + x + (x + 4) = 48$

$4x + 8 = 48$

Take 8 from both sides: $4x = 40$

Divide each side

Therefore the width is 10 cm

b) A cup cost x shillings

A cone costs $(x - 3)$ shillings

$x + 2(x - 3) = 54$

$x + 2x - 6 = 54$

$3x - 6 = 54$

Add 6 to both sides: $3x = 60$

Divide each side by 3: $x = 20$

Therefore a cup of ice cream costs 20 shillings

Example 3.16

The sum of two numbers is 120 and their difference is 18. Find the two numbers.

Solution

Let the smaller number be x.

Then, the larger number is $x + 18$.

Sum of the two numbers = $x + (x + 18)$.

i.e. $x + (x + 18) = 120$

$x + x + 18 = 120$

$2x + 18 = 120$

$2x = 102$

$\therefore \ x = 51$

Thus, the smaller number is 51 and the larger number is $51 + 18 = 69$.

Example 3.17

Tom and Mary share 450 FRW so that Mary gets 54 FRW less than Tom. Find their shares.

Solution

Let Tom's share be x FRW.

Then, Mary's share is $(x - 54)$ FRW.

Together, they have 50 FRW.

$$\therefore\ x + x - 54 = 450$$
$$2x = 450 + 54$$
$$2x = 504$$
$$\therefore\ x = 252$$

Thus, Tom's share is 252 FRW and Mary's share is 54 FRW less i.e.
(252 – 54) FRW = 198 FRW.

Example 3.18

Jack has 116 FRW and Jane has 64 FRW. How much must Jane give Jack so that Jack shall have 4 times as much as Jane?

Solution

Suppose Jane gives Jack x FRW.

Then Jack has (116 + x) FRW and June has (64 – x) FRW.

116 + x is 4 times as big as 64 – x,

$$\text{i.e. } 116 + x = 4(64 - x)$$
$$116 + x = 256 - 4x$$
$$x + 4x = 256 - 116$$
$$5x = 140$$
$$x = 28$$

Thus, June must give Jack 28 FRW.

Example 3.19

When 55 is added to a certain number and the sum is divided by 3, the result is 4 times the original number. What is the original number?

Solution

Let the number be x.

Adding 55 and dividing the sum by 3 gives

$$\frac{x + 55}{3} = 4x$$
$$\therefore\ x + 55 = 12x$$
$$55 = 12x - x$$
$$55 = 11x$$
$$x = 5$$

Thus, the number is 5.

Exercise 3.13

Find an answer to each of the following problems by forming an equation and solving it.

1. When I double a number and add 17, the result is 59. What is the number?
2. When a number is added to 4 times of itself, the result is 30. Find the number.
3. The difference of two numbers is 5 and their sum is 19. Find the two numbers.
4. Mr Ali has 7 marbles less than Mohammed and they have 29 between them. How many does each boy have?
5. When a number is doubled and 4 added, the result is the same as when it is tripled and 9 subtracted. Find the number.
6. The length of a rectangle is 3 times as long as it is wide. The total length round its boundary is 56 cm. Find its length and width.
7. Erick is twice as old as Peter, John is 3 years younger than Erick. The sum of their ages is 32. Find their individual ages.
8. Mr Chiwa and Mr Dziko share 1 470 FRW such that Dziko receives 190 FRW less than Mr Chiwa. Find their individual shares.
9. Find a number such that when it is divided by 3 and 2 added, the result is 17.
10. A number is such that when 3 is subtracted from three-quarters of it, the result is two thirds of the number. Find the number.

3.3 Inequalities

We are familiar with the equal sign, =. Remember that an algebraic statement which has an equal sign, e.g. $3a = 6$, is called an **equation**.

In real life, we often compare quantities and objects. This is so because the quantities exist either as "greater than" or "smaller than" or "more than" or "less than". For example:

- Mary runs faster than John.
- My pair of shoes is smaller than that of Mayamke.
- A tin of maize costs less than a bag.

3.3.1 Inequalities symbols

In Primary 6, you were introduced to simple inequalities with one unknown. In this section we will review inequality symbols and representation of inequalities on a number line.

Statements involving phrases such as "less than", "greater than" and "smaller than" can be written using inequality mathematical symbols.

In this section, we are concerned with statements which involve the following symbols:

$>$ meaning 'greater than', e.g. $6 > 2$ means 6 is greater than 2.

$<$ meaning 'less than', e.g. $3 < 7$ means 3 is less than 7.

$\geq$ meaning 'greater than or equal to' (or 'not less than').

$\leq$ meaning 'less than or equal to', (or 'not greater than').

Statements containing these symbols are called **inequalities** or **inequations**.

For example, $x < 2$, $x \geq 3$, $y \leq {}^{-}3$, $y > 10$ are inequalities and can be illustrated on a number line as shown below.

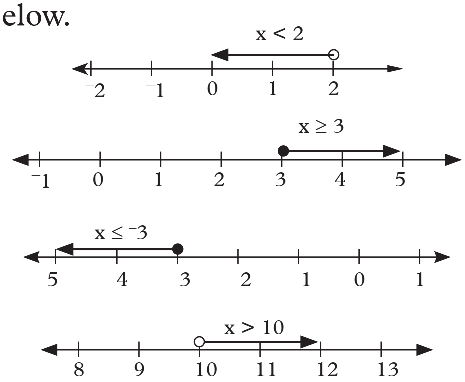


Fig. 3.32

These are also called **simple inequality statements**.

A statement such as $x > 2$ means 'all numbers that are greater than 2', which is a range of values. Just as we represent individual numbers on a number line, we can also represent such a range of numbers on a number line as shown in the following examples.

Example 3.20

Illustrate each of the following on a number line:

(a) $x > 3$ *(b)* $x \geq 3$

(c) $x < {}^{-}2$ *(d)* $x \leq {}^{-}2$

Solution

(a) $x > 3$:

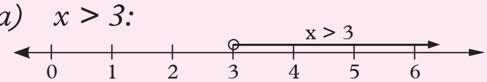


Fig. 3.33

In Fig. 3.33, the number 3 is not included in the list of numbers to the right of 3. The heavy arrow shows that the values of x go on without end. The open dot (o) is used to indicate that 3 is not included.

(b) $x \geq 3$:

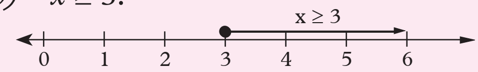


Fig. 3.34

In Fig. 3.34, number 3 is included in the list of the required numbers. The closed dot (•) is used to show that 3 is part of the list.

(c) $x < {}^{-}2$:

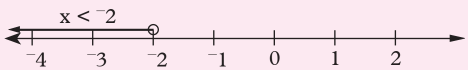


Fig. 3.35

In Fig. 3.35, the number $^{-}2$ is not included.

(d) $x \leq {}^{-}2$:

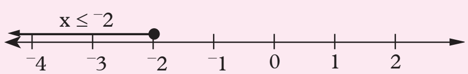


Fig. 3.36

In Fig. 3.36, the number $^{-}2$ is included.

We can formulate inequalities from its representation on a number line.

Example 3.21

Write the following using inequality signs (Fig. 3.37)

(a)

(b)

(c)

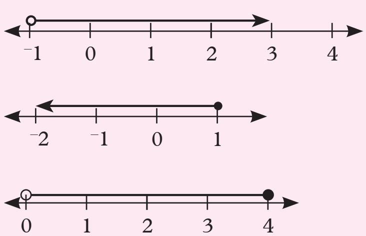


Fig. 3.37

Solution

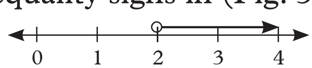
(a) $x > {}^{-}1$ *(b)* $x \leq 1$ *(c)* $0 < x \leq 4$

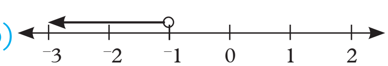
Exercise 3.14

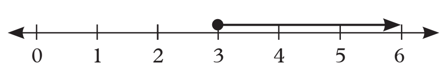
1. Rewrite each of the following statements using either $<$, $\leq$, $>$ or $\geq$ instead of the words.
   (a) 5 is less than 7
   (b) 5 is greater than 2
   (c) $^{-}1$ is less than 0
   (d) $^{-}2$ is greater than $^{-}3$
   (e) x is greater than or equal to 4
   (f) y is less than or equal to $^{-}5$
   (g) a is not less than 3
   (h) b is not greater than 0
   (i) $^{-}1$ is not less than p
   (j) 10 is not greater than q

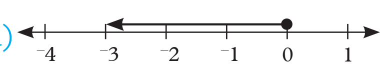
2. Copy and complete each of the following statements by inserting the correct symbol, $<$ or $>$.
   (a) $7 + 3 \square 11$ (b) $6 - 3 \square 5$
   (c) $13 + 2 \square 16$ (d) $12 - 4 \square 3$
   (e) $3 \times 4 \square 8$ (f) $^{-}4 \times 3 \square 12$
   (g) $^{-}6 \times {}^{-}2 \square {}^{-}6 \times 2$
   (h) $39 \div 3 \square 39 \div {}^{-}3$

3. Illustrate each of the following inequalities on a number line.
   (a) $x > {}^{-}5$ (b) $x < 0$ (c) $x \geq {}^{-}3$
   (d) $x \leq 4$ (e) $x > {}^{-}0.5$ (f) $x \leq 2.5$

4. Write each the following using inequality signs in (Fig. 3.38).
   (a)
   

   (b)
   

   (c)
   

   (d)
   

   (e)
   

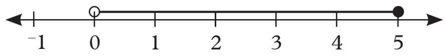
(f) 

Fig. 3.38

3.3.2 Compound statements

Activity 3.22

Illustrate the pairs of inequalities (a) $1 < x$ and $x \leq 5$ (b) $-5 \leq x$ and $x < 2$ (c) $0 < x$ and $x \leq 5$ on a number line. Combine each pair of inequality into a simple inequality. Consider (a) $-4 < x \leq 5$ (b) $-1 \geq x > -3$ (c) $3 \leq x < 8$. Illustrate these inequalities on separate number lines.

Sometimes, two simple inequalities may be combined into one **compound statement** such as $a < x < b$. This statement means that $a < x$ and $x < b$ or $x > a$ and $x < b$.

Example 3.22

Write the following pairs of simple inequality statements as compound statements and illustrate them on number lines.

(a) $x \leq 3, x > {}^-3$

(b) $x > {}^-1, x < 2$

Solution

(a) $x \leq 3, x > {}^-3$ *becomes* ${}^-3 < x \leq 3$ *(Fig. 3.39).*

$\therefore$ *x lies between* $^-3$ *and 3, and 3 is included.*

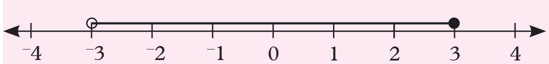


Fig. 3.39

(b) $x > {}^-1, x < 2$ *becomes* ${}^-1 < x < 2$ *(Fig. 3.40).*

$\therefore$ *x lies between* $^-1$ *and 2.*

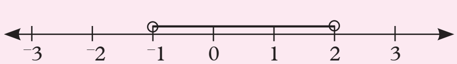


Fig. 3.40

Exercise 3.15

Write each of the following pairs of simple statements as a compound statement and illustrate the answer on a number line.

1. (a) $x > 2, x < 4$
   (b) $x < 3, x \geq 0$
   (c) $x \leq 5, x > {}^-2$
   (d) $x \geq {}^-1, x \leq 1$
   (e) $x < 1.5, x \geq {}^-0.5$
   (f) $x \leq 2.2, x \geq 1.8$

2. (a) $x < \frac{1}{4}, x \geq 0$
   (b) $x \geq \frac{3}{4}, x < 2\frac{1}{4}$
   (c) $x < 3\frac{1}{2}, x > 2\frac{1}{2}$
   (d) $x > \frac{1}{5}, x < \frac{2}{3}$
   (e) $x \geq {}^-0.75, x \leq 0.75$
   (f) $x < 4\frac{1}{2}, x > \frac{-1}{2}$

3.3.3 Solution of linear inequalities in one unknown

Solving an inequality means obtaining all the possible values of the unknown which make the statement true. This is done in much the same way as solving an equation.

Remember

We defined the inequality "x is greater than y written as $x > y$" to mean $x - y$ is positive.

Activity 3.23

Copy and complete the following:

- $x > y \Leftrightarrow x - y$ is ____________
  $\Leftrightarrow (x + a) - (y + a)$ is ______ where a is any real number
  $\Leftrightarrow (x + a)$ ______ $(y + a)$

- if a is a positive number, then
  $x > y \Leftrightarrow ax$ ——— ay
- if a is a negative number, then
  $x > y \Leftrightarrow ax$ ——— ay
- $a > b$ and $x > y \Rightarrow a + x$ ——— $b + y$.
- $x > y,\ y > z \Rightarrow x$___ z.

Now, do a similar exercise with the inequalities $<, \leq, \geq$ one at a time.

Now, you are ready to summarize the basic rules for manipulating inequalities.

The basic rules for manipulating inequalities are:

1. We may add the same number to both sides of the inequalities.
2. We may multiply both sides of an equality by the same positive number.
3. If both sides of an inequality are multiplied by the same negative number, the inequality is reversed.
4. We may add corresponding sides of the same type i.e. if $x > y$ and $a > b$, then $x + a > y + b$.
5. Inequalities of the same type are **transitive** i.e. $a > b$ and $b > c$ means $a > c$.

3.3.4 Manipulating algebraic inequalities

In order to solve inequalities, we must familiarize ourselves with the inequality rules that we have just discussed in this section.

Activity 3.24

Choose any four positive numbers a, b, x and y such that $a < b$ and $x < y$.

Determine $a - x$ and $b - y$.

What can you say about the inequality relating $a - x$ and $b - y$?

Is it true that if $a < b$ and $x < y$ then $(a - x) < (b - y)$?

Observations

You should have observed that if:

i) $a < b$ and $x < y \Rightarrow a - x < b - y$

ii) if $a < b$ and $x < y$ $\Rightarrow a - x < b - y$

Example 3.23

Solve the inequality $2(x - 1) \geq x + 2$ and represent your solution on a number line.

Solution

$2(x - 1) \geq x + 2$ ⟵ *expand LHS*

$2x - 2 \geq x + 2$ ⟵ *add + 2 to both sides*

$2x \geq x + 4$

$2x - x \geq x - x + 4$ ⟵ *subtract x from both sides*

$x \geq 4$

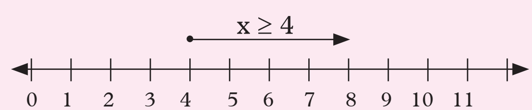


Fig. 3.41

On the number line, 4 has been shaded because it is part of the inequality.

Example 3.24

Solve the following inequalities.

(a) $x - 3 < 7$ *(b)* $x + 5 > 11$

Solution

(a) $x - 3 < 7$

$\Rightarrow x - 3 + 3 < 7 + 3$ *(Adding 3 to both sides)*

$\Rightarrow x < 10$

Thus, $x < 10$ is the solution of the inequality $x - 3 < 7$.

(b) $x + 5 > 11$

$\Rightarrow x + 5 - 5 > 11 - 5$ *(subtracting 5 from both sides)*

$\Rightarrow x > 6.$

Example 3.25

Solve the following inequalities.

(a) $3x - 4 \geq 5$ (b) $\frac{1}{4}x + 5 \leq 14$

Solution

(a) $3x - 4 \geq 5$

$\Rightarrow 3x - 4 + 4 \geq 5 + 4$

$\Rightarrow 3x \geq 9$

$\Rightarrow \frac{3x}{3} \geq \frac{9}{3}$ *(Dividing both sides by 3)*

$\Rightarrow x \geq 3$

(b) $\frac{1}{4}x + 5 \leq 14$

$\Rightarrow \frac{1}{4}x + 5 - 5 \leq 14 - 5 \Rightarrow \frac{1}{4}x \leq 9$

$\Rightarrow \frac{1}{4}x \times 4 \leq 9 \times 4 \quad \Rightarrow x \leq 36$

Example 3.26

Solve the inequality $\frac{x+6}{2} > \frac{x-2}{3} + 5$

Solution

In order to solve the inequality, we need to remove the denominator and then simplify the terms in the inequality.

Thus $\frac{x+6}{2} > \frac{x-2}{3} + 5$ *multiply all the terms by the LCM*

$6\left(\frac{x+6}{2}\right) > 6\left(\frac{x-2}{3}\right) + 5\ (6)$ *LCM = 6*

$3(x + 6) > 2\ (x - 2) + 30$

$3x + 18 > 2x - 4 + 30$ *expand the terms*

$3x + 18 > 2x + 26$

$3x - 2x > 26 - 18$

$x > 8$

Exercise 3.16

Solve the following inequalities and represent the solutions on number lines.

1. (a) $x + 4 > 11$ (b) $x - 6 \leq 5$
2. (a) $2x - 8 \leq 4$ (b) $3x + 4 > 19$
3. (a) $3 > 4x - 2$ (b) $7 \leq 5x + 12$
4. (a) $3 - 2x < 5$ (b) $4 - 5x \geq {}^{-}11$
5. (a) $\frac{1}{3}x - 3 > 4$ (b) $\frac{1}{5}x + 2 < 1$
6. (a) ${}^{-}4 > 2 - \frac{1}{7}x$ (b) $-\frac{2}{3}x + 4 \leq {}^{-}6$
7. (a) $4m - 3 < 7m$ (b) $2m + 1 \geq 5m - 10$
8. (a) $2 - 2p > 13 - 3p$
   (b) $\frac{1}{2} + p < 4p + \frac{1}{4}$
9. (a) $\frac{1}{3}q > 2 - 4q$
   (b) $\frac{1}{4}q + 2 < 8 - \frac{2}{3}q$
10. (a) $-\frac{1}{9}r < 5$
    (b) $4 - \frac{3}{4}r \geq 3 - \frac{1}{3}r$
11. (a) $2(1 + x) + 3(x - 2) \geq 25$
    (b) $3(4 - 3x) - \quad (5x - 3) \leq 2$

3.3.5 Solving simultaneous inequalities

Inequalities that must be satisfied **at the same time** are called simultaneous inequalities.

Activity 3.25

- Solve the inequalities $2x < x + 5$ and $x + 4 > 3$.
- Represent the two solutions on the same number line.
- What can you say about the two solutions?

- Express these solutions as a single inequality.

From the activity above, the solution of the inequality is x < 5.

That of x + 4 > 3 is x > -1.

Fig 3.42 shows the solutions marked on a number line.

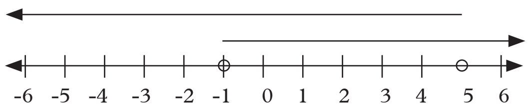


Fig. 3.42

The two solutions overlap between -1 and 5.

Thus -1 < x < 5.

This means the inequalities 2x < x + 5 and x + 4 > 3 have been solved simultaneously i.e. the values between -1 and 5 satisfy both the inequalities at the same time.

Example 3.27

Solve the following pair of simultaneous inequalities.

$3 - x < 5,\ 2x - 5 < 7$

Solution

$3 - x < 5$

$\Rightarrow \quad {}^{-}x < 2$

$\Rightarrow \quad x > {}^{-}2 \quad \ldots\ldots\ldots\ldots (i)$

Also $2x - 5 < 7$

$\Rightarrow \quad 2x < 12$

$\Rightarrow \quad x < 6 \quad \ldots\ldots\ldots\ldots (ii)$

Combining (i) and (ii), we have ${}^{-}2 < x < 6$.

Thus, x lies between ${}^{-}2$ *and 6.*

This is represented on a number line as in Fig. 3.43.

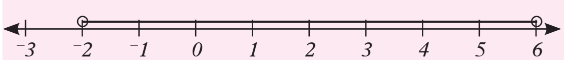


Fig. 3.43

Example 3.28

Solve the inequality:

$3x - 2 < 10 + x < 2 + 5x$

Solution

$3x - 2 < 10 + x < 2 + 5x$

Split the inequality into two simultaneous inequalities as:

$3x - 2 < 10 + x \quad \ldots\ldots\ldots\ldots (i)$

and $10 + x < 2 + 5x \quad \ldots\ldots\ldots\ldots (ii)$

Solve each separately.

$3x - 2 < 10 + x$

$\Rightarrow \quad 3x - x < 10 + 2$

$\Rightarrow \quad 2x < 12$

$\Rightarrow \quad x < 6 \quad \ldots\ldots\ldots\ldots\ldots (iii)$

$10 + x < 2 + 5x$

$\Rightarrow \quad 10 - 2 < 5x - x$

$\Rightarrow \quad 8 < 4x$

$\Rightarrow \quad 2 < x \quad \ldots\ldots\ldots\ldots\ldots (iv)$

Combine (iii) and (iv) to get $2 < x < 6$

This is represented on a number line as in Fig. 3.44.

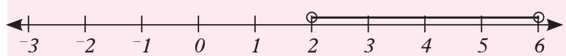


Fig. 3.44

Exercise 3.17

Solve the following simultaneous inequalities and represent each solution on a number line.

1. (a) $2x < 10,\ 5x \geq 15$
   (b) $3x \leq 9,\ 2x > 0$
2. (a) $x + 7 < 0,\ x - 2 > {}^{-}10$
   (b) $x \geq 3,\ 2x - 1 \leq 13$
3. (a) $4x - 33 < {}^{-}1,\ {}^{-}2 < 3x + 1$
   (b) $2x - 5 < 22 \leq 5x - 6$

4. (a) $3x - 4 < 8 + x < 2 + 7x$
   (b) $6x + 2 < 3x + 8 < 27x - 1$

3.3.6 Forming inequalities from word statements

Inequality symbols can be used to change word statements into algebraic statements.

Activity 3.26

- Research from any suitable source the process of forming and solving linear inequalities.
- What key words must you be on the lookout for in the given problems?
- Do your knowledge and skills in forming and solving equations help you?
- Discuss your findings with your partner.

Example 3.29

Mary has 25 books. Jane has more books than Mary. Write an algebraic statements for this.

Solution

If Jane has b books then $b > 25$.

Example 3.30

The distance from town A to town B is 260 km and that from town A to town D is 80 km.

If from town C to town A is a shorter distance than from town A to town B, and from town C to town A is a longer distance than town A to town D, write an algebraic statement for this.

Solution

Let the distance from town A to town C be x km.

Then, $x < 260$ *and* $x > 80$

Hence, $80 < x < 260$

Example 3.31

The area of a square is greater than 36 cm^2*. Write an inequality for*

(a) the length

(b) the perimeter of the square.

Solution

We must first define our variables, just as we do when forming equations.

Let the length of the square be x cm.

Area of the square $= x^2$.

(a) $x^2 > 36$

$\Rightarrow \sqrt{x^2} > \sqrt{36}$

$\Rightarrow x > 6$

(b) *Perimeter* $= 4x$

Since $x > 6$,

then $4 \times x > 4 \times 6$

i.e. $4x > 24$

Exercise 3.18

Write down each of the following statements as mathematical sentences using appropriate symbols.

1. When two is added to a certain number, the result is greater than ten.
2. When five is added to a number, the result is less than twice the number.
3. Multiplying a number by six, then adding five, gives a greater result than multiplying the number by five, then adding six.

4. The sum of three consecutive whole numbers is more than 260.
5. The area of a square is less than its perimeter.
6. The square of a number is less than the number cubed.
7. The radius of a circle is not more than 4 cm. What can you say about the circumference?

3.3.7 Application of inequalities in life

Inequality can be applied in real life situations.

Activity 3.27

- Think of a situation in your life where you have had to use your knowledge and skills of inequalities to solve a problem.
- Share your experience with your partner.
- Discuss with your partner whether there was any other way to handle your problem.
- Would you like to share the same with the rest of the class.
- You are now ready to go through the examples and the exercise.

Example 3.32

Lucy was given mangoes. She gave away three of them. When she divided by 2 the number of the remaining mangoes, the number was less than 17. What is the maximum number of mangoes that she could have been given?

Solution

Let the total number of mangoes be m

She gave away 3 mangoes

$\therefore$ *Number of mangoes remaining* $= m - 3$

Dividing $(m - 3)$ *by 2 gives us,* $\frac{m-3}{2}$

This number is less than 17.

Thus, $\frac{m-3}{2} < 17$

Multiplying by 2; $\frac{m-3}{2} \times 2 < 17 \times 2$

$m - 3 = < 34$

Adding 3: $m < 37$

$\therefore$ *Maximum number of mangoes must be 36.*

Example 3.33

Peter wants to order for some special tools for his business. The cost of each tool is 2 398 FRW. The cost of delivering the whole order is 1 499 FRW. How many of these tools can he order, if he has only 15 000 FRW to spend?

Solution

Cost of each tool = 2 398 FRW

Let the number of tools ordered be n

$\therefore$ *Cost of x tools is 2398 FRW × n = 2 398 FRW n*

Cost of delivering this tools is 1 499 FRW

$\therefore$ *Total cost will be 2 398 FRW n + 1 499 FRW*

Since total cost cannot exceed 15 000 FRW

we have $2\,398\,n + 1\,499 \leq 15\,000$

$\Rightarrow 2398\,n \leq 15\,000 - 1\,499$

$2398\,n \leq 13501$

$n \leq \frac{13\,501}{2398} = 5.63 \text{ (2 dp)}$

Since one cannot buy part of a tool, then the number of tools used must be a whole number

$\therefore n = 5$

Exercise 3.19

1. Mohammed scored 60% in a mathematics quiz. When a physics score is doubled, the score is still less than the score in mathematics. What is the maximum marks she could have scored in physics?

2. On a trip to united kingdom a manager had five times as much money as the secretary. When the manager spent £100, he still had more than the secretary. What is the least amount that the secretary was having. (Pounds (£) is the currency used in UK, 1£ = 0.00123 FRW).

3. Joan's age is 25 less than her father's age. If her father is 50 years old, what is the maximum age (to the nearest whole number) that she could have been seven years ago.

4. Abraham has 5 000 FRW in his savings. He wants to have at least 2 000 FRW at the end of the season. He withdraws 255 FRW each week for use.
   (a) Write an inequality to represent this situation.
   (b) In how many weeks can he withdraw the money.

5. One taxi in Rwanda charges 2US$ basic charge plus 1.5 US$ per km travelled to foreigners touring Rwanda. Judy has 200 US$ for travels and cannot spend more this for the taxi. (US$ stands for US dollar, currency used in USA. 1 US$ = 0.001337 FRW).
   (a) Write an inequality for the situation.
   (b) How many km can she travel without exceeding the limit (to the nearest km).

6. A company manufactures one specific sparepart. The total cost of producing n spare parts is 3 600 £ plus 1.4 £ per spare part produced. If each part is sold at 4.2 £ what is the maximum number of spare parts that must be produced to make a profit? (£ stands for UK pound, 1£ = 0.00123 FRW).

7. Jimmy charges 7 500 FRW for renting his premises for parties. He also charges 825 FRW per person attending the party. Jane has 22 500 FRW and wants to have a party at Jimmy's premises. What is the maximum number that can attend Jane's party?

8. Mutoni keeps dogs and cats in her place. The number of cats is three times the number of dogs. What is the greatest number of cats if the total number does not exceed 58?

9. Kalisa plans to buy a car 21 months from now. At present he has saved 450 000 FRW. The cheapest car he can buy costs 3 250 000 FRW. What is the minimum amount (whole number) that must he save per month for this period of time to be able to buy such a car?

Summary

1. Linear function/equation – Is the relationship between two variables, say x and y whose graphs represent a straight line.

2. Number line – is a graph/picture of all the negative and positive numbers.

3. Plotting of points is the marking of points in the Cartesian plane.

4. The general equation of a line is written in the form $y = mx + c$ where m and c are constants and c is the y intercept.

5. y-intercept is the point where the line which are neither vertical nor horizontal cross the y-axis.
6. x-intercept is the point where the lines that are neither vertical nor horizontal cuts the x-axis
7. The gradient of a straight line is the ratio, of $\frac{\text{Vertical distance}}{\text{Horizontal distance}}$ i.e the measure of steepness or slope

   Gradient = $\frac{\text{Vertical distance}}{\text{Horizontal distance}}$
8. From the general equation of the line $y = mx + c$, c is the y-intercept and from $x = ky + c$, c = the x-intercept.
9. Equation is a mathematical sentence with a symbol =.
10. A conditional equation is an equation that is satisfied by at least one value of the unknown e.g $x + 11 = 15$.
11. An identity – is an equation which is true regardless of what values are substituted for the unknown.
12. Inequalities/inequations – these are mathematical statements that involves phrases such as “less than”, “greater than”, “smaller to”, “greater or equal to”.

Unit Test 3

1. Determine the gradient of a line passing through the following pairs of points:
   (a) $(8, 3), (4, {}^{-}1)$
   (b) $({}^{-}3, {}^{-}3), ({}^{-}4, 1)$
   (c) $(4, {}^{-}2), (4, 3)$
   (d) $(2, 9), (0, {}^{-}4)$
2. Find the gradient of each of the following lines whose equations are:
   (a) $y = \frac{^{-}4}{5}x - 3$
   (b) $\frac{2}{3}y - \frac{2}{5}x = \frac{1}{6}$
   (c) $3x - 2y - \frac{1}{2} = 0$
   (d) $2y = 5$
   (e) $x = 9$
   (f) $\frac{4}{7}x - \frac{3}{5}y = \frac{1}{2}$
3. Solve the equations
   (a) $3(2x + 1) - 5(x - 2) = 2(3 - 2x)$
   (b) $\frac{3x - 1}{-2} = \frac{2x + 1}{3}$
   (c) $\frac{2x - 5}{-3} = \frac{4x + 1}{5}$
   (d) $5 + \quad (4 + x) = 8$
4. (a) Using any suitable scale, draw the graph of $y + 1 = 2x$ from $x = -3$ to $x = 3$.
   (b) On the same axes, draw the line which passes through the points $(-2, 5)$ and $(2, 1)$.
   (c) Find the coordinates of the point of intersection of the two lines.
5. (a) Draw the graph of $2x - y = 4$.
   (b) Calculate the gradient of your line.
   (c) State the coordinates of the point where the line meets
      (i) the x-axis
      (ii) the y-axis
6. (a) Without drawing the graph, find the gradient and the y-intercept of the line $3y - \frac{3}{5}x = 6$.
7. Find a number such that when it is divided by 3 and 2 added, the result is 17.

8. A man earns three times as much as his wife earns. After spending three fifths of their combined income, the couple have 10 000 FRW left. How much does the man earn?

9. The difference of two numbers is 5 and their sum is 19. Find the two numbers.

10. Solve the simultaneous linear inequalities.

    (a) $x - 4 < 3x + 2 < 2(x + 5)$

    (b) $-5 \leq 2x + 1 < 5$

11. Solve the simultaneous linear inequalities $x + y \leq 4$, $y \leq 2x$, $2y - x > 1$. List the points with integral coordinates that satisfy the inequalities.

12. Solve the inequalities

    (a) $3 - 7x \leq 2x + 21$

    (b) $2(3x + 1) \geq 4(x - 1) > 12$

4 PERCENTAGE, DISCOUNT, PROFIT AND LOSS

Key unit competence

By the end of this unit, I should be able to solve problems that involve calculating percentage, discount, profit and loss and other financial calculations.

Unit outline

- Percentages
- Discounts
- Commission
- Loans and saving

Introduction

In our day to day lives, we get involved in activities of either buying or selling goods or services using money. Such activities are called transactions. There are many terms that are used to describe various aspects of such transactions. In this unit, we will learn a number of these terms and how to calculate their values in financial transactions.

4.1 Percentages

Activity 4.1

1. You must have come across the term percentage in lower classes. Tell your class partner its meaning. Let both of you agree on the correct definition of the term.
2. Discuss with your partner on how to convert the following into percentages:

   (a) $\frac{5}{12}$ (b) 0.75

   (c) 450 FRW out of 1 000 FRW
3. Discuss and list down business related terms which are expressed and referred to in terms of percentages.

Learning point

A **percentage** is a fraction whose denominator is 100. The symbol for percentage is %. For example 47% means $\frac{47}{100}$, 23% means $\frac{23}{100}$ and so on.

We can express the decimal number 0.25 as a percentage as follows:

$$0.25 = \frac{25}{100} = 25\%$$

Similarly, to express the fraction $\frac{3}{20}$ as a percentage, we multiply the numerator and denominator by 100.

$$\frac{3}{20} = \frac{3}{20} \times \frac{100}{100} = \frac{15}{100}$$
$$= 15\%$$

Alternatively,

Just multiply the fraction $\frac{3}{20}$ by 100% (which is equal to $\frac{100}{100}$)

$$\frac{3}{20} \times 100 = 15\%$$

Example 4.1

Express each of the following as a percentages:

(a) $\frac{17}{20}$ *(b)* *0.24*

Solution

(a) $\frac{17}{20} = \frac{17}{20}$ *of 100%*

$= \frac{17}{20} \times 100 = 85\%$

(b) 0.24 = 0.24 of 100% = 24%

Example 4.2

What is $12\frac{1}{2}$% of 28000 FRW?

Solution

$12\frac{1}{2}\ \% = 12.5\ \%$

$$12.5\% \text{ of } 28\,000 \text{ FRW} = \frac{12.5}{100} \times 28\,000 \text{ FRW}$$

$$= 3500 \text{ FRW}$$

Example 4.3

Express 4 000 cm^3 of milk as a percentage of 8 litres of milk.

Solution

Express the quantities in the same units. For example, express both quantities in cm^3.

$1 \text{ litre} = 1\,000\ cm^3$

$8 \text{ litres} = 8\,000\ cm^3$

$\frac{4\,000\ cm^3}{8\,000\ cm^3}$ *as a percentage is*

$\frac{4\,000}{8\,000} \times 100 = 50\%$

Hence 4 000 cm^3 of milk is 50% of 8 litres of milk.

Exercise 4.1

1. Express the following fractions as percentages:
   (a) $\frac{5}{12}$ (b) $\frac{3}{4}$ (c) $\frac{27}{50}$
   (d) $\frac{3}{5}$ (e) $\frac{37}{40}$ (f) $\frac{13}{25}$
   (g) $\frac{7}{8}$
2. Express the following decimals as percentages:
   (a) 0.36 (b) 0.45
   (c) 0.81 (d) 0.04
   (e) 0.125 (f) 0.08
   (g) 0.025 (h) 0.8
3. Express the first quantity as a percentage of the second quantity in each of the following:
   (a) 20 cm, 50 cm
   (b) 30 cm, 5 m
   (c) 54 mm, 25 cm
   (d) 6 cm^2, 24 cm^2
   (e) 105 mm^2, 42 cm^2
   (f) 400 g, 1 kg
   (g) 9 000 cm^3, 4 litres
   (h) 25 m, 2 km
   (i) 3 m, 12 m
4. Find the value of:
   (a) 30% of FRW 350
   (b) 12% of 9 km
   (c) 84% of 450 g
   (d) $33\frac{1}{3}$% of 252 m^2
   (e) $12\frac{1}{2}$ % of 4.88 cm
   (f) 21% of 40 kg
5. Mutanguha scored 27 marks out of the possible 45 in a mathematics test. What percentage was this?
6. In a survey of 100 cars, 45 were white, 25 were blue and 30 were red. Express each of these numbers as a percentage of the total.
7. $\frac{7}{10}$ of the surface of the earth is water. Express this as a percentage.
8. A farmer has 150 birds in a flock. 80 of them are female. What percentage of birds are:
   (a) male? (b) female?
9. A businessman keeps a record of the number of items he sells in one day. Of the 200 items he sold, 45% were newspapers, 26% were books, 15% were sweets and the remaining were other assorted items. Calculate the number of each item he sold.

10. From the world bank statistics of the year 2014, our country Rwanda had a forest cover of 18.4% while Zimbabwe had 38.7%. If the areas of the two countries are 26 338 km^2 and 386 350 km^2 respectively, determine the area under forests in each country.

> Trees are very important for our survival. They facilitate rainfall formation, provide food, purify the air. Let's keep increasing our forest cover by planting more trees and caring for them.

4.2 Discount

Activity 4.2

You may have heard or even used the term discount. Do a research from business books and the internet on:

1. The actual meaning of the term discount and the quantities used to calculate the discount.
2. The possible reasons why a business person or service provider may decide to give some discount to the customer.
3. Present your findings to your classmates in a class.

Discussion

It is obvious that a price is set for any good or service meant for sale. This price is known as the marked price. By a simple definition, the marked price, also known as the list price, is the price initially quoted by the seller to the buyer.

However, there are cases where the seller may sell the good or service to a buyer at a price lower than the marked price.

The amount reduced from the marked price is known as discount. The exact price for which an item is sold after a discount is known as the sale price.

Thus,

Learning points

Discount = marked price – sale price.

From which we get:

Sale price = marked price – discount

And;

Marked price = sale price + discount

Usually, discount is expressed as a percentage of the marked price as,

$$\text{Percentage discount} = \frac{\text{Discount}}{\text{Marked price}} \times 100\%$$

The percentage discount is also called the rate of discount. Thus, actual discount = marked price × rate of discount.

Some reasons why sellers may give a discount to a product includes:

- To encourage the customers to buy more.
- To encourage the customers to buy in cash.
- To encourage the debtors to pay within a short credit period.

Example 4.4

Find the sale price of a watch whose marked price is 3 000 FRW if 20% discount is given.

Solution

Marked price = 3 000 FRW,

Discount rate = 20%

Discount = marked price × rate of discount

$$= 3\,000 \text{ FRW} \times \frac{20}{100}$$

$$= 600 \text{ FRW}$$

Sale price = *marked price – discount*

= *3 000 FRW – 600 FRW*

= *2 400 FRW*

OR

Sale price = *(100 – 20)% of marked price*

$= \frac{80}{100} \times 3\ 000$ *FRW*

= *2 400 FRW*

Example 4.5

Find the rate of discount of an item that sells for 3 800 FRW and is on sale for 3 400 FRW.

Solution

Marked price = *3 800 FRW,*

Sale price = *3 400 FRW*

Discount = *marked price – sale price*

= *3 800 FRW – 3 400 FRW*

= *400 FRW*

$\text{Rate of discount} = \frac{\text{Discount}}{\text{marked price}} \times 100\%$

$= \frac{400 \text{ FRW}}{3\ 800 \text{ FRW}} \times 100\%$

= *10.53%*

Example 4.6

A trader allows a 5% discount on cash purchases. How much will a customer pay for a suit priced at 59 500 FRW?

Solution

Discount = *marked price × rate of discount*

$= 59\ 500 \text{ FRW} \times \frac{5}{100}$

= *2 975 FRW*

Sale price = *marked price – discount*

= *59 500 FRW – 2 975 FRW*

= *56 525 FRW*

OR

Sale price = *(100 – 5)% of marked price*

$= \frac{95}{100} \times 59\ 500$ *FRW*

= *56 525 FRW*

Activity 4.3

A stationary dealer offers two discount options for anyone buying exercise books.

1. 11% discount on the total cost of the books purchased.
2. Gives one free book for every 9 books purchased.

In groups of two, student, discuss the best discount option between the two options.

Exercise 4.2

1. The marked price for a sofa set is 24 000 FRW. If it is sold at a discount of 20%, calculate
   (a) The discount.
   (b) The sale price.
2. A radio costs 2 400 FRW. If it is sold at a discount for 1 920 FRW, find the:
   (a) Discount.
   (b) Rate of discount.
3. A shirt is sold for 4 000 FRW when a discount of 30% is allowed. Find the:
   (a) Marked price.
   (b) Discount allowed.
4. A television costs 54 000 FRW. If a customer paid 43 200 FRW after being allowed a discount, find the:
   (a) Discount allowed.
   (b) Rate of discount.
5. The selling price for a wardrobe is 192 000 FRW. A discount of 20% is allowed for cash payment. Calculate the marked price.

6. The marked price of a jacket is 7 440 FRW. In the winter sale, the price is reduced by 15%. What is the sale price of the jacket?
7. A coat is priced at 5 400 FRW. During a sale offer, 2 400 FRW is allowed as a discount. What percentage discount was allowed on the sale of the coat?
8. A motorcycle dealer bought two motorcycles at 680 000 FRW and 504 000 FRW after being allowed 15% and 10% discount respectively. Find the:
   (a) total discount given.
   (b) selling price of the second motorcycle if he made a profit of 20%.

4.3 Commission

Activity 4.4

- Suppose you are the manager in charge of sales in a company.
- Discuss in your group, ways you would use to reward your sales people who sell more than the target given them, without necessary increasing their monthly basic salary or retainer.
- How would you ensure that they get the extra reward for the extra sales they bring? What name would you give to such a form of payment.

Commission is the money paid to sales or agents representative for the sales made. For example, an insurance agent gets a commission for selling insurance policies. Company sales representative gets commission for selling the company's goods. The more the sales, the more the commission a sales representative gets.

Example 4.7

An insurance sales lady sold insurance policies worth 440 000 FRW. If her commission is 9%, of the total sales, how much money did she get?

Solution

Commission $= \frac{9}{100} \times 440\ 000$

$= 39\ 600$ FRW

Example 4.8

A company's sales representative sold goods worth 6 760 000 FRW in a certain month. The representative earns a salary of 150 240 FRW and gets a commission of 10% of sales above 5 200 000 FRW. Calculate how much the sales representative earned that month.

Solution

Commission

$= \frac{10}{100} \times (6\ 760\ 000 - 5\ 200\ 000)$ FRW

$= \frac{10}{100} \times 1\ 560\ 000$ FRW

$= 156\ 000$ FRW

Total earning = salary + commission

$= (150\ 240 + 156\ 000)$ FRW

$= 306\ 240$ FRW

Example 4.9

A saleslady received a commission of 5% for the first sale of 80 000 FRW and 6% for sales above 80 000 FRW. In one month she made sales amounting to 168 000 FRW. Find her total commission that month.

Solution

Commission for the first 80 000 FRW

$= \frac{5}{100} \times 80\ 000$ FRW $= 4\ 000$ FRW

Commission for excess of 80 000 FRW

$= \frac{6}{100} \times (168\,000 - 80\,000)$ *FRW*

$= \frac{6}{100} \times 88\,000$ *FRW*

= 5 280 FRW

Total commission earned that month

= (4 000 + 5 280) FRW

= 9 280 FRW

Exercise 4.3

1. Peter receives a monthly salary of 120 000 FRW plus a commission of 12% on all sales. Last month he made sales worth 1 200 000 FRW. How much did he earn that month?
2. Mrs. Uwamahoro sells charity tickets. She gets 160 FRW for every 8 tickets she sells. How much will she get for selling 480 tickets?
3. A mobile money agent received a commission of 40 000 FRW for a transaction worth 1 280 000 FRW. Find the rate of his commission.
4. Jane is an insurance agent and receives a monthly salary of 128 000 FRW and a commission of 15% on all the policies she sells in a month. In a certain month she sold policies worth 14 762 400 FRW. What was her total earnings that month?
5. After touring Rwanda, a tourist exchanged 680 000 FRW at the bank into sterling pounds. If the exchange rate was 1 008 FRW = UK£1 and the bank charged a commission of 5%, find:
   (a) how much was the bank's commission?
   (b) how much the tourist received to the nearest UK£?

4.4 Profit and Loss

Activity 4.5

1. Discuss what you understand by the words profit and loss.
2. In the following transactions, identify the buying price and the selling price. Determine whether a profit or loss was made in each of these transactions.
   (a) Habanabashaka bought a radio for 14 000 FRW and sold it for 17 500 FRW.
   (b) Charloite sold a dress for 24 500 FRW. She had bought it for 26 700 FRW
3. Let one of you present your finding in step 1 and 2 to the rest of the class

Learning points

If one buys an item and sells it at a higher price than the buying price, then we say that a profit has been made. Thus, profit is the extra amount gained after selling a commodity at a price higher than the buying price i.e.

Profit = selling price (SP) – buying price (BP)

On the other hand, if one sells an item at a lower price than he/she bought it, we say that the person has made a loss. Thus, loss refers to the amount of money lost when a property is sold below the actual buying price. i.e.

Loss = buying price (BP) – selling price (SP)

Example 4.10

A salesman bought a TV for 122 500 FRW sold it at 147 000 FRW. How much profit did he make?

Solution

Buying price = 122 500 FRW

Selling price = 147 000 FRW

Profit = selling price – buying price

= 147 000 – 122 500

= 24 500 FRW

Example 4.11

A dealer sold 1 000 bottles of water at 140 FRW each, to clear the stock. What profit or loss did she incur if she had bought each bottle at 175 FRW.

Solution

Buying price = 175 FRW,

selling price = 140 FRW

This means she made a loss

Loss = Buying price – selling price

(175 – 140) FRW = 35 FRW per bottle.

Total loss = 35 FRW × 1 000

= 35 000 FRW

Example 4.12

A profit of 75 FRW is obtained for each litre bottle of soda sold at a price of 450 FRW. If there are 100 bottles, find the price at which the bottles were bought.

Solution

Each bottle has a profit of 75 FRW. The total profit for 100 bottles is 75 × 100 FRW

= 7 500 FRW.

The selling price for each bottle is 450 FRW.

The total selling price for 100 bottles is

450 × 100 = 45 000 FRW.

Buying price = (45 000 – 7 500) FRW

= 37 500 FRW

Thus, the bottles were bought at a price of 37 500 FRW.

4.5 Percentage loss and percentage profit

Percentage profit is the ratio of profit to the buying price expressed as a percentage.

$$\text{Percentage profit} = \frac{\text{Profit}}{\text{Buying price}} \times 100\%$$

Similarly, percentage loss is the ratio of loss to the buying price expressed as a percentage.

$$\text{Percentage loss} = \frac{\text{Loss}}{\text{Buying price}} \times 100\%$$

Example 4.13

Calculate the percentage loss for an item bought at 2 000 FRW and sold at a price of 1 850 FRW.

Solution

Buying price = 2 000 FRW,

Selling price = 1 850 FRW

Loss = buying price (BP) – selling price (SP)

= (2 000 – 1 850) FRW

= 150 FRW

$$\textit{Percentage loss} = \frac{\textit{loss}}{\textit{buying price}} \times 100\%$$

$$= \frac{150}{2\ 000} \times 100$$

= 7.5% loss

Example 4.14

Jane bought 12 pairs of shoes at 1 400 FRW each and sold them at 2 000 FRW each. Calculate

(a) Her percentage profit.

(b) *The percentage loss she would have made if she sold each shoe at 1 200 FRW.*

Solution

(a) *There are 12 pairs of shoes.*

The total buying price

$= 12 \times 1\ 400$ FRW

$= 16\ 800$ FRW

The total selling price is

$2\ 000 \text{ FRW} \times 12 \text{ pairs} = 24\ 000$ FRW

Profit = selling price – buying price

$= 24\ 000 \text{ FRW} - 16\ 800$ FRW

$= 7\ 200$ FRW

Percentage profit

$$= \frac{\text{Profit made}}{\text{Buying price}} \times 100\%$$

$$= \frac{7\ 200}{16\ 800} \times 100\%$$

$= 42\%$ or 42.86%

(b) *Total selling price would be:*

$1\ 200 \text{ FRW} \times 12 = 14\ 400$ FRW

Loss = Buying price – Selling price

$= 16\ 800 \text{ FRW} - 14\ 400$ FRW

$= 2\ 400$ FRW

$$\text{Percentage loss} = \frac{\text{Loss}}{\text{Buying price}} \times 100\%$$

$$= \frac{2\ 400}{16\ 800} \times 100\%$$

$= 14.29\%$ loss

Example 4.15

The total percentage profit of each cell phone bought at 48 000 FRW is 20%.

(a) *At what price does the business person sell each cell phone?*

(b) *Find the total amount of money the business person will get after selling all the stock of 200 cell phones.*

Solution

(a) *Percentage profit = 20%,*

Buying price = 48 000 FRW

$$\text{Percentage profit} = \frac{S.P - B.P}{B.P} \times 100\%$$

$$20\% = \frac{S.P - 48\ 000}{48\ 000} \times 100\%$$

$$\frac{1}{5} = \frac{S.P - 48\ 000}{48\ 000}$$

By cross multiplication

$48\ 000 \times 1 = 5(\text{selling price} - 48\ 000)$

(Dividing by 5 both side)

$9\ 600 = \text{selling price} - 48\ 000$

$\text{Selling price} = 9\ 600 + 48\ 000$

$= 57\ 600$ FRW

(b) *The total amount of money obtained is 57 600 FRW × 200*

$= 11\ 520\ 000$ FRW

Exercise 4.4

1. A student bought 10 Mathematics books from a book store selling each book at a price of 400 FRW. If each book was bought at a price of 260 FRW, find the percentage profit.
2. An item was bought at 2 000 FRW and sold at 2 600 FRW. Find the percentage profit.
3. A company buys and sell cars. During a certain month, the company sales were 80% of the buying price. Find the amount of money invested if the total sales were twenty-eight million Rwandan Francs.
4. Neza bought 50 bottles of water at 200 FRW each and 100 bars of soap at 120 FRW each. If he sold each bottle at 340 FRW and each bar of soap at 116 FRW each, how much profit or loss did he get?

5. 28 000 FRW was obtained from selling 1 000 pieces of one metre electrical wires. If the percentage loss from this sale was 15%, find the buying price.
6. Gahigi bought an item at 100 000 FRW. He sold it making a loss which was $\frac{1}{4}$ of the buying price. Find the percentage loss.
7. The percentage loss for an item sold at 9000 FRW is 12%. Find the buying price and the loss obtained.
8. How much money should a business person invest so as to make a profit of 40000 FRW equivalent to a percentage profit of 25%?

4.6 Loans and savings

Activity 4.6

You are aware that successful people, businesses or institutions usually save money in banks and take loans to finance their operations.

Discuss:

1. Advantages of saving money. Why is it better to save money in a bank and financial institutions rather than keeping it elsewhere?
2. The advantages and the disadvantages of taking loans.
3. Some ways of managing your finances well.

In our daily life, we need money to carry out most of our financial engagements. However, this money may not be available hence we resort to borrowing from financial institutions. The financial institution charges us a fee for using this borrowed money. This charge is known as *interest*

On the other hand, we may deposit (save) our surplus money with the bank for a specific period of time. The bank then pays us a fee for using our money called interest.

Activity 4.7

Tell your class partner the meaning of the following terms: principle, rate, time of investment and interest as used in the determination of loans and savings.

The fee paid for borrowing or depositing money for a specific period of time is called interest, denoted by I.

The total amount of money borrowed from a financial institution or deposited to earn interest is called the principal, denoted by P.

The ratio of interest earned to the principal borrowed or deposited for a specific period of time is called the rate denoted by R. Rate is usually expressed as a percentage per annum (p.a) or year.

The duration over which a given sum of money is borrowed or deposited to earn some interest is known as time, denoted by T.

4.7 Simple interest

Activity 4.8

(a) Remind your class partner what simple interest is and its formula.

(b) Listen to your partner as he/she reminds you and in turn correct/confirm what he/she says.

(c) Discuss how you would find the simple interest. Lucy borrowed 2 000 000 FRW from bank for three years at an interest of 15% p.a.

Simple interest is the amount charged when one borrows money or loan from a financial institution which acrue yearly.

This interest is a fixed percentage charged on money/loan that is not yet paid.

This interest is calculated based on the original principal or loan and is paid at regular intervals.

Example 4.16

John borrowed 4 800 FRW from a local bank at 10% p.a to start a small-scale business. Find the simple interest for the first and second year. Hence find the total interest for the 2 years.

Solution

Principal = 4 800 FRW, Rate = 10%

Interest for the first year $= \frac{10}{100} \times 4\ 800$

$= 480$ *FRW*

Total interest for two years

$= (480 + 480)$ *FRW*

$= 960$ *FRW*

From, above example, we see that when the principal (P), rate in percentage (R) and time in year (T) are given, then simple interest (I) for the given period is given by:

$$I = P \times \frac{R}{100} \times T = \frac{PRT}{100}$$

The total amount (A) paid back by the borrower or the financial institution on the expiry of the interest period is the sum of the principal and the interest earned.

Thus, amount (A) = principal (P) + interest (I).

Example 4.17

Find the simple interest earned from 3 400 FRW borrowed for 3 years at the rate of 10% p.a.

Solution

Interest, $I = \frac{PRT}{100} = \frac{3\ 400 \times 10 \times 3 \text{ FRW}}{100}$

$= 1\ 020$ *FRW*

Example 4.18

Gatete borrowed 32 000 *FRW* from a lending institution to start a business. If the institution charged interest at a rate of 8% p.a, calculate the simple interest and the total amount she eventually paid back after 4 years.

Solution

Interest, $I = \frac{PRT}{100} = \frac{32\ 000 \times 8 \times 4 \text{ FRW}}{100}$

$= 10\ 240$ *FRW*

Amount, A $= 32\ 000 + 10\ 240$ *FRW*

$= 42\ 240$ *FRW*

Example 4.19

Calculate the principal that will earn a simple interest of 32 400 FRW in 3 years at the rate of 6% p.a.

Solution

$32\ 400 \text{ FRW} = \frac{P \times 6 \times 3}{100}$

$32\ 400 \text{ FRW} = \frac{18P}{100}$

$P = 180\ 000$ *FRW*

Example 4.20

For how long, to the nearest year, must 160 000 FRW be invested in a financial institution to earn a simple interest of 19 200 FRW at 4.5% p.a.?

Solution

$19\ 200\ FRW = \frac{160\ 000 \times 4.5 \times T}{100}$

$1\ 920\ 000\ FRW = 720\ 000T$

$T = 2.666$

$= 3$ years

Example 4.21

Calculate the rate in percent per annum, at which 14 400 FRW will earn 3 240 FRW in 3 years.

Solution

$3\ 240\ FRW = \frac{14\ 400 \times 3R}{100}$

$324\ 000\ FRW = 43\ 200R$

$R = 7.5\%$

Exercise 4.5

1. Find the simple interest earned on each of the following:
   (a) 24 000 FRW for 3years at 8% p.a.
   (b) 28 860 FRW for 5 years at 6% p.a.
   (c) 26 400 FRW for 1 year 3 months at 4% p.a.
   (d) 44 000 FRW for $2\frac{1}{2}$ years at 5% p.a.
2. Calculate the amount earned on each of the following:
   (a) £8 400 for 2 years at 10% p.a.
   (b) 26 928 FRW for 3years at 8% p.a.
   (c) 48 000 FRW for $3\frac{1}{2}$ % p.a.
3. Calculate the rate of simple interest on each of the following:
   (a) 33 600 FRW earning 2 520 FRW in 2 years.
   (b) £162 earning £43.20 in 4 years.
   (c) 72 000 FRW earning 2 400 FRW in 6 years.
   (d) 9 600 FRW earning 960 FRW in 3 years.
4. Calculate the principal that will earn an interest of:
   (a) 28 000 FRW in 2 years at 14% p.a.
   (b) 8 000 FRW in 6 months at 10% p.a.
   (c) 24 000 FRW in $1\frac{1}{2}$ years at 8% p.a.
   (d) 60 800 FRW in 3 years at 8% p.a.
5. Find the time to the nearest year in which:
   (a) 33 800 FRW will earn 2 000 FRW at 8% p.a.
   (b) 6 600 FRW will earn 24 000 FRW at 10% p.a.
   (c) 100 000 FRW will earn 20 800 FRW at $8\frac{1}{2}$ % p.a.
   (d) 300 000 FRW will earn 48 000 FRW at 7% p.a.
6. If 720 000 FRW is invested for 9 months at an annual simple interest rate of 15%.
   (a) How much interest will be earned?
   (b) What is the amount of investment after 9 months?
7. A businessman obtained a loan of 200 000 FRW from a bank that requires him to pay back 24 000 FRW after 6 months. What is the interest rate?

8. Calculate the value of P from the equation: $I = \frac{PRT}{100}$, if I = 51 200 FRW, T = 16 years and R = 16%.

4.8 Tax

Activity 4.9

In groups of 4 learners, discuss the following:

1. (i) What are taxes?
   (ii) How many taxes do you know? Give examples of each category.
2. What happens when you import goods from another country into Rwanda? What taxes are you expected to pay? How are the tax rates determined.
3. What is the name of tax charged on products like alcoholic beverages and tobacco. How are they charged on?

Taxes are two types:

(a) Direct taxes e.g. income tax and pay as you earn.

(b) Indirect taxes e.g withholding taxes, excise duty, value added tax and so on.

(a) Indirect taxes

Indirect taxes are levied on goods and services. Examples of such taxes include: value added tax (VAT), excise tax, stamp duty, withholding tax etc. The details concerning rates of these taxes are outlined in the country's tax guidelines. Other sources of taxable income include loyalties, dividends, interest and professional fees.

Customs duty: This is a tax levied on goods and services. This tax is collected by the customs authority and comprise of import duty, import excise duty, VAT other assessed taxes etc.

Custom duty is payable on imports of foreign goods. There are guidelines on the criteria of assessing tax. Just like any other tax, this tax is used to raise revenue for the government expenditure and also as a means to restrict/reduce imports to protect local industries.

Custom duty is charged depending on the value of the goods, weight, dimensions and source of those goods. Incase of cars and machinery, tax is charged depending on the size of the engine. This tax is a certain percentage of cost of goods.

Excise duty: It is a tax charged on specific goods produced or consumed within the country. Petrol and intoxicating liquor and tobacco, perfumes etc are some examples of goods subject to excise duty. **Assessed taxes** are duties not strictly in the category of excise duty but are classified under that heading because they are levies on particular services within a country e.g. motor vehicles licenses, trade licenses etc. The rates and the amounts to be paid are determined by the tax authority.

Example 4.22

A car worth 900 000 FRW was subjected to the import duty (Custom duty) at a rate of 40% followed by a VAT at 18%. Calculate the total amount of tax charged.

Solution

Customs duty $= \frac{40 \times 900\,0\cancel{00}}{1\cancel{00}}$

$= 360\,000$ FRW

New value of the car is:

$= 900\,000 + 360\,000$

$= 1\,260\,000$ FRW

$VAT = \frac{1\,260\,0\cancel{00}}{1\cancel{00}} \times 18$

$= 226\,800$ FRW

Total cost of the car

$= 1\ 260\ 000 + 226\ 800$

$= 1\ 486\ 800$ *FRW*

Total tax $= 1\ 4\ 86\ 800 - 900\ 000$

$= 586\ 800$ *FRW*

Total tax can also be obtained by adding the two taxes i.e. Customs duty + VAT

$= 360\ 000 + 226\ 800$

$= 586\ 800$ *FRW*

(b) Withholding tax

This is a type of tax deducted from certain specified payments. The deduction is done by the person doing the payment and then remitted to the administration. Currently, this tax is charged on dividends, interest, royalties, technical service fee among others. Tax on dividends is charged at 10%; tax on interest and loyalties at 15% as an advance tax.

WITHHOLDING TAX RATES	
Nature of payment	**Rate in percentage (%)**
Royalties	15
Rent in excess of 1 000 000 FRW	30
Payment of any supplies to traders and institutions e.g food	30
Commission	10
Payment of carriage and haulage	20
Payment to contractors/ subcontractors	4
Payment for public entertainment	15
Payment in excess of FRW 30 000 casual labour	15
Payment of services	15
Payment of tobacco sales	3
Bank interest in excess FRW 10 000	15
Fees	10

Table 4.1

(c) VAT (Value Added Tax)

Activity 4.10

Carry out the following discussion

- Tell your partners what Value Added Tax is.
- Obtain a supermarket receipt of goods bought. What is the rate of V.A.T from the receipt.
- Are there goods that are exempted from V.A.T?
- Who pays V.A.T? How does V.A.T get to Rwanda Revenue Authority?

This is an indirect tax on goods calculated by adding a percentage to the value added to a product at each stage of production. The whole cost of the tax is eventually passed on to the consumer of the finished products. Currently the standard rate of VAT in Rwanda is 18%. VAT is payable by those who supply taxable goods and services and those who make a turnover of taxable goods of 20 million FRW or more per annum. VAT is also charged on some imported goods and services as may be guided by the tax authority. The importer is mandated to charge VAT and then remit it to the tax department.

A withholding tax of 5% of the value of goods imported for commercial use is paid at custom on the cost insurance and freight value before the goods are released by customs.

A withholding tax of 3% of the sum of invoice excluding value added tax is retained on payments or by public institutions to the supply of goods and services based on public tenders.

Example 4.23

A manufacturer produces a product at a cost of 20 000 FRW and sold it at 28 000 FRW to a wholesaler. Calculate the value of VAT that he paid to the tax authority.

Solution

Cost of manufacturing and materials

$= 20\ 000$ FRW

Selling price of the product

$= 28\ 000$ FRW

Value added $= 28\ 000 - 20\ 000$ FRW

$= 8\ 000$ FRW

Therefore, VAT is charged on 8 000 FRW at the rate of 18%

$$\text{Therefore, VAT} = \frac{(8\ 0\cancel{00} \times 18)}{1\cancel{00}} \text{ FRW}$$

$$= (80 \times 18) \text{ FRW}$$

$$= 1\ 440 \text{ FRW}$$

Example 4.24

An author earned a royalty of 980 250 FRW before an advance tax was deducted. If the tax was charged at a rate of 20%, calculate:

(a) How much tax was he charged?

(b) His earning after the tax.

Solution

(a) Advance tax means withholding tax

The rate of 20% means for every 100 FRW, the tax charged is 20 FRW

Tax on 100 FRW = 20 FRW

$$\text{Tax on } 980\ 250 \text{ FRW} = \frac{980\ 25\cancel{0} \times 2\cancel{0}}{1\cancel{00}}$$

$$= 196\ 050 \text{ FRW}$$

(b) Earning before tax $= 980\ 250$ FRW

Tax $= 196\ 050$ FRW

Earning after tax

$= 980\ 250 - 196\ 050$

$= 784\ 200$ FRW

Exercise 4.6

1. A machine is sold at 8 000 FRW plus VAT. If VAT is charged at 18%, how much would one pay for the machine?
2. The price of three seats set inclusive of VAT is 58 250 FRW. What is the price of those seats exclusive of VAT?
3. A washing machine costs 48 000 FRW exclusives of VAT. Given that VAT is charged at 18%, how much will the machine cost including VAT?
4. If the value added tax is charged at 18%, find the tax payable if a commodity is priced at 5 000 FRW inclusive of VAT.
5. A machine costs 699 000 FRW with a VAT at 16.5% inclusive. Calculate the cost of the machine before VAT was added.
6. A town council imposes different taxes on different fixed assets as follows:

Commercial property	25% per year
Residential property	15% per year
Industrial property	20% per year.

An investor owns a residential building on a plot all valued at 80 000 000 FRW an industrial plot worth 75 000 000 FRW and a commercial premise worth 12 500 000 FRW. How much tax does the investor pay annually?

7. A company declares a final dividend of 10.54 FRW per share. Given that a shareholder holds 18 000 FRW shares, calculate:
   (a) His gross dividend.
   (b) The withholding tax charged on the dividend at 15%
   (c) The net amount due.

8. **a)** An author paid a withholding tax of 1 080 000 FRW on his royalty. Calculate:
   i) Her gross royalty
   ii) Her net royalty

   b) An author earned a royalty of 11 763 000 FRW after tax. Calculate:
   i) His gross royalty.
   ii) The tax charged on the royalty.

4.9 Insurance

(a) What is insurance?

Activity 4.11

Discuss the following:

1. What is insurance.
2. Why do people insure goods or themselves?
3. What are some of the mandatory forms of insurance in Rwanda?
4. Give examples of other forms of insurance.
5. Are these benefits of insurance that you know of?

A home is probably one of the most expensive purchases a person can make. To enable a person to protect himself/herself against risks of heavy loses, the owner makes annual payments to an insurance company so that the company can compensate the owner should the risk occur. The owner is paid a sum of money at which the damage or loss is assessed. The sum to cover the risk is paid annually and is called the **premium**. The contract signed between the company and the insured is called the **insurance policy**. The premium is calculated based on the value insured and the likelihood of the risk occurring.

Some forms of insurance are compulsory. For example, the owner of a car must insure against **third party risks** so that anyone who is injured, or has his or her car damaged as result of his negligent driving may receive compensation.

Another principal form of insurance is life insurance. This may be classified as:

(i) **A whole life policy**, which in return for an annual premium secures a lump sum payment at death. This form of insurance benefits the dependent of the policy holder.

(ii) **An endowment policy**, which in return for an annual premium for a fixed period of time secures a lump sum payment at a definite age e.g. at retirement. In this case, the policy holder is the beneficiary.

However, if the insured person dies before the policy matures, the whole or a fraction of the amount paid in premiums is returned to the insured person's estate.

(b) Benefits of insurance

In return for the premiums that the insured person pays during the period of insurance, there are certain benefits that he/she enjoys, for example, when a car is insured against third party risks or even comprehensively, there is possible compensation in case of an accident. Similarly, in life assurance, the sum assured accrues a bonus annually which is paid together with the assured amount when the policy matures.

In asset management, the invested funds accrue interest which is payable annually. The investor has a choice to receive his interest at the end of the year or to re-invest it to earn more.

There are many other schemes that offer specific benefits depending on the individual companies and the products that they offer.

Points to note

1. It is illegal to knowingly insure a property for more than its value.
2. It is illegal to insure against a risk in which the insurer has no financial interest.

Other forms of insurance include: medical, burglary and loss of household property and so on.

Example 4.25

A businessman insures goods valued at 426 250 000 FRW at the rate of 13 640 FRW for every 3 410 000 FRW worth of goods. What annual premium does he pay?

Solution

Premium for 3 410 000 FRW is 13 640 FRW

Premium for 426 250 000 FRW

$$= \frac{426\ 250\ 000}{3\ 410\ 000} \times 13\ 640$$

$$= 1\ 705\ 000 \text{ FRW}$$

He pays an annual premium of 1 705 000 FRW

Example 4.26

A man took a life insurance policy for 2 728 000 FRW on his 25th birthday. He paid a premium of 35 740 FRW yearly. He died at the age of 65 years. The company paid into his estate the whole amount of his insurance policy. How much more than his premium contribution did the company pay into the man's estate?

Solution

From age 25 years to age 65; 40 premiums were paid. Amount paid in premiums: $35\ 740 \times 40 = 1\ 429\ 600$

Policy value was 2 728 000 FRW

Difference: $2\ 728\ 000 - 142\ 960 = 1\ 298\ 400$ *FRW*

The company paid 1 298 400 FRW above his contribution premiums.

Example 4.27

The annual insurance on a car is 180 000 FRW. The owner is allowed a 20% non-claim bonus because he had not made any claims during the previous year. How much premium did he pay for the year.

Solution

$$\text{No-claim bonus} = \frac{20}{100} \times 180\ 000$$

$$= 36\ 000 \text{ FRW}$$

The owner will pay in premium less the no-claim bonus.

$$\text{Insurance premium} = 180\ 000 - 36\ 000$$

$$= 144\ 000 \text{ FRW}$$

Exercise 4.7

1. Goods in transit worth 95 480 FRW were insured against damage at 682 FRW in every 27 200 FRW worth of goods. Find the premium for insuring the goods.
2. John takes out a 20-year 8 360 000 FRW endowment policy and pays 71 600 FRW in 1 672 000 FRW of the coverage. Find the total premiums.
3. Anne's policy is 5 270 FRW per year for each 1 672 000 FRW of goods coverage. If the goods are estimated to have a replacement value of 11 787 600 FRW, find the premium.
4. A property sells for 810 920 000 FRW. The new owner has taken a policy coverage of 418 000 000 FRW. What would be your advice to him regarding the coverage?
5. A business premises was insured for 24 500 000 FRW. The insurance company promised to pay 50% of the sum insured incase the premises got destroyed. How much money will the company pay if the premises is destroyed?
6. A businesswoman insures her business at 14%. If the business was worth 1 250 000 FRW. What annual premium does she pay?
7. A home valued at 300 000 000 FRW is to be insured. The insurance company quotes a premium of 3 000 FRW per 1 000 000 FRW. How much will the owner pay in premiums per year?

Summary

1. Percentage is a fraction whose denominator is 100.
2. Discount is the amount reduced from the marked price of a commodity.
3. Commission is the money paid to sales agents or representatives for sales made.
4. Profit is the extra amount gained after selling a commodity at a price higher than the buying price.
5. Loss is the amount of money lost when a commodity is sold below the actual buying price.
6. Percentage loss is the ratio of loss to the buying price expressed as a percentage.
7. Percentage profit is the ratio of profit to the buying price expressed as a percentage.
8. Interest is the fee paid for borrowing or depositing money for a specific period of time. It is a fixed percentage charged on unpaid loans.
9. Principal is the total amount of money borrowed from a financial institution or deposited to earn interest.
10. Rate is the ratio of interest earned to the principal borrowed or deposited for a specific period of time.
11. Simple interest is the amount charged when one borrows money or loan from a financial institution which accrue yearly.

Unit Test 4

1. Convert into percentage:

   (a) $\frac{17}{20}$ (b) 0.35 (c) $\frac{13}{27}$

   (d) $\frac{2}{11}$ (e) $\frac{5}{7}$

2. (a) What is $7\frac{1}{2}$ % of 50 000 FRW?

   (b) 22% of a given area is 99 cm^2. What is the complete area?

3. A trader borrowed money from a bank at a rate of 12% per annum. He was to repay the money in 3 years. If his total interest for this duration was 1 800 000 FRW, what amount of money had he borrowed?
4. A businessman bought 100 pairs of shoes at 2 000 000 FRW and made a loss of 7%. Calculate the selling price of each pair of shoes.
5. A businessman imported a car valued at 1 000 000 FRW. An import duty was levied at 50% and VAT on the value after duty at 16.5%. He then sold the car making a profit of 30%. How much money did he receive?
6. Calculate the rate of simple interest on each of the following:
   (a) 8 200 000 FRW earning 4 200 FRW in 6 years.
   (b) 2 421 00 FRW earning 2100 FRW in 2 years.
   (c) 31 000 FRW earning 370 FRW in 4 years.
   (d) 40 800 000 FRW earning 4 000 FRW in 7 years.
7. A pharmacist bought medicine worth 27 000 000 FRW and sold them at 29 800 000 FRW. What was his percentage profit?
8. A man takes out a whole life insurance policy for $ 100 000 on his 25th birthday by paying a premium of $ 1 310 a year. If he dies at the age of 64 years and 8 months, how much more does the company pay his estate than he has paid in premiums to the company? ($ = Us dollar. 1 US $ = 1.0013377 FRW)
9. A car owner pays an annual premium of 90 000 FRW. He is allowed a non-claim bonus of $25\frac{1}{2}$%. How much does he pay in premiums that year?
10. Mrs Mitako had a beauty shop in Kigali. She employed a sales agent to sell the beauty products around the town. The sales agent earned a salary of 49 000 FRW and a commission of 6% for all the goods sold. In one month, the sales from the agent were 50 429 000 FRW. How much did she pay the agent.
11. The marked price of a blouse in a shop was 7 000 FRW. The seller offered a discount of 7.5% on sales above 3 blouses. A customer bought 8 pieces of the blouses. How much did the customer pay?
12. Calculate the selling price of a car whose discount was 8% and a marked price of 7 029 000 FRW.
13. A sales agent earns a commission of 15% for selling goods worth 2 220 000 FRW and above. During a certain month, goods worth 82 101 200 FRW were sold. How much was the agents commission that month?
14. A shareholder who holds a total of 30 000 shares he received a net dividend of 13 500 FRW in a certain year. Given that dividends are taxed at 15%, Calculate:
    a) The shareholder's net dividend.
    b) The dividend declared by the company per share.
    c) The amount of money he paid in withholding tax.

5 RATIO AND PROPORTION

Key unit competence

By the end of this unit, I should be able to solve problems involving ratio and proportion.

Unit outline

- Ratio, proportion and sharing.
- Applying ratio and proportion in practical and everyday contexts.
- Direct and indirect proportions

Introduction

There are many situations in real life that involves ratios and proportions. For example, two people who jointly invest some money into a common project in a particular ratio share the proceeds (or profit) in the same ratio. In primary 6, you were introduced to ratios and proportions. In this unit, we will learn about them and solve problems involving them.

5.1 Ratios

To help us remind ourselves what ratios are, let us go through Activity 5.1.

Activity 5.1

1. Tell your partner what a ratio is.
2. Through a discussion with your partner, express the following relationships with ratios.
   (a) John and Lucy partnered to save money for some time and later buy a taxi. For every 800 FRW that John saved, Lucy saved 120 FRW. In what ratio were their contribution? What another simplest ratio is same as this?

   > ! Let us learn to save and accumulate resources to do a bigger thing in future.

   (b) Jane and David sold milk to a vendor in the morning. Jane sold 4 500 ml while David sold 7.5 litres. In what ratio are their milk sales?

A **ratio** is a mathematical statement which shows how two or more quantities or numbers compare.

For example, if the age of Joseph is 21 years, and his father is 63 years old, we say the ratio of Joseph's age to his father's age is 21 to 63 written as 21 : 63.

Ratios can also be expressed as fractions.

For example, the ratio of Joseph's age to his father's age as a fraction is $\frac{21}{63} = \frac{1}{3}$. This means that the father is 3 times as old as the son.

Note:
A ratio compares quantities of the same kind, and the units for all the quantities involved must be the same. Ratios are usually expressed in their lowest form and have no units.

In general, if **a** and **b** are two quantities of the same units, then the ratio of **a** to **b** is written as **a** : **b** and can be expressed as a fraction as $\frac{a}{b}$.

Example 5.1

A business lady mixes every fifteen kilograms of sorghum flour with nine kilograms of *cassava flour in order to increase quality and maximize profit. Represent her mixture of flour with a ratio.*

Solution

Sorghum flour : Cassava flour

15 kg : 9 kg (are in the same units)

In ratio form we write as 15:9 (note that a ratio has no units).

Example 5.2

150 cm^3 of water are contained in 15 litres of a chemical. Find the simplest ratio of water to the chemical.

Solution

Since quantities of a ratio must be of the same unit, we need to convert litres to cubic centimetres.

$$1 \text{ litre} = 1\,000 \text{ cm}^3$$

$$15 \text{ litres} = 1\,000 \times 15 = 15\,000 \text{ cm}^3$$

$$\text{Required ratio} = \frac{\text{Amount of water}}{\text{Amount of chemicals}} = \frac{150 \text{ cm}^3}{15\,000 \text{ cm}^3} = \frac{1}{100}$$

The ratio is 1:100.

5.1.1 Simplifying Ratios

Activity 5.2

Diane and Alexis had 2 heaps of 10 mangoes each. They shared the mangoes as follows:

- From the first heap of 10 mangoes, Diane picked 3 mangoes for every 2 mangoes that Alexis picked until the heap was finished.
- From the other heap of 10 mangoes, Diane took 6 mangoes for every 4 mangoes that Alexis took.

(a) How many mangoes did each of them get from each heap?

(b) In what ratios did they share the mangoes in the two heaps?

(c) What can you say about the two ratios?

In simplifying ratios, the two quantities of a ratio may be multiplied or divided by the same number without changing the value of the ratio.

For example, 6 : 8 can be multiplied by 3 to get $3 \times (6 : 8) = 18 : 24$.

Similarly, the ratio 4 : 26 can be divided by 2 to get $\frac{4}{2} : \frac{26}{2} = 2 : 13$.

Example 5.3

Express the ratio 15 : 45 in its simplest form.

Solution

15 : 45

Divide both values with their GCD

The GCD of 15 and 45 is 15

$$15 : 45 = \frac{\cancel{15}^{\,1}}{\cancel{45}_{\,3}} = \frac{1}{3} = 1:3$$

Therefore, the ratio 15:45 can be simplified to 1:3.

Example 5.4

Express the ratio $\frac{2}{3} \div \frac{3}{4}$ as a fraction in its simplest form.

Solution

The given ratio is $= \frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$

Multiply both quantities of the ratio, by the LCM of the denominators i.e; 12. :

$\frac{2}{3} \times 12 : \frac{3}{4} \times 12$

$= 8 : 9 = \frac{8}{9}$

Example 5.5

Complete the following ratios

(a) $4:7 = 12: ?$ (b) $\frac{9}{13} : \frac{?}{52}$

Solution

If two ratios are equal, one ratio is a multiple of the other. We use the multiplying factor to find the missing value.

(a) $4:7 = 12 : ?$

multiplying factor $= \frac{12}{4} = 3$

missing value $= 7 \times 3 = 21$

$4 : 7 = 12 : 21$

(b) $\frac{9}{13} : \frac{?}{52}$

multiplying factor $= \frac{52}{13} = 4$

missing value $= 9 \times 4 = 36$

$\frac{9}{13} : \frac{36}{52}$

Example 5.6

Given the ratio a : b = 5 : 2, and ratio b : c = 3 : 4, find the ratio a : b : c.

Solution

Rewrite the ratio a : b : c as follows

$a : b : c$

$5 : 2$

$3 : 4$

Since b is a quantity in both ratios, we make the value of b the same in both ratios so as to join the two ratios as one.

We do this by multiplying the first ratio with the value of b in the second ratio and then we multiply the second ratio with the value of b in the first ratio.

$a : b : c$

$(5 : 2) \times 3$

$2(3 : 4)$

$a : b : c$

$15 : 6$

$6 : 8$

since b is same, $a : b : c$

$15 : 6 : 8$

Activity 5.3

Individually, Match the ratios that are equal

4 : 7	39 : 24
45 : 54	11 : 9
13 : 8	15 : 18
55 : 35	28 : 49

Exercise 5.1

1. Express the following ratios in their simplest forms:
   (a) 28 : 42
   (b) 30 : 50
   (c) 24 kg : 30 kg
   (d) 150 cm to 3 m
   (e) 1 litre to 250 m*l*
   (f) 45 min : $1\frac{1}{2}$ hours
   (g) 2.6 kg to 130 g
   (h) 160 cm^3 to 2 litres
   (i) 28 days to 2 weeks
2. Simplify the following ratios:
   (a) 2 : 0.4 (b) 0.9 : 0.18
   (c) 0.3 : 0.12 (d) $\frac{3}{4} : 10$
   (e) $\frac{3}{4} : \frac{3}{5}$ (f) $3\frac{1}{2} : 2\frac{1}{2}$
3. Mutanguha had 120 chicken in his farm. He sold 30 of them to buy a dairy goat. What is the simplest ratio

of the remaining number of chicken to the original number of chicken?

4. A church has 325 males and 390 females. Find the simplest ratio of:
   (a) the number of females to the number of males.
   (b) the number of males to the total number of people in the church.
   (c) the number of females to the total number of people in the church.

5. Complete the following ratios:
   (a) 3 : 8 = 9 : _?_
   (b) 4: _?_ = 28 : 63
   (c) _?_ : 30 = 5 : 6
   (d) 45 : _?_ = 15 : 39
   (e) $\frac{7}{?} = \frac{21}{24}$
   (f) $\frac{35}{20} = \frac{7}{?}$

6. (a) If p : q = 4 : 3 and q : r = 5 : 3, find p : q : r.
   (b) If x : y = 3 : 4 and y : z = 5 : 2, find x : y : z.

5.1.2 Sharing quantities using ratios

Activity 5.4

Suppose two old men from your village have come to you to arbitrate after they disagreed over how to share 7 000 FRW such that for every 2 FRW that the first man gets, the other one gets 3 FRW.

(i) In what ratio would you share the money between them?

(ii) Tell your partner how you would share the money and how much each would get.

Let us all be crusaders of peace wherever we are, rather than be the ones to fuel disagreements.

There are many cases in real life where people or organizations need to share items or resources unequally but in a given ratio. For example, a father may want to share 24 acre of land among his two sons. One of them who is disabled gets double of what the other son gets. In such a case, we share the quantity using the given ratio. For example the father would share the land in the ratio of 2:1. Assume the whole land is first subdivided into equal parts whose number is equal to the sum of the two values in the ratio i.e. 2+1 = 3 parts. The disabled son gets 2 parts out of 3 parts of the whole.

i.e. $\frac{2}{3}$ of 24 acres = $\frac{2}{\cancel{3}_1} \times \cancel{24}^{8}$ acres = 16 acres

The other son gets 1 part of 3 parts of the whole.

i.e. $\frac{1}{3}$ of 24 acres = $\frac{1}{\cancel{3}_1} \times \cancel{24}^{8}$ acres = 8 acres.

Notice the two proportions add up to the whole 18 + 6 = 24 acres.

We should always take care of the needs of the disabled persons and facilitate and empower them where we can.

In general,

To **share a quantity into two parts in the ratio a : b**, the quantity is split into a + b equal parts. The required parts become $\frac{a}{a+b}$ and $\frac{b}{a+b}$ of the quantity.

Example 5.7

Share 38 400 FRW between Linda and Jean in the ratio 5:7 respectively.

Solution

38 400 FRW is to be shared in the ratio 5:7. It is split into 12 equal parts i.e 5 + 7 = 12 equal parts.

The amount Linda receives

$\frac{5}{12} \times 38\,400 \text{ FRW} = 16\,000 \text{ FRW}$

Jean receives

$\frac{7}{12} \times 38\,400 \text{ FRW} = 22\,400 \text{ FRW}$

Example 5.8

Ingabire, Mugenzi and Shamarima have jointly invested in buying and selling of shares in the Rwanda stock exchange market. In one sale, they realised a gain of 1 080 000 FRW and intend to share it in the ratio 2:3:4 respectively. How much did Mugenzi get?

Solution

Mugenzi's share

$$= \frac{3}{2+3+4} \text{ of } 1\,080\,000 \text{ FRW}$$

$$= \frac{3}{9} \times 1\,080\,000 \text{ FRW}$$

$$= 360\,000 \text{ FRW}$$

Exercise 5.2

1. Divide the following quantities in the given ratio:
   (a) 1600 FRW (3:5)
   (b) 24 ha (2:3:5)
   (c) 327 kg (1:5:6)
   (d) 775 chicken (14:11)
2. When 5 280 FRW is shared by Anne, Shyaka and Elise in the ratio 2:4:5 respectively, what is the difference between Elise's and Anne's share?
3. 8 720 000 000 FRW allocated to a certain department of a ministry from the national budget is to be shared between two sections A and B, in the ratio 3 : 5 respectively. How much will each department get?
4. A man and his wife share a sum of money in the ratio 3 : 2. If the sum of money is doubled, in what ratio should they divide it so that the man still receives the same amount?
5. Ann, Ben and Mike share 192 000 FRW. Ann has twice as much as Ben and Ben has three times as much as Mike. In what ratio are their shares? How much did Ben receive?
6. 55% of students in a school are boys.
   (a) What is the ratio of boys to girls?
   (b) How many girls are there if the school has 400 students?

5.1.3 Application of ratios in scale drawing

Activity 5.5

On a plan, the length of 20 m of a house wall is represented by a length of 5 cm. Discuss with your partner and determine the scale of the plan to the actual house in ratio form in its simplest form.

One way of representing the scale between a drawing of an object or map and the actual object is by use of a ratio. For example, a scale of 1cm represents 5 m can be written in ratio form as 1cm represent 500 cm (same units) hence 1:500.

Example 5.9

A pupil drew the plan of her school to a scale of 1:500.

(a) *If the football field is 90 m by 70 m, find its length and width on the drawing.*

(b) *If in the scale drawing a classroom block is drawn as a rectangle 9 cm by 2.2 cm, find its actual length and width.*

Solution

1:500 means that 1 cm represents 500 cm or 1 cm represents 5 m.

(a) *5 m is represented by 1 cm*

90 m is represented by $\frac{1}{5} \times 90$ *cm* $= 18$ *cm*

70 m is represented by $\frac{1}{5} \times 70$ *cm* $=$ *14 cm.*

The football field is 18 cm long and 14 cm wide.

(b) *1 cm represents 5 m*

9 cm represents 5×9*m* $= 45$ *m and*

2.2 cm represents 5×2.2 *m* $= 11$ *m*

The classroom block is 45 m long and 11 m wide.

Exercise 5.3

1. Find the scale drawing length that represents:
   (a) 73 m when the scale is '1 cm represents 10 m'.
   (b) 73 m when the scale is '1 cm represents 5 m'.
   (c) 1 km when the scale is '10 cm represents 1 km'.
   (d) 1.53 km when the scale is '5 cm represents 2 km'.
2. Find the actual length represented by a scale drawing length of,
   (a) 7.3 cm when the scale '1 cm represents 3 m'.
   (b) 8.5 cm when the scale is '2 cm represents 4 km'.
   (c) 5.3 cm when the scale is '5 cm represents 1 km'.
   (d) 15.6 cm when the scale is '2 cm represents 100 m'.
3. A physical planner drew the plan of a dump site to a scale of 1 : 1 000. If the site measures 60 m by 45 m, find the dimensions of the site on the plan.
4. A triangular plot measuring 200 m by 180 m by 300 m is drawn on a map and measures 4 cm by 3.6 cm by 6 cm respectively. Determine the scale used to draw the map.

5.2 Proportion

Activity 5.6

Consider the ratios given below.

(a) 3 : 5 (b) 9:15

Write ratios (a) and (b) in their simplest form. What do you notice about the size of ration (a) and (b)?

Discussion

The two ratios are equal

The ratio 3 to 5 is equal to the ratio 9 to 15. This can be written as $\frac{1}{5} = \frac{9}{15}$ or 3 : 5 = 9 : 15.

When the equivalence of two ratios is written in this form it is called a **proportion**, and it is read as '3 is to 5 as 9 is to 15'. i.e a proportion is a relationship between four numbers or quantities in which the ratio of the first pair equals the ratio of the second pair, and is written as a:b = c:d and is read as "a is to b as c is to d"

Read these proportions:

(a) $\frac{2}{3} = \frac{4}{6}$ (b) $12 : 6 = 2 : 1$

(c) $\frac{5}{3} = \frac{x}{6}$

Write these as proportions:

(a) 4 is to 7 as 12 is to 21

(b) 2 is to x as 6 is to 3

5.2.1 Properties of proportion

In a proportion, the second and third values are called **means** and the first and fourth values are called **extremes**.

extremes

e.g. $3 : 5 = 6 : 10$ $\frac{3}{5} = \frac{6}{10}$

means means extremes

(a) Cross product property

Activity 5.7

Consider the following ratio

$4 : 5 = 8 : 10$ $\frac{4}{5} = \frac{8}{?}$

means extremes

What is the product of the means?

What is the product of the extremes?

What do you notice about the product of the exchange and means?

In a true proportion, the product of the means is equal to the product of the extremes.

$4 \times 10 = 8 \times 5 = 40$. In general, $\frac{a}{b} = \frac{c}{d}$ is same as $a \times d = b \times c$

This property is called cross product property.

Example 5.10

Solve these proportions

(a) $\frac{3}{2} = \frac{6}{x}$ (b) $\frac{5}{10} = \frac{2}{x}$

(c) $\frac{10}{8} = \frac{x}{4}$

Solution

(a) $\frac{3}{2} = \frac{6}{x}$ (b) $\frac{5}{10} = \frac{2}{x}$

$x = 4$ $x = 4$

(c) $\frac{10}{8} = \frac{x}{4}$

$x = 5$

Example 5.11

Tomatoes are selling at 3 tomatoes for 100 FRW. How much will 12 tomatoes cost?

Solution

Proportion: $\frac{3}{12} = \frac{100}{x}$

Solve: $3x = 12\ \ 100$ *FRW*

$3x = 1\,200$ *FRW*

$x = 400$ *FRW*

i.e. 12 tomatoes will cost 400 FRW.

(b) Means or extremes switching property

Activity 5.8

Consider the following ratios:

$2 : 3 = 4 : 6$

$\frac{2}{3} = \frac{4}{6}$

Write the numerators in fraction form.

Write the denominators in fraction form.

Simplify the fractions in their simplest form.

What do you notice about the size of the fractions?

Discussion

$\frac{2}{4} = \frac{3}{6} = \frac{1}{2}$

The new fractions are equal.

In general, $\frac{a}{c} = \frac{b}{d}$ is the same as

This is called Means or Extremes switching property.

Example 5.12

If $\frac{8}{10} = \frac{4}{5}$, what is the value of $\frac{8}{4}$?

Solution

By means or extremes switching property

$\frac{8}{4} = \frac{10}{5}$ or $\frac{4}{8} = \frac{5}{10}$

Example 5.13

If $\frac{x}{5} = \frac{y}{4}$, find the ratio of $\frac{x}{y}$

Solution

Using switching property of proportion

$\frac{x}{y} = \frac{5}{4}$

(c) Inverse property of proportions

Activity 5.9

Consider the following ratios

$2 : 3 = 4 : 6$

$\frac{2}{3} = \frac{4}{6}$

Interchange the numerator with denominator in each side of the equation.

Simplify the proportion.

What do you notice about the value of the simplest form of the fractions?

Discussion

Interchanging the numerator and denominators we have,

$\frac{3}{2} = \frac{6}{4}$ $\frac{6}{4} = \frac{3}{2}$

The simplest form of $\frac{6}{4} = \frac{3}{2}$ hence the proportions are equal.

In general, $\frac{a}{b} = \frac{c}{d}$ is same as $\frac{b}{a} = \frac{d}{c}$

Example 5.14

If $\frac{9}{a} = \frac{5}{b}$, find the ratio $\frac{a}{b}$

Solution

Applying inverse property

$\frac{a}{9} = \frac{b}{5}$

Applying switching property

$\frac{a}{b} = \frac{9}{5}$

(d) Additional property of proportions

Activity 5.10

Consider the following:

$\frac{a}{b} = \frac{c}{d}$

Add 1 on both sides

Simplify the sum on both sides.

How do we call such property?

Discussion

Adding 1 both sides we get

$\frac{a}{b} + 1 = \frac{c}{d} + 1$

Simplifying we get $\frac{a+b}{b} = \frac{c+d}{d}$

Similarly, $\frac{a-b}{b} = \frac{c-d}{d}$.

This is called additional property of proportions

Example 5.15

If $\frac{5}{8} = \frac{x}{y}$, find the value of $\frac{13}{8} = ?$

Solution

Applying additional property

$\frac{5}{8} + 1 = \frac{x}{y} + 1$

$\frac{5+8}{8} = \frac{x+y}{y}$

$\frac{13}{8} = \frac{x+y}{y}$

Exercise 5.4

1. Which of these are true proportions?

(a) $\frac{2}{3} = \frac{4}{6}$ (b) $\frac{8}{10} = \frac{2}{3}$

(c) $\frac{8}{6} = \frac{6}{4}$ (d) $3:2 = 3:5$

(e) $6:4 = 2:1$ (f) $1:3 = 6:18$

2. Solve the proportions:

(a) $\frac{3}{7} = \frac{x}{14}$ (b) $\frac{8}{x} = \frac{16}{4}$

(c) $\frac{6}{4} = \frac{x}{12}$ (d) $3:4 = x:8$

(e) $3:4 = 9:x$ (f) $x:12 = 6:8$

3. Find the unknown values in the following proportions.

(a) $6:a = 18:15$

(b) $8:7 = \frac{b}{28}$

(c) $\frac{2}{3} = 20:50$

(d) $21:16 = \frac{P}{48}$

4. Given that $x:y = 3:5$, find the ratios

(a) $(2x - y):(2x + 3y)$

(b) $(x + y):(2x - y)$

5. Given that $a:b = \frac{1}{3}:\frac{1}{4}$

(a) $(a + b):(a + b)$

(b) $(a - 2b):(2a - 3b)$

5.2.2 Direct proportion

Activity 5.11

Consider Table 5.1 below which shows the relationship between the number of pens and the cost of pens.

No. of pens	1	2	3	4	5
Cost (FRW)	120	240	360	480	600

Table 5.1

1. What happens to the cost when the number of pens is doubled.
2. If you divide the number of pens by two, what do you notice about the cost?.
3. What can you say about the relation between the cost of pens and the quantity of pens?

From Activity 5.11, you realize that two quantities are such that, when one quantity increases through a particular ratio, the other quantity increases in the same ratio, and vice versa. Such quantities are said to be **directly proportional**.

e.g., $\frac{2}{1} = \frac{240}{120}$; $\frac{3}{2} = \frac{360}{240}$; $\frac{5}{3} = \frac{600}{360}$

Activity 5.12

1. Individually, use the values in table 5.1 to draw the graph of the number of pens (N) against cost (C).
2. Describe the graph you have drawn in 1 above.

From Activity 5.12, quantities N and C are directly proportional. The graph of N against C is a straight line passing through the origin as shown in Fig. 5.1.

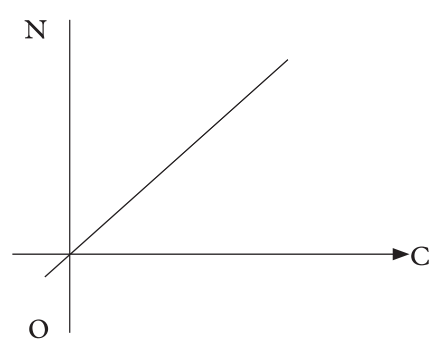


Fig. 5.1

Problems of direct proportionality can be solved by ratio or unitary methods.

Example 5.16

The cost of 5 kgs of sugar is 2 880 FRW. Find the cost of 3 kgs of the same sugar.

Solution

We can use ratios or unitary method to solve this problem.

Ratio method

Let the cost of 3 kg be x

Then x : 2 880 = 3:5 (the ratio of the cost is equal to the ratio of the sugar in kgs)

Using the ratios in fraction form: $\frac{x}{2880} = \frac{3}{5}$

$x = \frac{3}{5} \times 2\ 880$ FRW $= 1\ 728$ FRW

Hence x, the cost of 3 kg = 1 728 FRW

Unitary method

5 kg cost 2 880 FRW

1 kg cost $\frac{2\ 880}{5}$ *FRW*

3 kg cost $\frac{2\ 880 \times 3}{5}$ *FRW = 1 728 FRW*

Notice that a decrease in the number of kg of sugar bought leads to a decrease in the cost in the same ratio.

Example 5.17

A train travels 120 km in 2 hours. How long will it take to cover a distance of 480 km?

Solution

Ratio method

Let the time taken be x hours

Then x : 2 = 480 : 120

Using the ratio in fraction form; $\frac{x}{2} = \frac{480}{120}$

$x = \frac{48}{12} \times 2 = 8$ *hours*

Unitary method

It takes 2 hours to cover 120 km

1 km is covered in $\frac{2}{120}$ *h*

So 480 km will take $\frac{2 \times 480}{120} = 8$ *h*

Notice that an increase in distance covered leads to an increase in the time taken in the same ratio.

Example 5.18

Table 5.2 shows the distance (d) in kilometres covered and the time taken at constant speed.

Distance(d) covered in km	50	100	150	200	250
Time(t) taken in hours	1	2	3	4	5

Table 5.2

Use a graph to find:

(a) *The distance covered in 2hr 12 min.*

(b) *The time taken to cover a distance of 180 km.*

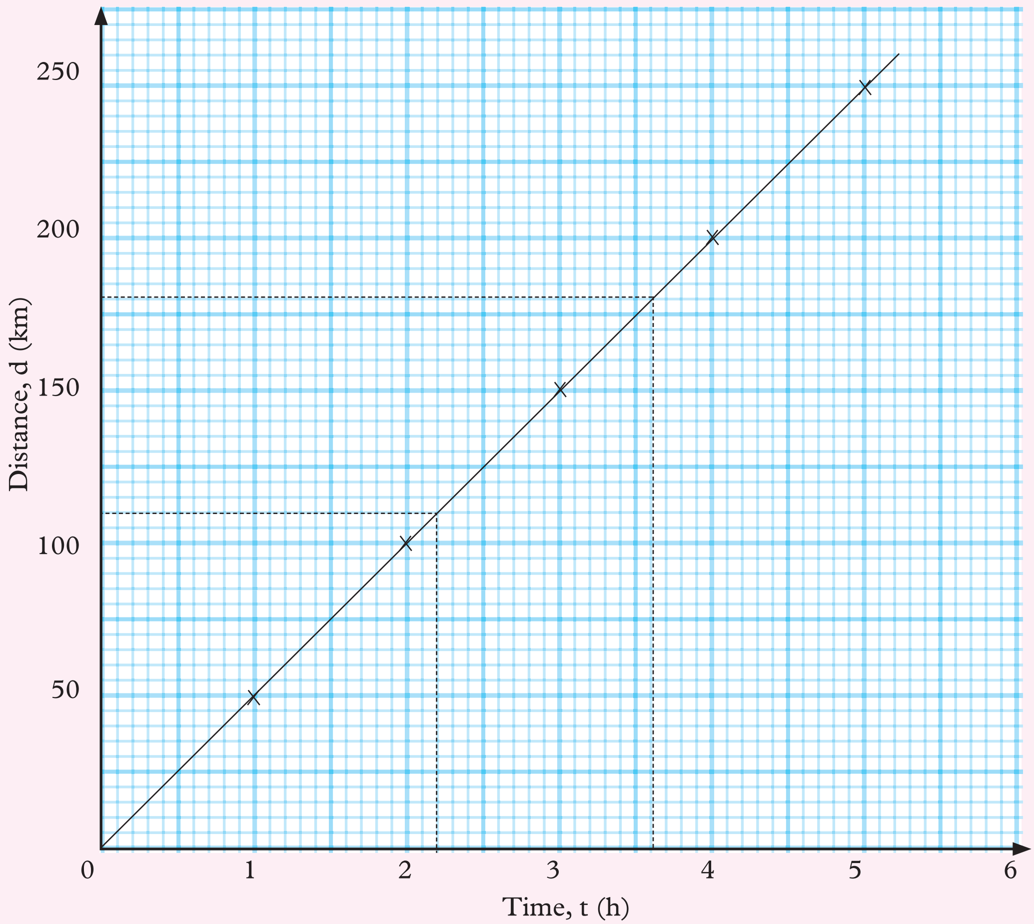


Fig. 5.2

Solution

(a) *From the graph, we see that the distance covered in 2h 12 min is 110 km.*

(b) *From the graph we see that a distance of 180 km will take 3 hours 36 minutes.*

Exercise 5.5

1. At a steady speed, a car uses 5 litres of petrol to travel 85 km. How much petrol is needed to travel 70 km at the same speed?

2. In a spring balance, the extension in the spring is proportional to the load. If the extension is 3.5 cm when the load is 10 newtons, what is the extension when the load is 4.5 newtons?

3. If the mass of 16 cm^3 of a metal is 20 g, what is the mass of 30 cm^3?

4. A man earns 27 000 FRW in a 5-day week. What is his pay for 3 days?

5. Nine milk bottles contain $4\frac{1}{2}$ litres of milk. How much do six bottles hold?

6. Given that x and y are directly proportional, copy and complete Table 5.3 below.

x	1	2	4	5	10
y		10		25	

Table 5.3

7. It takes two hours for a motorbike to cover a distance at 50 km/h. How long will it take to cover the same distance at 75 km/h?

8. The extension of a spring is directly proportional to the force applied on it. Table 5.4 shows the values of force and the corresponding extensions for a spring obtained by a Senior 1 student in an experiment.

Force (N)	0	0.3	0.6	0.9	1.2
Extension (mm)	0	12	24	36	48

Table 5.4

(a) Plot a graph of force against extension.

(b) Use your graph to find:

(i) The extension produced by a force of 0.4 N.

(ii) The force required to produce an extension of 40 mm.

5.2.3 Inverse proportion

Activity 5.13

Consider the relationship between the speed and time taken by a car to cover a fixed distance of 320 km. (Table 5.5).

Speed (km/h)	20	40	80	160
Time (h)	16	8	4	2

Table 5.5

Take 20 km/h to be the original speed.

(a) What do you notice when the speed is doubled?

(b) Plot the graph of speed against time.

(c) Describe the graph you drew to your friend.

From the graph you drew in Activity 5.13, you should have noticed that; two quantities are such that if one quantity increases in the ratio $\frac{a}{b}$ the other quantity decreases in the ratio $\frac{b}{a}$ and vice versa. The two quantities are said to be **inversely proportional**.

The graph of speed (S) against hour (h) is a curve of the form shown in Fig. 5.3.

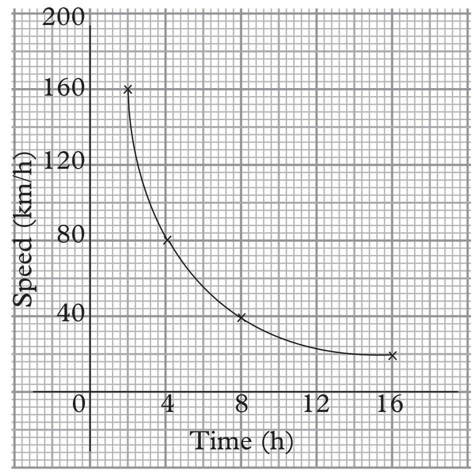


Fig. 5.3

Problems on inverse proportion are solved using either unitary, ratio or graphical method.

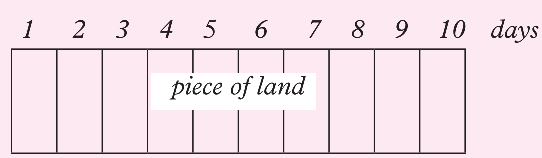
Example 5.19

Four men working at the same rate can dig a piece of land in ten days. How long would it take five men to do the same job?

Solution

Unitary method

4 men can dig in 10 days



4 men dig $\frac{1}{10}$ of the land in 1 day.

1 man digs ($\frac{1}{10} \div 4 = \frac{1}{40}$) of the land in 1 day.

1 man takes 40 days to dig the entire land.

For 5 men = $\frac{40}{5}$ days (taken by one man)

$= 8$ *days*

Ratio method

Five men take shorter time than four men so we use the inverse ratio.

The number of men increase in the ratio 5:4

The days decrease in the ratio x : 10

Using inverse ratio, $\frac{x}{10} = \frac{4}{5}$

$x = \frac{4}{5} \times 10$

$= 8$ *days.*

Example 5.20

9 men working in a factory produce a certain number of pans in 6 working days. How long will it take 12 men to produce the same number of pans if they work at the same rate?

Solution

Ratio method

The number of men increase in the ratio 12 : 9 = 4 : 3 (a : b)

The number of days thus decrease in the ratio 3 : 4 i.e (b : a)

Let the number of days be x.

$x : 6 = 3 : 4$

$x = \frac{3 \times 6}{4} = 4\frac{1}{2}$ *working days*

Unitary method

9 men work for 6 days

1 man works for 9 × 6 days = 54 complete working days

12 men will work for $\frac{9 \times 6}{12} = 4\frac{1}{2}$ *working days.*

Example 5.21

Table 5.6 shows the time taken to cover a distance of 120 km at various speeds.

Speed, x(km/h)	20	30	40	60	80	120
Time, t(h)	6	4	3	2	1.5	1.0

Table 5.6

(a) *Draw a graph of speed against time taken.*

(b) *Use your graph to determine:*

(i) *The time taken to cover the same distance at a speed of 75 km/h.*

(ii) *The speed required to cover the same distance in 2.4 h.*

Solution

(a)

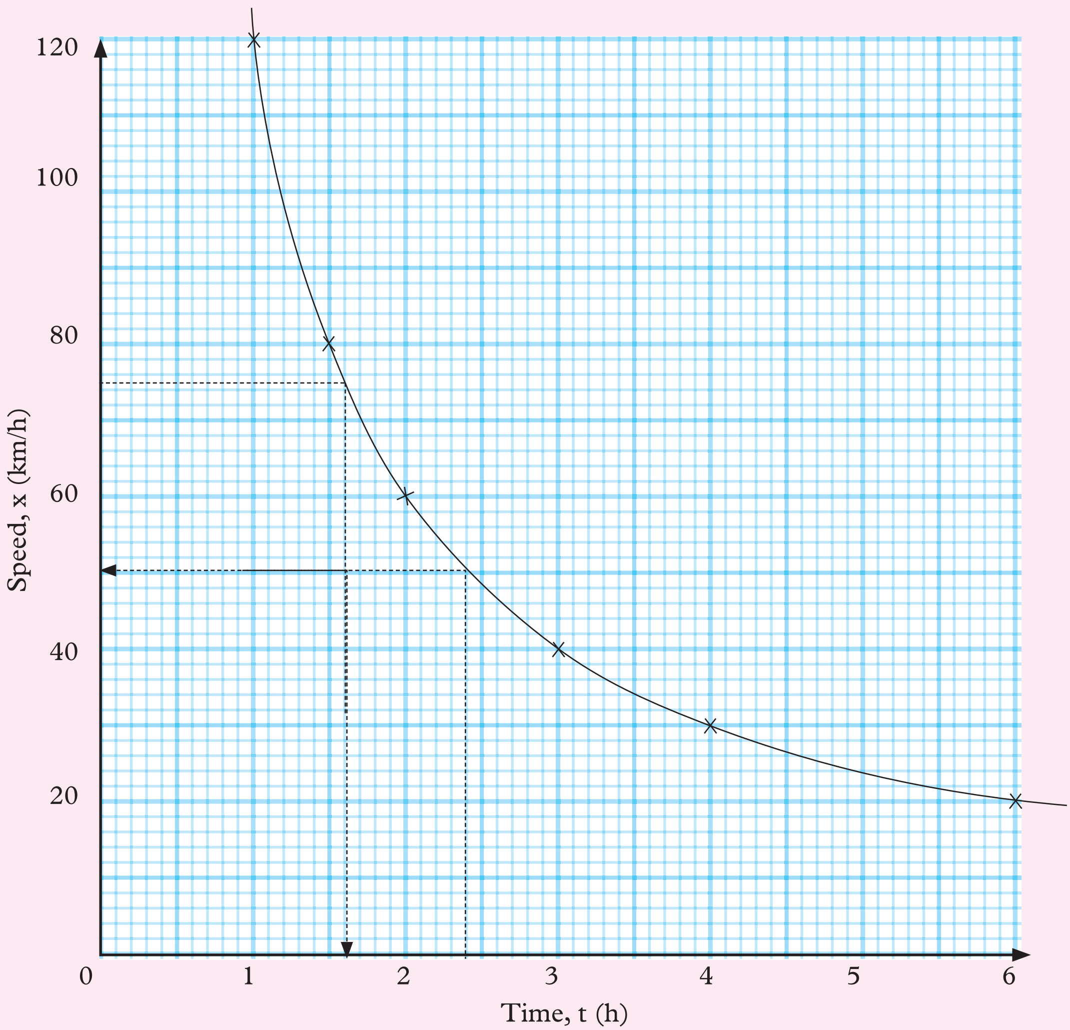


Fig 5.4

(b) *From the graph:*

(i) *The time taken to cover the same distance at a speed of 75 km/h is 1.6 h.*

(ii) *The speed required to cover the same distance in 2.4 h is 50 km/h.*

Exercise 5.6

1. 12 taps can fill a tank in three hours. How long would it take to fill the tank if only 4 taps are working?
2. If 8 people can cultivate a piece of land in 6 days. How long will it take 12 people to cultivate the same piece of land?
3. Three men can build a wall in 10 hours. How many men would be needed to build the wall in $7\frac{1}{2}$ hours?
4. If it takes 6 men 4 days to dig a hole 5 m deep, how long will it take 10 men to dig a hole 8 m deep?
5. Given that x is inversely proportional to y, copy and complete the following table.

x	1	2	4	8	16	32	64
y		16		4		1	

Table 5.7

6. A school bought 120 books costing 960 FRW each. If the price of each book was reduced by 280 FRW, how many more books could be bought?
7. A cyclist averages at a speed of 27 km/h for 4 hours. At what average speed would he need to cycle to cover the same distance in 3 hours?
8. The volume of a fixed mass of a gas at constant temperature is inversely proportional to its pressure. Table 5.7 below shows the volume of a given mass of gas at different pressures when the temperature is constant.

Pressure, P (Pascals)	750	500	400	300	250	200	150	100
Volume, V (m^3)	2	3	3.75	5	6	7.5	10	15

Table 5.8

   (a) Draw a graph of V against P.
   (b) Using the graph, find:
      (i) P when V = 8 m^3.
      (ii) V when P = 350 pascals.
   (c) Calculate V when P = 600 pascals.

9. Table 5.9 shows the acceleration of different masses when a certain constant force is applied to them. The masses m (kg) and the acceleration a (m/s^2) are shown.

m (kg)	20	15	12	10	8	6	5	2
a (m/s^2)	1	1.33	1.67	2	2.5	3.33	4	10

Table 5.9

   (a) Draw a graph of m against a
   (b) Use the graph to find:
      (i) m when $a = 3$ m/s^2.
      (ii) the acceleration when the mass is 9 kg.
   (d) The mass when acceleration is 4.5 m/s^2.

Summary

1. **Ratio** – It is a mathematical statement of how two or more quantities or numbers compare.
2. **Simplifying ratios** – This is where two quantities of a ratio may be multiplied or divided by the same number without changing the value ratio.
3. **Sharing** – To share a quantity into two parts in the ratio a : b is where the quantity is split into a + b equal parts and the required parts become $\frac{a}{a+b}$ and $\frac{b}{a+b}$ of the quantity.

4. Direct proportion – It is the proportion in which two quantities are such that, when one quantity increases through a particular ratio, the other quantity increases in the same ratio and vice versa.
5. Inverse proportion – It is the proportion in which two quantities are such that, when one quantity increases in the ratio $\frac{a}{b}$, the other quantity decreases in the ratio $\frac{b}{a}$.

Unit Test 5

1. It takes 32 men to complete a piece of work in 10 days. How many extra men are required to complete the same work in 8 days?
2. A teacher shares sweets among 10 students so that each gets 6 sweets. How many sweets would each student have if there are 12 students?
3. The electric current (I) through an electric conductor is directly proportional to the voltage (V) applied across the conductor. Table 5.10 shows the corresponding values of the applied voltage (V) and the current (A) through the conductor obtained in an experiment.

Voltage (V)	0	1.0	2.4	3.0	4.0	4.8	6.0
Current (A)	0	0.3	0.8	1.0	1.3	1.6	2.0

Table 5.10

   (a) Draw a graph of voltage (V) against current (A).
   (b) Use your graph to determine:
      (i) The voltage required to drive a current of 1.2A through the conductor.
      (ii) The correct current passing through the conductor when a voltage of 3.5 is applied across it.
4. In a hall, chairs are arranged in 35 rows of 18 chairs each.
   (i) How many rows would there be with 21 chairs in a row?
   (ii) How many chairs would there be in each row if there were 15 rows?
5. Find the ratio a : b : c in each of the following:
   (a) a : b = 3 : 7 and b : c = 9 : 5
   (b) a : b = 2 : 9 and b : c = 3 : 11
   (c) a : b = 4 : 5 and b : c = 2 : 5
   (d) a : b = 1 : 3 and b : c = 9 : 13
6. A dispensary has 78 workers. 28 of them are women and the rest are men. Find the simplest ratio of:
   (a) The number of female to the number of male.
   (b) The number of male to the number of all workers.
   (c) The number of female to the total number of workers.
7. Simplify the following ratios
   (a) 5 : 0.75 (b) $3\frac{1}{3} : 6\frac{1}{6}$
   (c) 0.4 : 0.8 (d) $\frac{2}{5} : \frac{3}{7}$
8. The angles of a triangle are in the ratio 2 : 5 : 3. Calculate the size of each angle.
9. A farmer mixed two chemicals A and B in the ratio 4:5. If the farmer used 20 litres of chemical A in the mixture, what was the total quantity of the mixture?
10. An architect drew a house plan to a scale of 1 : 5 000. If the actual room measured 6 m by 5 m, find the dimensions of the room on the plan.

6 POINTS, LINES AND ANGLES

Key unit Competence

By the end of the unit, I should be able to construct mathematical arguments using the angle properties of parallel lines.

Unit outline

- Segment and rays.
- Angles on a straight line.
- Angles at a point.
- Angles on a parallel line.

6.1 Points

Activity 6.1

With reference to a number line, define the following:

(a) a point

(b) a line

In each case, use a diagram to illustrate your findings.

Discussion

In geometry, *a point marks a particular position.* A point has no size or dimension, but for the sake of illustration it is marked with a fine dot and labelled with a capital letter.

Fig. 6.1 Illustrates a set of points on a plane.

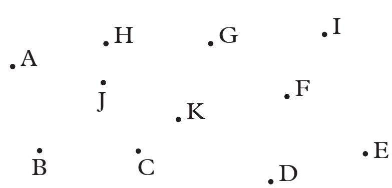


Fig. 6.1

6.2 Lines

A line is a set of points which are joined together. It is neither bent or curved. Fig. 6.2 shows different lines in different directions. The arrows are meant to show that the lines continue indefinitely.

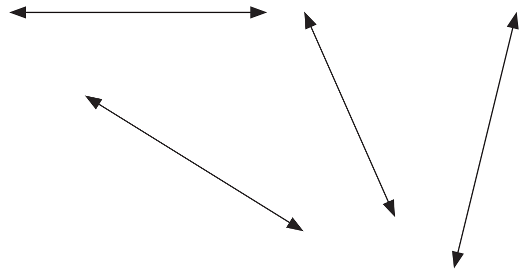


Fig. 6.2

Activity 6.2

Working in pairs, identify the differences between the diagrams given in Fig. 6.3

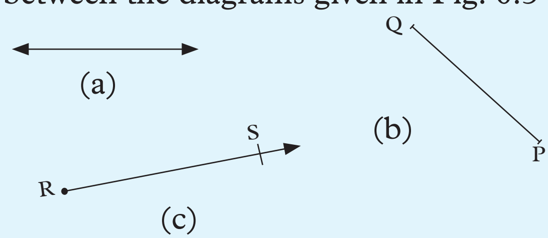


Fig. 6.3

Observations

From Activity 6.2, you should have observed that

(a) Fig. 6.3(a) represents a line, which is shown to continue in two directions indefinitely. There are two points on the line by which we can name the line. We can denote it as line AD.

(b) Fig. 6.3 (b) shows a fraction of a line that is defined by two points, P and Q. It has a beginning and an end. The portion of a line PQ is called a line segment. A line segment is also a set of points.

(c) Fig. 6.3(c) shows a portion of a line which has a beginning and no end. Such a portion of line is called a half line or a ray. A ray also is a set of points.

Note:

A line is unbounded. It extends into two directions indefinitely.

A line segment is bounded at both ends i.e. it has definite beginning and end.

A ray is bounded at one end. It has a definite beginning but has no ending.

Example 6.1

(a) Draw a straight line $\overleftrightarrow{PQ}$.

(b) Draw a half line $\overrightarrow{PQ}$.

(c) Draw a line segment $\overline{PQ}$.

Solution

Fig. 6.4 shows the required lines

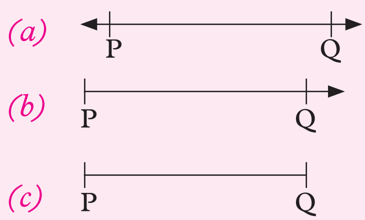


Fig. 6.4

Exercise 6.1

1. Study Fig. 6.5 below and answer the questions that follow.

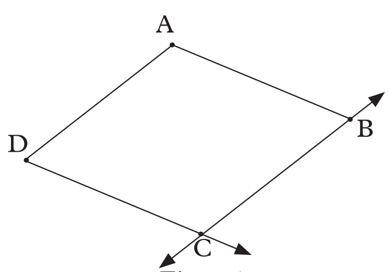


Fig. 6.5

(a) Name four points.
(b) Name the straight lines.
(c) Name the line segments.
(d) Name the rays.

2. Use Fig. 6.6 below to answer the following questions that follows.

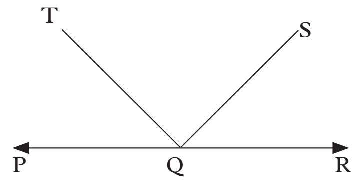


Fig. 6.6

(a) Name the half lines.
(b) Name the straight line or lines.
(c) Name the line segments.

6.3 Angles

In Primary 4, we learnt about angles. The following activities reminds us of some types of angles.

6.3.1 Types of Angles

(a) Right Angles

Activity 6.3

1. Tell your partner what you think a right angle is.
2. Obtain a rectangular plain piece of paper.
3. Fold the paper into two equal halves. Fold it again into two equal halves as shown in Fig. 6.7 below.

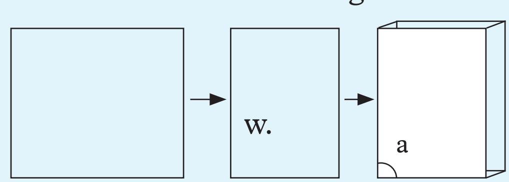


Fig. 6.7

What is the name given to the angle formed (angle a)?

4. Use the paper folding you obtained in step 3 to measure the angle formed by two edges of your maths textbook as shown in Fig. 6.8 below.

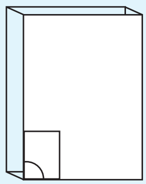


Fig. 6.8

What do you notice? Sketch the angle formed in your exercise book.

5. Obtain a wall clock. Set the time at 3.00 o'clock as in Fig. 6.9.

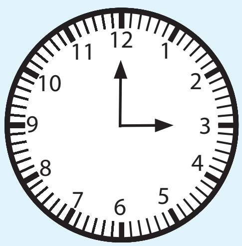


Fig. 6.9

Compare the type of angle formed between the minute hand and the hour hand and the angle formed on the paper folding you made in step 3.

What do you notice?

How do the two angles compare?

Discussion

The angle formed by the square corner of a paper folding is called a right angle. Similarly, when a clock is set at 3.00 o'clock, the type of angle formed between the minute hand and the hour hand is a right angle.

Learning points

A right angle is equivalent to 90° on a protractor as shown in Fig. 6.10 below.

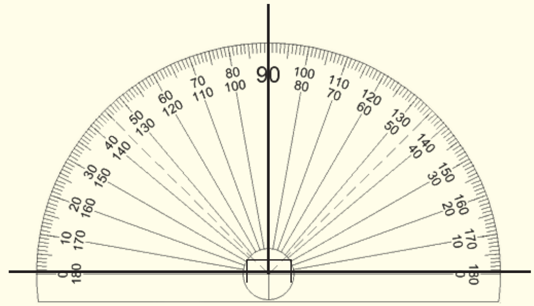


Fig. 6.10

The symbol for an angle of 90° or a right angle is ∟ e.g. ◺ right angled triangle.

(b) Acute angle

Activity 6.4

1. Set the clock at 2.00 o'clock.

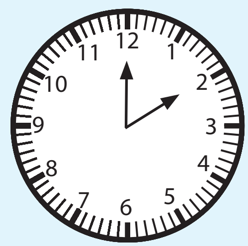


Fig. 6.11

What type of angle is formed by the space between the minute hand and the hour hand?

What is the name given to this type of angle?

2. Open your classroom door slightly as shown in Fig. 6.12

Fig. 6.12

What type of angle is formed at the

top between the frame and the top edge of the open door?

3. Compare this type of angle with the angle formed by setting the clock at 1.00 o'clock. Comment on the type of angle.
4. Compare the type of angle formed in steps 1 and 2 with the right angle. What do you notice?

Learning points

An angle which measures less than 90° is called an acute angle e.g 60°, 20°, 88°, 12° and so on. Fig. 6.13

Angle ABC < 90°

$\Rightarrow x^{o} < 90^{o}$

i.e. $0 < x^{o} < 90^{o}$

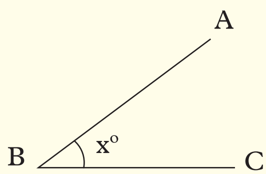


Fig. 6.13

(c) Straight angle

Activity 6.5

Carry out the following activities:

1. Set your clock at 6.00 o'clock as shown in Fig. 6.14.

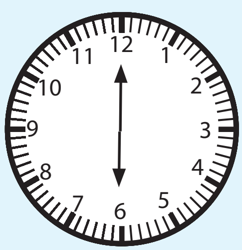


Fig. 6.14

(a) What type of angle is formed between the minute hand and the hour hand?

(b) What is the name of the type of angle formed by the space between the minute hand and the hour hand?

Learning point

The angle between the hour hand and the minute hand at 6.00 o'clock is 180°

An angle which measures 180° is called a straight angle. e.g.

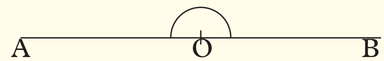


Fig. 6.15

The marked angle at 0 is equal to 180°
i.e. $\angle AOB = 180^{o}$

(d) Obtuse Angle

Activity 6.6

Carry out the following activities:

1. Set your clock at 4.00 o'clock as in Fig 6.16.

Fig. 6.16

What type of angle is formed between the minute hand and the hour hand?

What is the name given to such an angle? Sketch the angle formed in Fig. 6.16 above.

2. Obtain a rectangular paper. Fold it into two equal parts. Fold one of the corners of the paper slightly as shown in Fig 6.17 below.

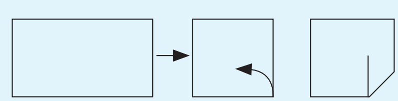


Fig. 6.17

What is the name of angle formed?

3. Sketch the angle and compare it with a right angle. What do you notice?

Discussion

How does the angle formed compare to a straight angle? Describe the angle formed in relation to the 90° angle and the straight angle.

Learning point

An angle greater than 90° but less than 180° is called an obtuse angle e.g 120°, 98°, 164°, 178°, 145° and so on. For example, in Fig. 6.18 angle x° is greater than 90° but less than 180°.

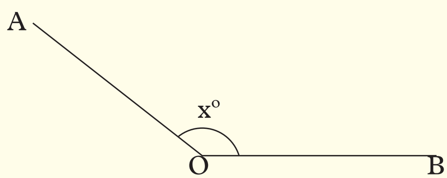


Fig. 6.18

$90° < x° < 180°$

(e) Reflex angle

Activity 6.7

Carry out the following activities.

1. Obtain a clockface or a wallclock. Set the time at 8.00 o'clock as shown in Fig. 6.19.

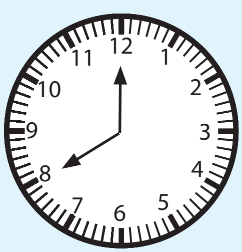


Fig. 6.19

What type of angle is formed between the minute hand and the hour hand in the clockwise direction?

Sketch the angle formed. What is the name given to such an angle?

Discussion

How does the angle compare with a straight angle?

Describe this angle in relation to the straight angle.

Learning point

An angle greater than 180° but less than 360° (full revolution) is called a reflex angle e.g. 270°, 200°, 168°, 344° and so on. For example,

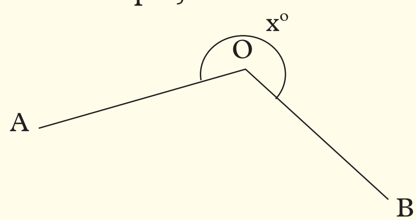


Fig. 6.20

Marked angle AOB = x°

$180° < x° < 360°$

Exercise 6.2

1. Indicate the type of angle marked with letters in each of the following (Fig. 6.21)

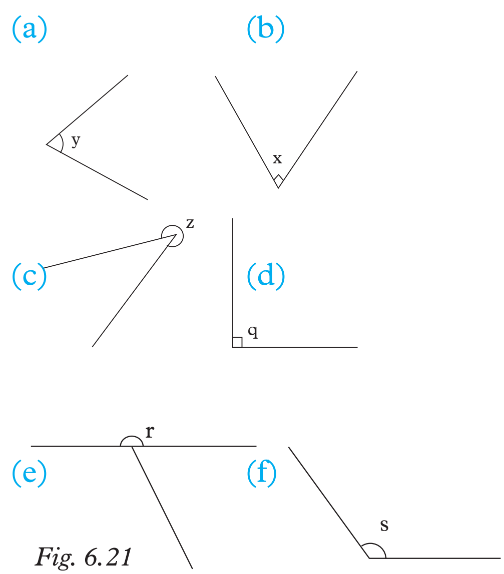


Fig. 6.21

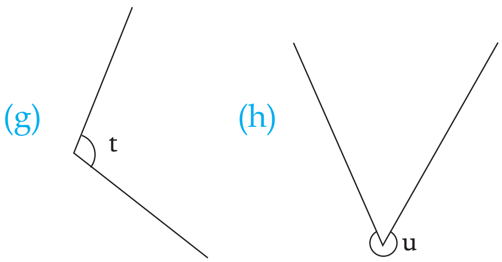


Fig. 6.21

2. State the type of angles marked with letters in each of the following (Fig. 6.22):

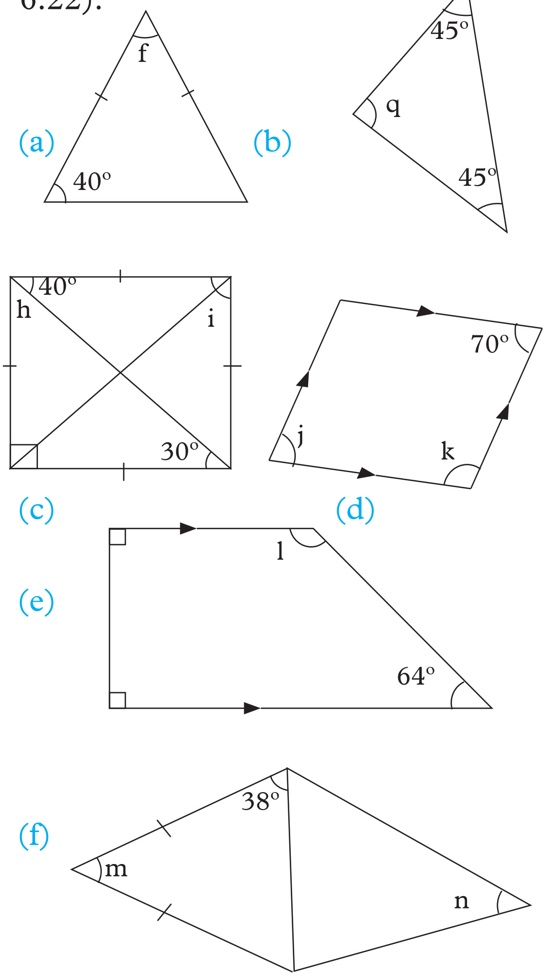


Fig. 6.22

3. Using a ruler and a protractor draw:
   (a) an acute angle of 35°
   (b) a right angle
   (c) an obtuse angle of 148°
   (d) a straight angle
   (e) a reflex angle of 295°

6.3.2 Angles on a straight line

Activity 6.8

Carry out the following activities.

1. Draw triangle ABC. Drop a perpendicular from point A to line BC at D. Cut out triangle ABC (Fig. 6.23).

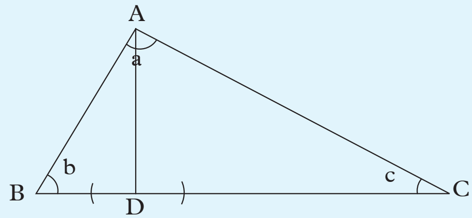


Fig. 6.23

2. Fold the paper cut out in such a way that vertex A lies exactly on D. Label the fold line EF as in Fig. 6.24 below.

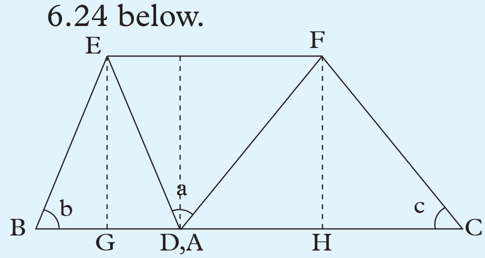


Fig. 6.24

What is the name of the figure 6.24?

3. Fold triangle BED along its vertical perpendicular such that that B lies on D. Fold triangle FDC (Fig. 6.24) along its perpendicular FH such that C lies on D as shown in Fig. 6.25 below.

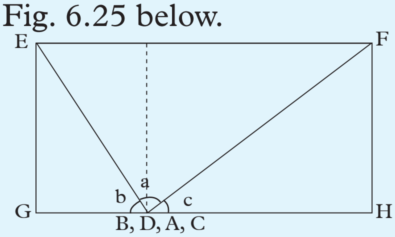


Fig. 6.25

(a) What figure did you obtain?

(b) Measure angles a, b and c. What is their sum?

(c) Comment on your answer.

Discussion

What is the sum of angles on a straight line?

What is the sum of the interior angles of a triangle?

Learning points

1. Angles on a straight line add up to 180°. To mark a straight angle, we use symbols such as:

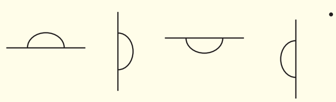


Fig. 6.26

2. Interior angles of a triangle add up to 180° i.e. a° + b° + c° = 180°. Fig. 6.27.

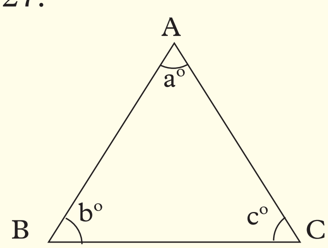
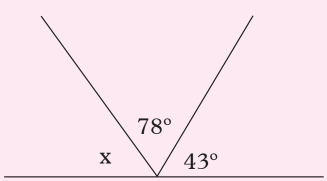


Fig. 6.27

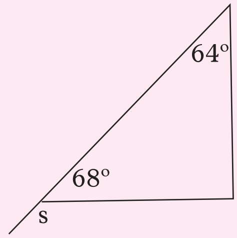
Example 6.2

Calculate the size of the angles marked with letters in each of the following figures.

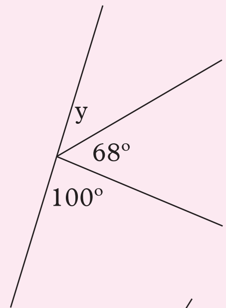
(a)



(b)



(c)



(d)

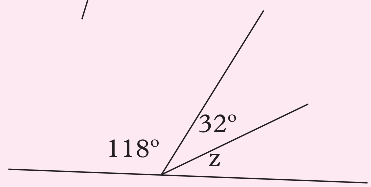


Fig. 6.28

Solution

(a) $x + 78° + 43° = 180°$ *(angles on a straight line)*

$x + 121° = 180°$

$x = 180° - 121°$

$x = 59°$

(b) $s + 68° = 180°$ *(angles on a straight line)*

$s = 180° - 68°$

$s = 112°$

(c) $100 + 68° + y = 180°$ *(angles on a straight line)*

$168° + y = 180°$

$y = 180° - 168°$

$y = 12°$

(d) $118° + 32° + z = 180°$ *(angles on a straight line)*

$150° + z = 180°$

$z = 180° - 150°$

$z = 30°$

Exercise 6.3

1. Calculate the size of each of the angles marked with letters in Fig. 6.29 below.

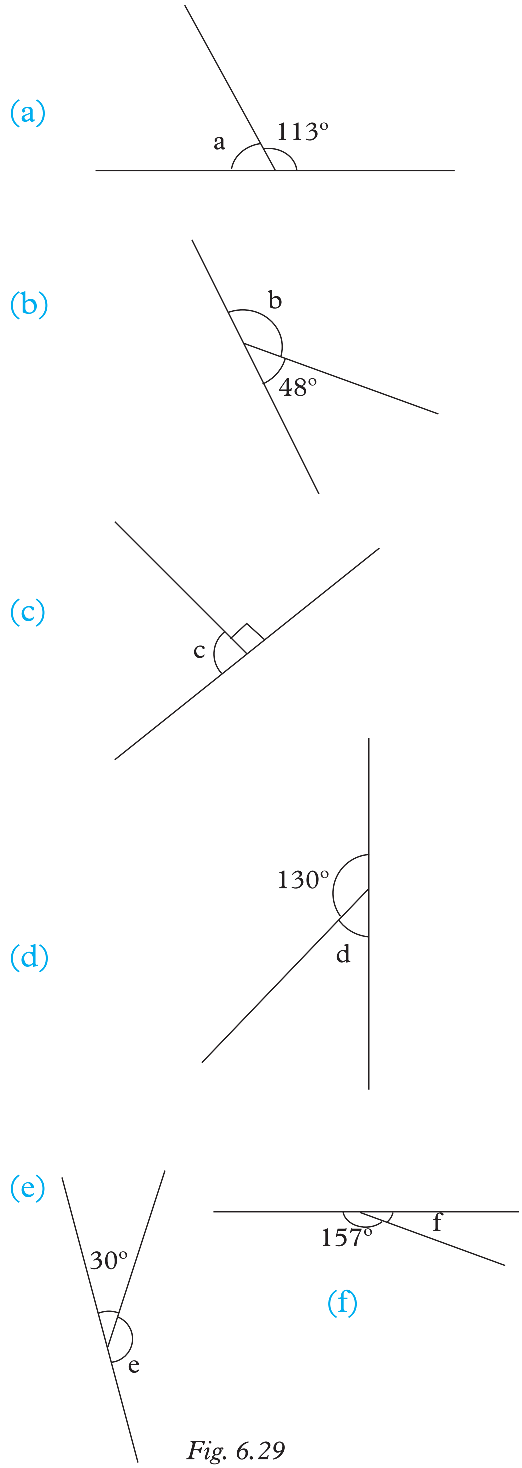


Fig. 6.29

2. In each of the following triangles, use the relationship between the angles of a triangle and the straight angle properties to fill in the gaps.

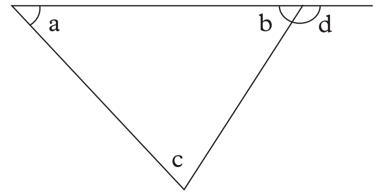


Fig. 6.30

(a) a + b + c =
b + d =

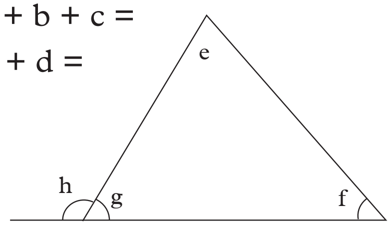


Fig. 6.31

(b) e + f + g = ___
h + ___ = 180°

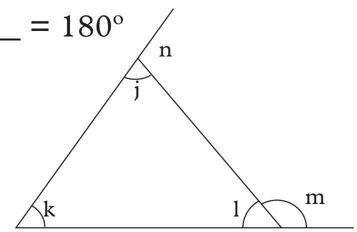


Fig. 6.32

(c) k + ___ + *l* = 180°
m + *l* = ___

6.3.3 Angles at a point

Activity 6.9

1. Draw and cut out two triangles of any measurement as shown in Fig. 6.33 (a) and (b).

(a)

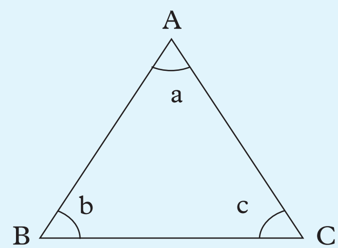


Fig. 6.33

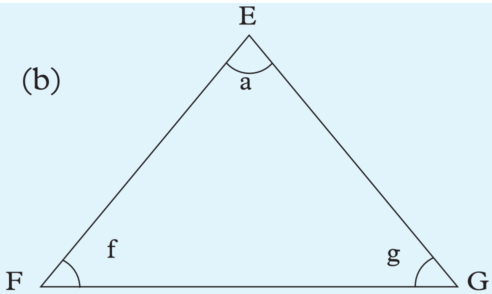


Fig. 6.33

2. On a plain piece of paper, mark a dot/point.

3. Cut off all the angles of triangles ABC and EFG Fig. 6.34. Fit all the six angles on the point y as marked in step 2 adjacent to each other without overlapping as shown in fig. 6.34.

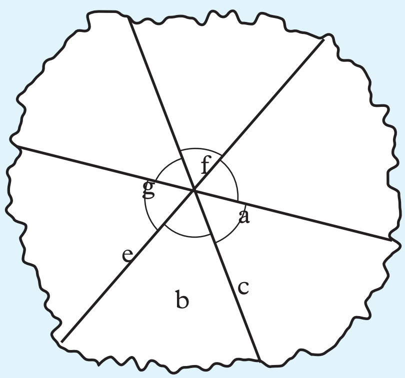


Fig. 6.34

(i) What do you notice? Discuss.

(ii) How do the six angles fit on one point?

(iii) What is the sum of all the six angles?

(iv) What is the sum of angles at a point?

Learning points

Angles a, b and c are interior angles of ΔABC i.e $a + b + c = 180°$.

Angles e, f and g are interior angles of ΔEFG i.e $e + f + g = 180°$.

Angles that share a common vertex and add upto 360° are called **angles at a point**.

The angles at the common point add upto $180° + 180° = 360°$.

Example 6.3

Find the angles marked with letters in each of the following.

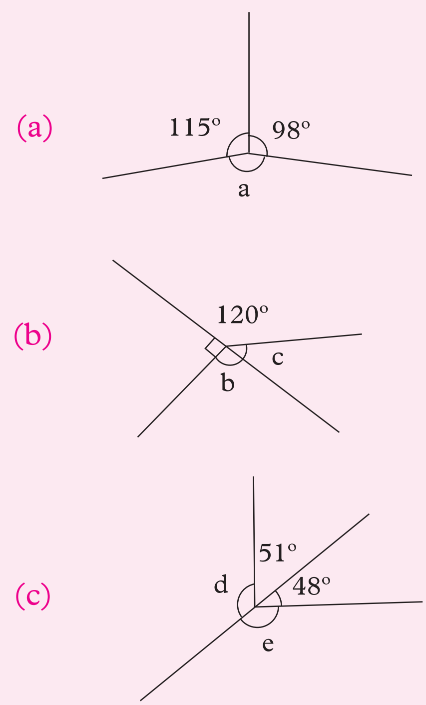


Fig. 6.35

Solution

(a) $115° + 98° + a = 360°$ *(angles at a point add up to 360°)*

$213° + a = 360°$

$a = 360° - 213° = 147°$

$\therefore\ a = 147°$

(b) $120° + 90° + b + c = 360°$

$210° + b + c = 360°$

But $b + 90° = 180°$ (angles on a straight line)

$\therefore\ b = 180° - 90° = 90°$

Thus, $210° + b + c = 360°$

$210° + 90° + c = 360°$

$\therefore\ c = 360° - 300°$

$c = 60°$

(c) $d + 51° = 180°$ (angles on a straight line)

$d = 180° - 51°$

$= 129°$

Similarly,

$e + 48° = 180°$ (angles on a straight line)

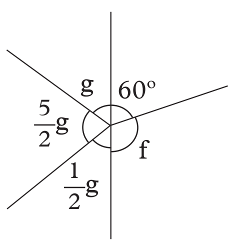
$\therefore\ e = 180° - 48°$

$= 132°$

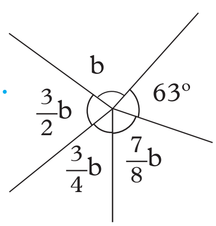
Exercise 6.4

Calculate the size of each of the following angles marked with letters in Fig 6.33.

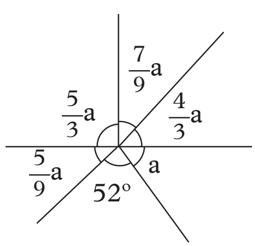
1.



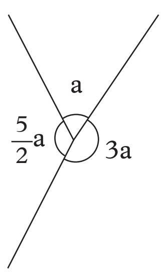
2.



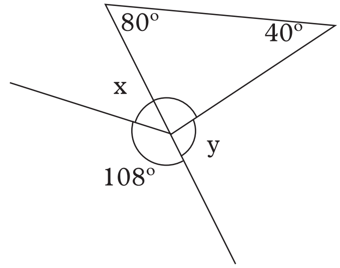
3.



4.



5.



6.

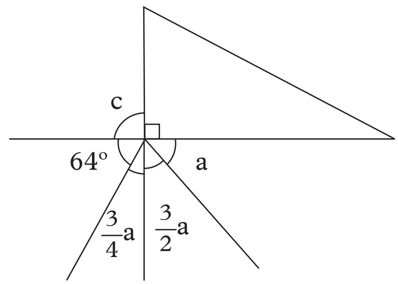


Fig. 6.36

6.3.4 Angles on a parallel line

Activity 6.10

1. Using the edges of a ruler, draw a pair of parallel lines as shown in Fig. 6.37. Put arrow heads at the centre of the line to show that the two lines are parallel.

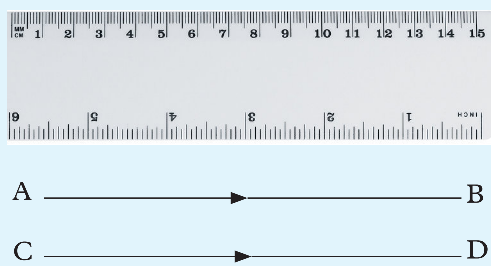


Fig. 6.37

2. Draw a straight line to cut lines AB and CD at points E and F respectively. Prolong this line (ST) on either sides of the parallel lines (Fig. 6.38).

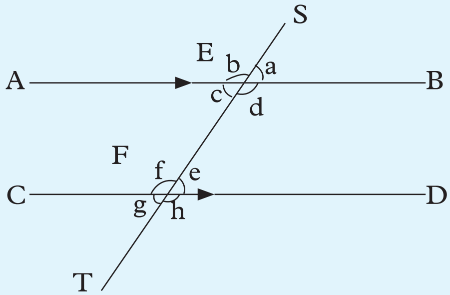


Fig. 6.38

What is the name of the line such as line ST above?

3. Using a paper, trace angles *a* and *b* as shown in Fig. 6.39.

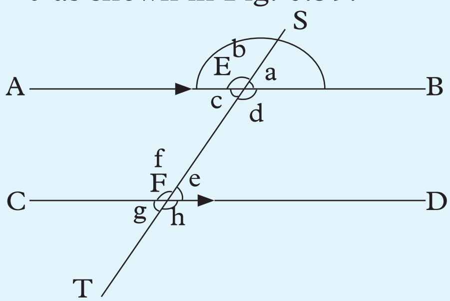


Fig. 6.39

4. Cut out the traced angles *a* and *b*.
5. Use the cut out angle to measure other angles on the diagram e.g. angles c, d, e, f, g and h.

Discussion

1. Compare the size of angle pairs *a* and *e*, *b* and *f*. What do you notice? What is the name of the angle pairs?
2. Compare the size of the angle pairs *d* and *f*, *e* and *c*. What do you notice? What is the name of the angle pairs?
3. Compare the size of the angle pairs *a* and *c*, *e* and *g*. What do you notice? What is the name of the angle pair?
4. What is the name of the angle pairs *a* and *b*, *c* and *d*, *e* and *f* and *h* and *g*?

Learning points

Consider Fig. 6.40 below.

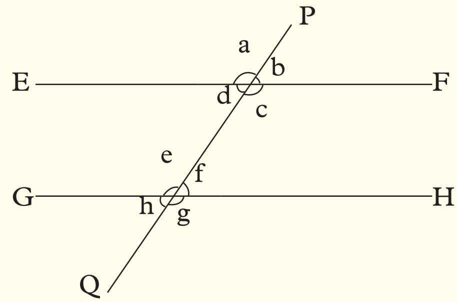


Fig. 6.40

1. The line PQ which cuts parallel lines EF and GH is called a transversal line. Transversal line is a straight line which cuts through two lines on the same plane at distinct points.
2. Angles that are on the same relative position when a transversal cuts through two points are called corresponding angles. When the two lines are parallel, the corresponding angles are equal. Examples of corresponding angles are as shown in Fig. 6.41.

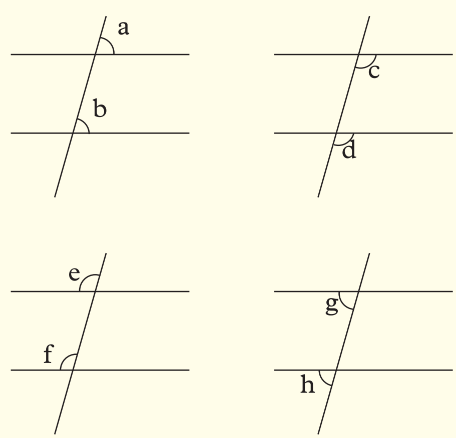


Fig. 6.41

Angles a and b, c and d, e and f, g and h are corresponding angles.

3. Pairs of interior angles on the opposite side of a transversal (one on each intersection) are called alternate angles. Examples of alternate angles are as shown in Fig. 6.42 below.

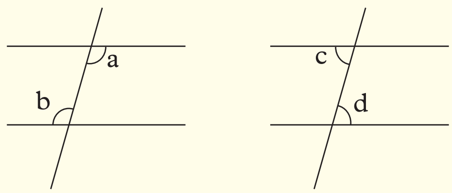


Fig. 6.42

Angles a and b, c and d are alternate angles. Alternate angles are equal.

4. Angles which are opposite each other where two straight lines intersect or cuts each other are called vertically opposite angles. Examples of vertically opposite angles are as shown in Fig. 6.43 below.

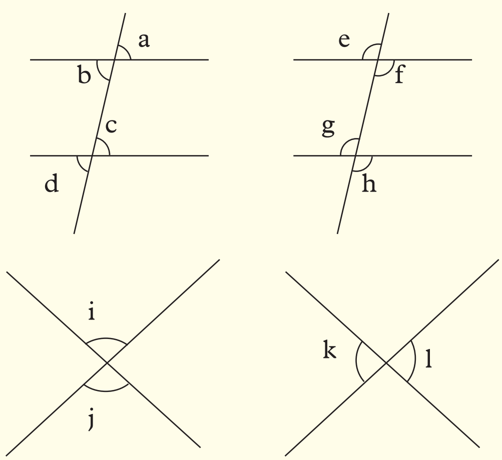


Fig. 6.43

Angles a and b, c and d, e and f, g and h, i and j, k and l are vertically opposite angles. Vertically opposite angles are equal.

5. Angles that add up to 180° are called supplementary angles e.g. (Fig. 6.44)

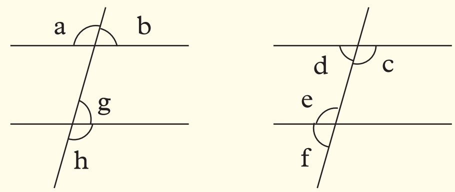


Fig. 6.44

Angles a and b, c and d, e and f and g and h are supplementary angles.

6. Pairs of interior angles on the same side of the transversal are called co-interior angles. Fig. 6.45.

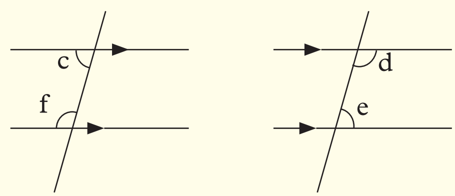


Fig. 6.45

c and f, d and e are pairs of co-interior angles.

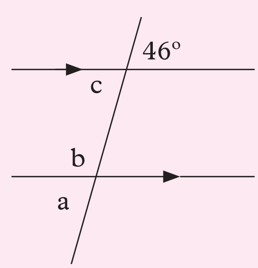
$c + f = 180°$; $d + e = 180°$

Therefore, co-interior angles are supplementary.

Example 6.4

Calculate the angles marked with letters in each of the following (Fig. 6.46)

(a)



(b)

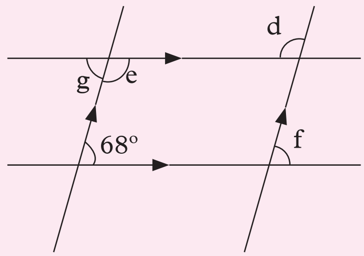


Fig. 6.46

Solution

(a) $c = 46°$ (vertically opposite angles)

$a = c = 46°$ (corresponding angles)

$b = 180° - a$ (angles on a straight line/ supplementary angles)

$b = 180° - 46°$

$b = 134°$

(b) $f = 68°$ (corresponding angles)

$g = 68°$ (alternate angles)

$e = 180° - g$ (supplementary angles)

$e = 180° - 68°$

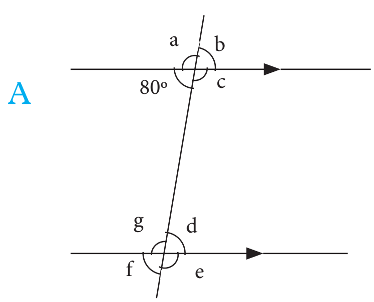
$e = 112°$

$d = e$ (alternate angles)

$d = 112°$

Exercise 6.5

1. Use Fig. 6.47 A to D to answer the questions that follow.



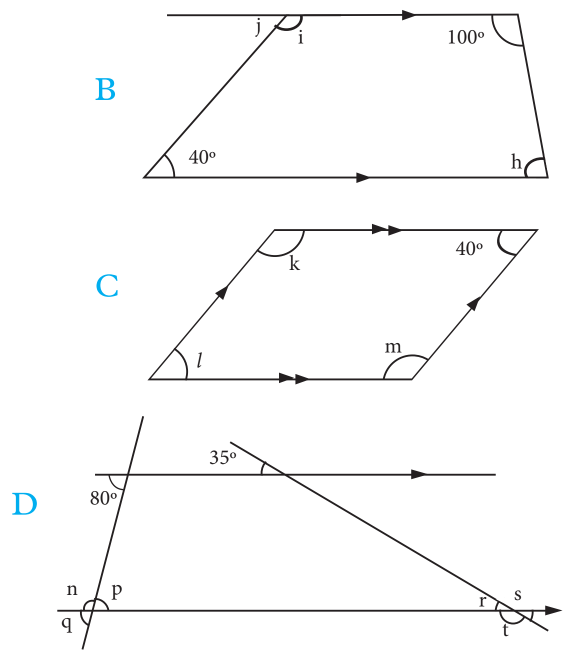


Fig. 6.47

(a) On each figure, write down all the pairs of:

 (i) corresponding angles.

 (ii) alternate angles.

 (iii) co-interior angles.

(b) On each figure write down the sizes of the angles marked with letters.

2. In Fig. 6.48, find pairs of parallel lines.

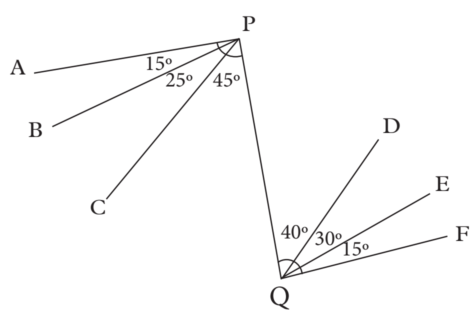
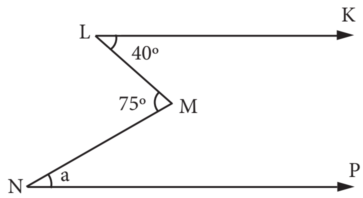


Fig. 6.48

3. Find the angles marked with letters in

Fig. 6.49 (a) and (b). [Hint: Copy the given figures and insert other parallel lines at M and C respectively].

(a)



(b)

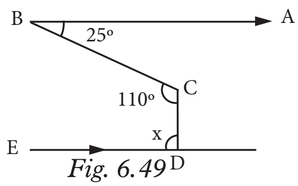


Fig. 6.49

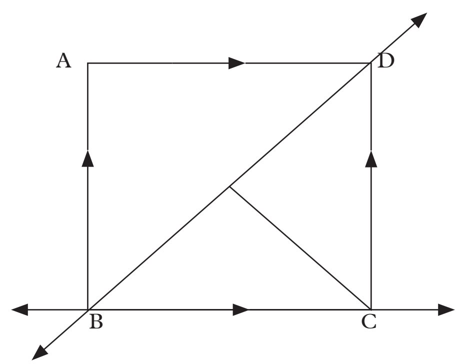
Summary

1. A ray is a straight line which starts from a fixed point and moves towards one direction without an end.
2. A line segment is a part of a straight line which has two fixed points, the starting and the ending points.
3. Acute angle is an angle measuring less than 90°.
4. A right angle is an angle equal to 90°.
5. A straight angle is angle equal to 180°.
6. An obtuse angle is angle measuring between 90° and 180°.
7. A reflex angle is an angle greater than 180° but less than 360°
8. Angles at a point are angles which share a common vertex and add upto 360°
9. A transversal line is a straight line which cuts through two lines in the same plane at two distinct points.
10. Corresponding angles are angles that occupy the same relative position when a transversal cuts through two straight lines.
11. Alternate angles are pairs of interior angles on the opposite side of a transversal (one on each intersection point).
12. Supplementary angles a pair of angles on a straight line that add up to 180°.
13. Co-interior angles are pairs of angles on the same side of a transversal. Such angles are supplementary.

Unit Test 6

1. Use Fig 6.50 to identify the following:
   (a) Rays
   (b) Line segments
   (c) Straight line

Fig. 6.50



2. Measure the angle indicated by letters in each of the following Fig 6.51. What type of angle is it?
3. Calculate the size of the angles

(a)

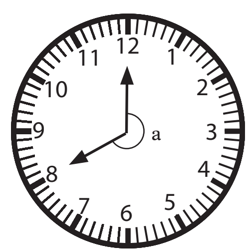
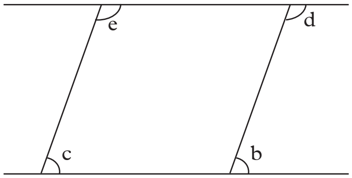
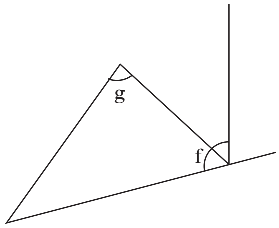


Fig. 6.51

(b)



(c)



(d)

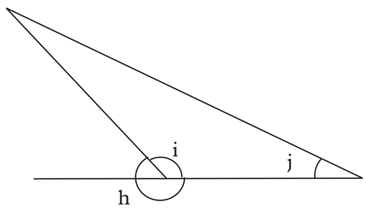


Fig. 6.51

marked with letters in the following figures. Fig. 6.52

(a)

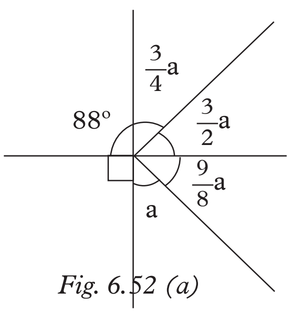
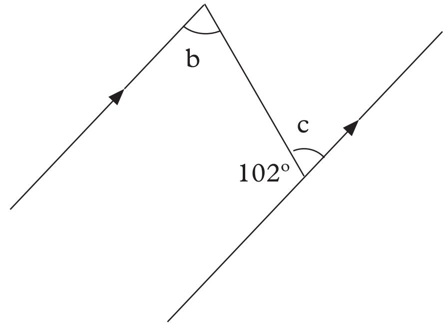
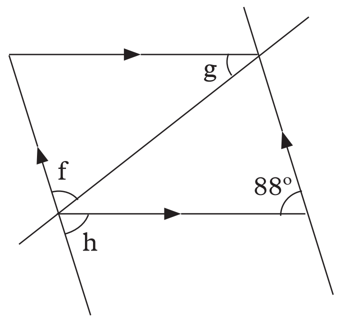


Fig. 6.52 (a)

(b)



(c)



(d)

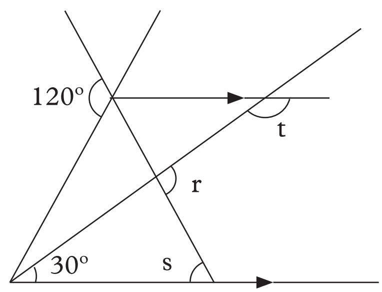


Fig. 6.52 (b) to (d)

7 SOLIDS

Key unit competence

By the end of this unit, I should be able to select and use formulae to find the surface area and volume of solids.

Unit outline

- Properties of solids.
- Surface area and volume of a prism, pyramid, cylinder, cone and sphere.
- Formulae for surface area.

In Primary 5 and 6, you were introduced to designs and construction of nets of cuboids and other prisms. In this unit, we are going to further our skills on determination of areas and volumes.

7.1 Properties of solids

Activity 7.1

1. Use a Mathematic dictionary or the internet to research the word polyhedra (singular polyhedron).
2. List the properties of polyhedra.
3. Name and describe some examples of polyhedra.
4. Discuss your findings with your partner.
5. Make a summary of your findings and then share it with the rest of the class.

From Activity 7.1, you should have learnt that polyhedron is any 3-Dimension figure that has a flat surface for each face. Prisms and pyramids are examples of polyhedral. The majority of packages you see in everyday life are examples of **polyhedra**.

These and other 3 – Dimension figures are called **solids**.

Activity 7.2

- Use your dictionary to find the meaning of the words: face, edge, vertex.
- Observe the closed cartons provided by your teacher. Identify the parts of the cartons and name them

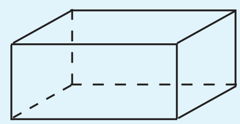


Fig. 7.1

- Use Fig 7.2 (a – j) below to investigate certain facts about some common solids as follows.

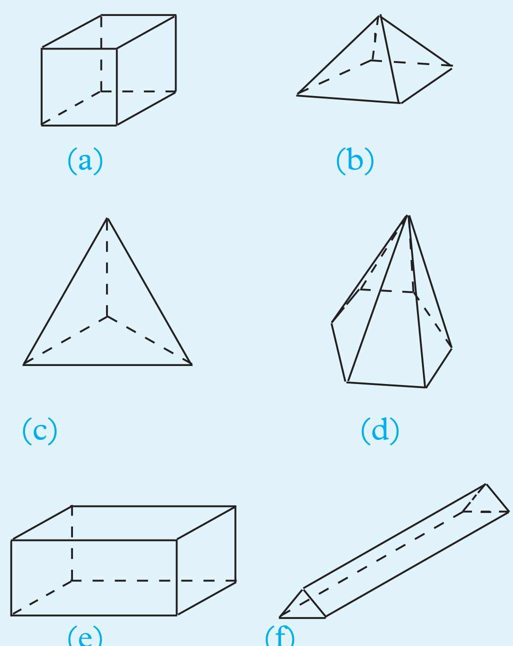


Fig. 7.2

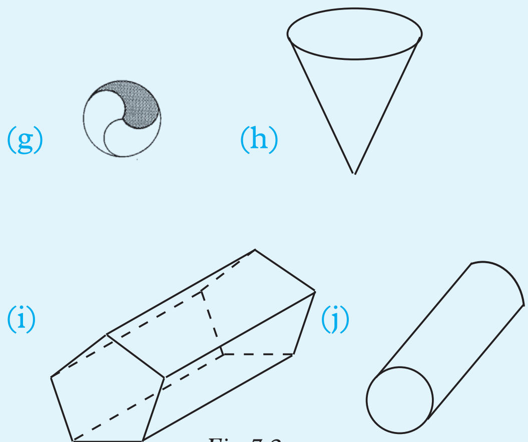


Fig. 7.2

(a) State the number of:

  (i) faces
  (ii) edges
  (iii) vertices that each figure has.

(b) Describe each shape and suggest a name for it.

(c) Which of the solids do not represent a polyhedron?

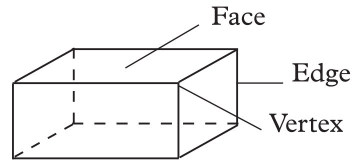
Discussion

1. Which of these solids are polyhedra?
2. List down all the prisms in Fig. 7.2.
3. List all the pyramids in the figure.
4. Which solids in Fig. 7.2 are hexahedra?
5. Are there any pentahedra? If so which are they?

From activity 7.2, you should have come up with observations similar to the ones listed below:

1. Cuboid

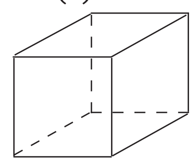
It is a solid bounded by three pairs of identical faces which are all rectangles. Sometimes, a cuboid is referred to as a 'rectangular box' or a 'rectangular block, Fig. 7.2 (e).'



2. Cube

It is a solid bounded by six identical faces which are all squares and equal.

A cube is a special type of cuboid, and it may sometimes be called a 'square box' Fig. 7.2 (a).

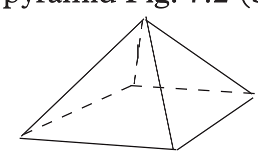


3. Pyramid

It is a solid figure with triangular slanting faces which meet at one point, called an apex or vertex, above a polygonal base.

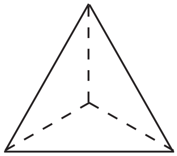
A pyramid is always named after the shape of its base. Thus, there are triangular-based pyramids (tetrahedra), square and rectangular-based pyramids, pentagonal-based pyramids, and so on.

If the pyramid has its vertex vertically above the centre of the base, it is called a right pyramid Fig. 7.2 (b) and (d).



4. Tetrahedron

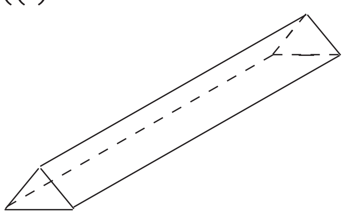
This is a solid figure with four faces which are all triangles. Therefore, it is a pyramid with a triangular base. Fig 7.2 (c).



5. Prism

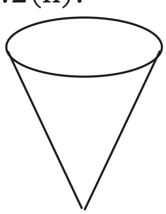
It is a solid figure with identical and parallel end faces. If we cut a prism through a plane **parallel** to an end face, the cut surface will be exactly the same as the end face.

A prism is named according to the shape of the end face. If the end face is a triangle, it is a triangular prism; if rectangular, it is a rectangular prism, and so on. The end face of a prism is called a cross-section (Fig 7.2 ((f).



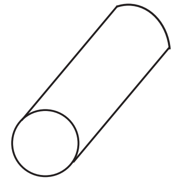
6. Cone

It is a solid figure which narrows to a point (a vertex) from a circular flat base. It is a pyramid with a circular base Fig 7.2(h).



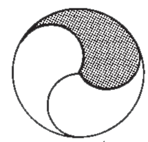
7. Cylinder

This is a solid figure with a uniform thickness and circular ends. It comprises of two identical circular end faces and a curved surface. A cylinder is a prism with a circular cross-section Fig 7.2(j).



8. Sphere

This is a solid figure that is entirely round, such as a football. Half of a sphere is called a **hemisphere.** A sphere has no edges and no vertices but has one curved surfaces Fig 7.2 (g).



Note

1. Plane shapes with many sides are called polygons (poly = many; gon = sides). Similarly, solids with many faces are called polyhedra (singular: polyhedron)
2. Just as polygons may be named according to the number of sides, polyhedra may also be named according to the number of faces that they have Table 7.1 shows some examples.

Number of faces	Name
4	Tetrahedron
5	Pentahedron
6	Hexahedron
7	Heptahedron
8	Octahedron
9	Nonahedron
10	Decahedron
12	Duodecahedron

Table 7.1

3. Some polyhedra have all their faces **equal**. Such polyhedra are said to be **regular**. An example of a regular polyhedron is a cube. A tetrahedron with all its faces being equilateral triangles is also a regular polyhedron.

Point of interest

There are five regular polyhedra also known as platonic solids. All their faces are identical. These are:

1. The cube (6 faces).
2. The regular tetrahedron (4 faces).
3. The regular octahedron (8 faces).
4. The regular dodecahedron (12 faces).
5. The regular icosahedron (20 faces).

Edges and vertices

Activity 7.3

Carry out the following activities:

1. Draw a cuboid, a cube and a square based pyramid.
2. From your drawing, how many edges and vertices are there in:
   (i) a cuboid? (ii) a cube?
   (iii) a square-based pyramid?

Now copy and complete Table 7.2. Draw the other solids to help you in filling in the table.

Solid	No. of faces(f)	No. of edges(e)	No. of vertices(v)	$f+v$
Cube	6	12	8	14
Cuboid	6	12	8	14
Tetrahedron	–	–	–	–
Square-based pyramid	5	8	5	10
Hexagon-based pyramid	–	–	–	–
Triangular prism	–	–	–	–
Pentagonal prism	–	–	–	–

Table 7.2

1. For each of the polyhedra, subtract the number of edges (e) from the sum of the number of faces (f) and the number of vertices (v). What do you notice?
2. Does your answer to question 1 above agree with those of other members of your class?
3. What is the least number of faces that a polyhedron can have?
4. What do you notice about $f+v$ and e?
5. Compare, $f+v$ and $e+2$? What do you notice?
6. What is the name of the relation? Is it true for all polyhedra?

For questions 1-6, which of the statements are true and which ones are false? For those that are false, explain why?

1. Every parallelogram is a polyhedron.
2. The edges of a polyhedron are line segments.
3. Each edge of a polyhedron is in two faces of a polyhedron.
4. A ball is an example of a polyhedron.

5. Every polyhedron has more vertices than edges.
6. Every polyhedron has more vertices than faces.

From the activity, you have learnt that;

The line at which two faces (f) of a polyhedron meet is called an edge (e).

Edges meet at a point which is referred to as a vertex (plural: vertices) (v).

Note: f + v = e + 2

The relation f + v = e + 2 is named after Leonhard Euler (1707–1783). Euler was a gifted Swiss Mathematician, and is reputed to have produced the highest number of works in Mathematics and History. He is rated as the first modern Mathematical universalist.

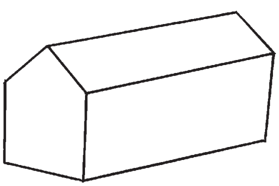
The relation is called Euler's relation.

Exercise 7.1

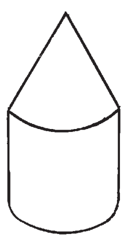
Working in pairs, discuss this exercise.

Identify all the basic solids that compose the shapes in Fig. 7.3.

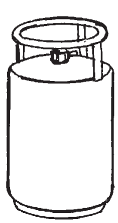
1.



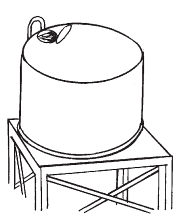
2.



3.



4.



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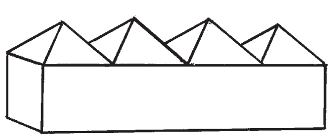

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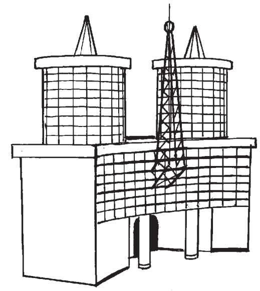

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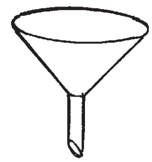


Fig. 7.3

7.2 Surface area of solids

You are already familiar with the use of accurate nets to determine area of cuboids.

(a) Surface area of a cuboid

Activity 7.4

- Draw the cuboid in Fig 7.4 below.

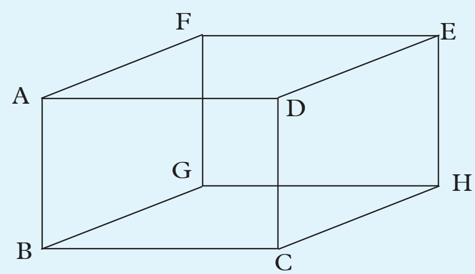


Fig 7.4

- Draw a well labeled net of the cuboid above.
- How many faces does the cuboid have from the net?
- How would you calculate the area of the cuboid using the net?
- Use the net to calculate its area in terms of l, w, and h.

From Activity 7.5, you have learnt that: the surface area of a cuboid of length l, width w and height h (Fig. 7.5), is given by $2lw + 2lh + 2wh$ *e.g.*

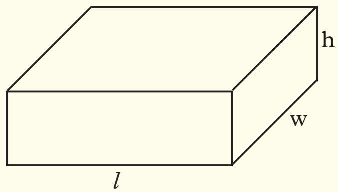


Fig. 7.5

The net of the cuboid will be as follows:

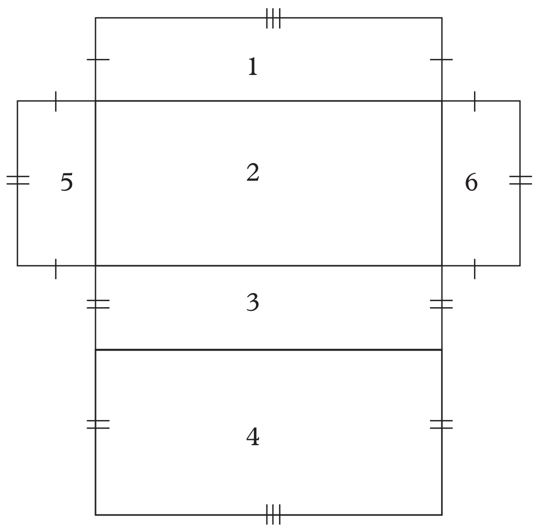


Fig 7.6

A cuboid has 6 faces.

The sum of the areas of the six faces gives the surface area of a cuboid.

Example 7.1

The net of a cuboid consists of a series of rectangles. How many rectangles are there? What is the surface area of the cuboid if it measures 6 cm by 3 cm by 2 cm?

Solution

Fig. 7.7 shows the sketch of the cuboid described in this example.

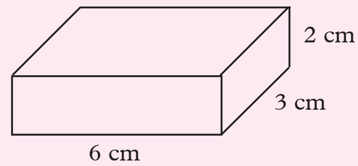


Fig. 7.7

Fig 7.8 shows a possible net of a cuboid which measures 6 cm by 3 cm by 2 cm.

It is composed of three pairs of rectangles i.e. 6 rectangles.

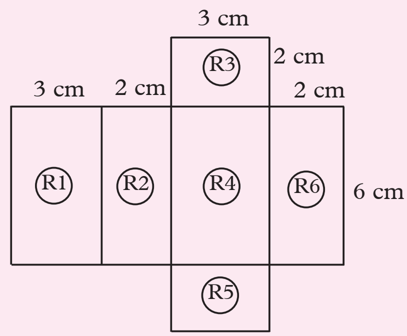


Fig. 7.8

The surface area of the cuboid = sum of the areas of all the rectangles that comprise the net.

Area of rectangle R1 $= 6\text{ cm} \times 3\text{ cm}$
$= 18\text{ cm}^2$

Area of rectangle R2 $= 6\text{ cm} \times 2\text{ cm}$
$= 12\text{ cm}^2$

Area of rectangle R3 $= 3\text{ cm} \times 2\text{ cm}$
$= 6\text{ cm}^2$

Area of rectangle R4 $= 6\text{ cm} \times 3\text{ cm}$
$= 18\text{ cm}^2$

Area of rectangle R5 $= 3\text{ cm} \times 2\text{ cm}$
$= 6\text{ cm}^2$

Area of rectangle R6 $= 6\text{ cm} \times 2\text{ cm}$
$= 12\text{ cm}^2$

Total surface area $= 18cm^2 + 12cm^2 + 6cm^2 + 18cm^2 + 6cm^2 + 12cm^2 = 72\ cm^2$

Surface area of the cuboid $= 72\ cm^2$

(b) Surface area of a cube

Activity 7.5

Draw the net of a cube of sides l unit.

Use your net to calculate the total surface area of the cube.

Compare your answer with those of other groups in your class.

Discussion

1. Describe a cube to your friend.
2. Compare the surface area of a cuboid to that of a cube. What do you notice?
3. Is there a shorter formulae to calculate the surface area of a cube?

Learning point

By definition, a cube has six faces that are identical. The faces are all squares. Therefore, total surface area = area of one face x 6. If the length of each face is l units,

Area $= l \times l \times 6$

$= 6l^2$ square units

Example 7.2

Find the surface area of a cube of sides 5 cm.

Solution

Fig. 7.9 is the sketch of the cube.

The cube has 6 identical faces each of area $5\ cm \times 5\ cm = 25\ cm^2$

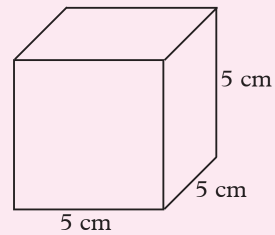


Fig. 7.9

$SA = 6l^2$

$= 6 \times 5\ cm \times 5\ cm$

$= 150\ cm^2$

Example 7.3

A cube has a total surface area of 96 square cm. Find the length of the side of the cube.

Solution

The length, the breadth and the height of a cube are all equal.

Let $l = b = h = x$ *units*

Fig 7.10 shows the cube

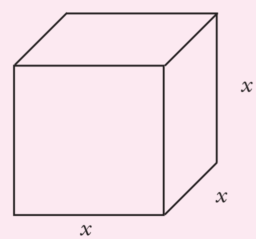


Fig 7.10

Number of faces of a cube $= 6$

Area of each face $= x \times x$

$\therefore$ *Total surface area*

$= x \times x \times 6$

$= 6x^2$

$\therefore\ 6x^2 = 96\ cm^2$

$x^2 = \frac{96}{6}$ *(divide both sides by 6)*

$x^2 = 16\ cm^2$

$\therefore\ x = 4\ cm$

$\therefore$ *The length of the side of the cube is 4 cm.*

(c) Surface area of a cylinder

Activity 7.6

- Obtain a rectangular piece of paper. Label it A, B, C, D.
- Find the area of the rectangle ABCD. Without folding, join edges AD onto CB using glue as shown in Fig. 7.11.

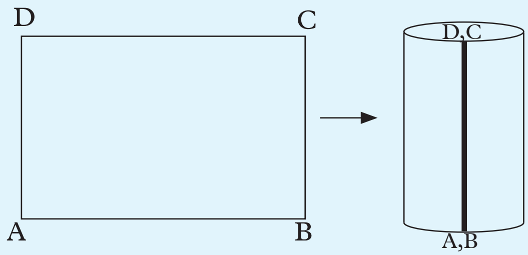


Fig. 7.11

- What is the name of the distance round DC and AB?
- Cut a paper that fits round the circumference of the cylinder you made.
- Trim any excess paper with a pair of scissors so that the circular cut out does not overlap.
- Calculate the area of the paper cut out. How did you find radius of these end faces?
- What is the total surface area of the can?
- Obtain a can and construct its net.
- Make appropriate measurements and record them on the net.
- Find the surface area of the can.

Discussion

Suggest a relationship that can help calculate surface area of a can.

To find the area of the end faces, you needed radius. How did you obtain this radius?

How did you find the area of the curved surface?

How does the area of the curved surface compare with the area of the rectangle?

Discuss your findings with your partner.

Learning Point

The net of a cylinder is made up of one rectangle whose length is equal to the height of the cylinder and the width is equal to the circumference of its circular cross-section, and the two ends of a closed cylinder faces. The surface area of a closed cylinder is composed of the sum of the curved surface and the two end faces.

Open cylinder = $\pi r^2 + 2\pi rh$

Closed cylinder = $2\pi r^2 + 2\pi rh$

Example 7.4

A closed cylinder whose height is 18 cm has a radius of 3.5 cm. Draw the net of the cylinder and use it to find the total surface area of the cylinder.

Solution

Fig. 7.12(a) shows the diagram of the cylinder and Fig. 7.12 (b) shows its net.

(a)

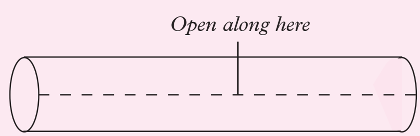


Fig. 7.12

(b)

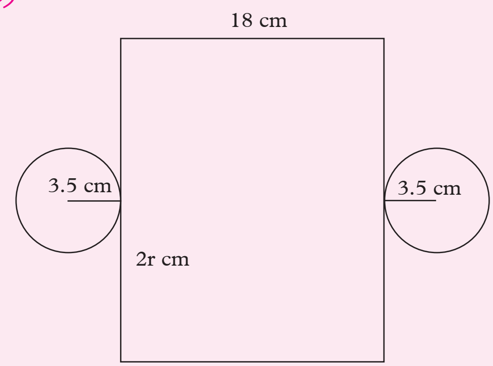


Fig. 7.12

Circumference of one the end face (circle)

$= 2\pi r$

$= 2 \times \frac{22}{7} \times \frac{7}{2}$ cm

$= 22$ cm

The net consists of two end circles, and a rectangle measuring 22 cm by 18 cm.

Total surface area $= 2 \times \pi r^2 + (2\pi r)h$

$= 2 \times \pi \times (3.5 \text{ cm})^2 + 22 \text{ cm} \times 18 \text{ cm}$

$= (76.98 + 396) \text{ cm}^2$

$= 472.98 \text{ cm}^2$

Example 7.5

A very thin sheet of metal is used to make a cylinder of radius 5 cm and height 14 cm. Using $\pi = 3.142$*, find the total area of the sheet that is needed to make:*

(a) *a closed cylinder*

(b) *a cylinder that is open on one end.*

Solution

(a) *Radius of the circular face of the cylinder*

$= 5$ cm

Area of a circular face

$= \pi r^2 = (3.142 \times 5^2) \text{ cm}^2$

Area of the two circular end faces

$= 2 \times \pi r^2 = (2 \times 3.142 \times 5 \times 5) \text{ cm}^2$

Recall that when a cylinder is opened up to form its net, the curved surface becomes a rectangle of length $2\pi r$ *(i.e. the circumference of the cylinder) and width h (the height of the cylinder).*

Thus, area of curved surface $= 2\pi r \times h$

$= (2 \times 3.142 \times 5 \times 14) \text{ cm}^2$

Now, total surface area of the metal sheet

$= ((2 \times 3.142 \times 5 \times 5) + (2 \times 3.142 \times 5 \times 14)) \text{ cm}^2$

$= 2 \times 3.142 \times 5(5 + 14) \text{ cm}^2$

$= 596.98 \text{ cm}^2$

(b) *Surface area of open cylinder*

$= \pi r^2 + 2\pi rh$

$= (3.142 \times 5 \times 5) + (2 \times 3.142 \times 5 \times 14)$

$= 3.142 \times 5(5 + 2 \times 14) \text{ cm}^2$

$= 3.142 \times 5 \times 33$

$= 518.43 \text{ cm}^2$

Note: From the working in examples above, we are reminded that:

Total surface area of a closed cylinder $= 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

(d) Surface area of a Prism

Activity 7.7

1. Tell your partner what a prism is.
2. Fig 7.13 shows a sketch of a tent which has a sewn-in ground sheet.

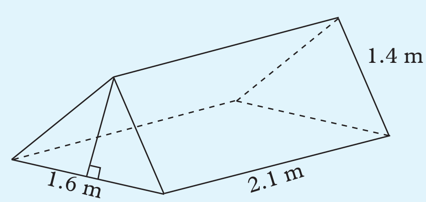


Fig 7.13

Find the amount of material used to make the tent. An extra 0.5 m^2 material is added for the same.

Discussion

1. How many faces does the prism has?
2. How many shapes can the cross-section of a prism take?
3. Construct the net of the prism above.

From Activity 7.7, you learnt that, a prism is a solid with a uniform cross-section. The cross-section may take the shape of any polygon such as triangle, square, rectangle, hexagon etc.

The faces of a prism other than the cross-section (or the bases) are called **lateral** faces of the prism. The edges joining pairs of lateral faces are called **lateral** edges.

To determine the surface area of a prism:

- Identify all the faces that compose the prism.
- Construct an appropriate net.
- Calculate the total surface area of the net.

Self test on properties of a prism

Which of the following statements are true and which ones are false? For those that are false, explain why.

1. Each prism has two bases.
2. All the lateral faces of a prism are parallel.
3. The planes determined by the bases of a prism are parallel.
4. Each triangular prism has three lateral faces and three lateral edges.
5. Each base of a triangular prism is a triangle.
6. Each vertex of a prism is in one of the two bases.
7. The bases of a prism are identical.
8. The lateral edges of a prism are equal and parallel.
9. Each base of a quadrangular prism is a parallelogram.
10. The lateral faces of a prism are perpendicular to the bases.

Answer the following questions

1. What is the least number of faces that a prism can have?
2. What is the total number of vertices of:
   (a) a triangular prism?
   (b) quadrangular prims?
   (c) hexagonal prism?
3. Generalize the total number of faces given the bases has n sides.

Explain how to find the total surface area of the prism in terms of n.

Learning points

- A prism has two parallel bases.
- A prism has as many lateral faces as the number of sides of its base.
- The bases of a prism are identical.
- The lateral edges of a prism are equal and parallel.
- The lateral faces of a prism are perpendicular to the bases.
- If the base of a prism has n sides, then the prism has n lateral faces and since any prism has two bases, the total number of faces is $2 + n$ lateral faces.

  $\therefore$ total number of faces = $(n + 2)$ faces
- Total surface area of a prism = 2 (area of one base) + areas of the n lateral faces.

Example 7.6

Construct the net of the prism in Fig. 7.14. Use it to find the surface area of the prism.

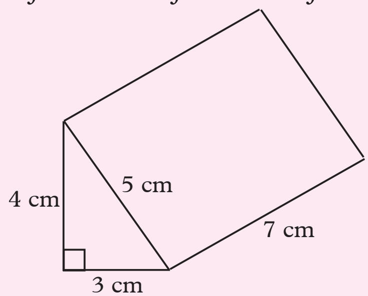


Fig. 7.14

Solution

Fig. 7.15 represent the net of the prism.

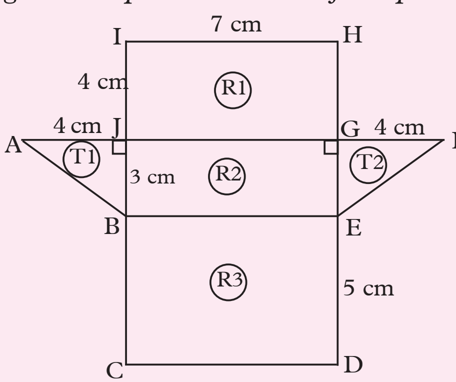


Fig. 7.15

The area of this prism is made up of 3 rectangles and 2 triangles.

Area of rectangle R1 $= 7\text{ cm} \times 4\text{ cm}$

$= 28\text{ cm}^2$

Area of rectangle R2 $= 7\text{ cm} \times 3\text{ cm}$

$= 21\text{ cm}^2$

Area of rectangle R3 $= 7\text{ cm} \times 5\text{ cm}$

$= 35\text{ cm}^2$

Area of triangle T1 $= \frac{1}{2} \times 3\text{ cm} \times 4\text{ cm}$

$= 6\text{ cm}^2$

Area of triangle T2 $= \frac{1}{2} \times 3\text{ cm} \times 4\text{ cm}$

$= 6\text{ cm}^2$

Total surface area of the prism $= 96\text{ cm}^2$

Note:

It is also possible to calculate the total surface area of a prism or any other solid without necessarily constructing the net.

But, we must be familiar with the shape and the size of all the faces. Also, we must be able to draw a sketch of the net in its true shape.

Exercise 7.2

1. Use the appropriate nets in each case to calculate the total surface area of the solids in Fig. 7.16. (All measurements are in cm.)

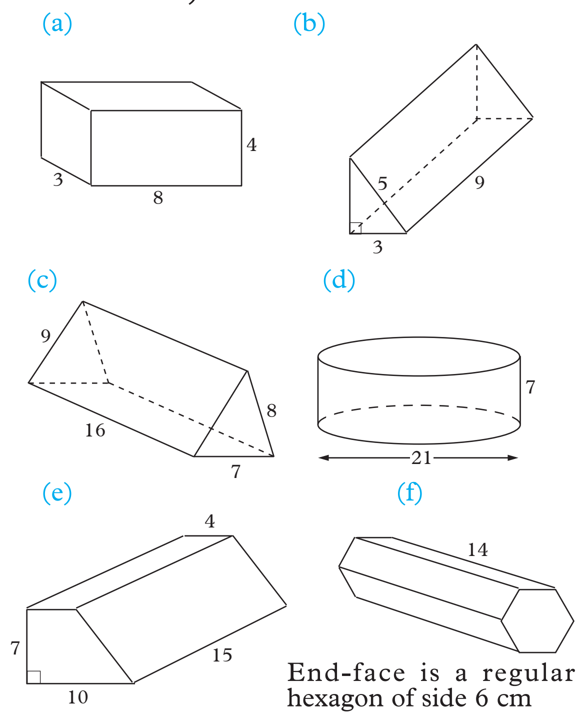


Fig. 7.16

2. Calculate the total surface area of a solid cylinder whose radius and height are 9 cm and 12 cm, respectively, leaving π in your answer.
3. A paper label just covers the curved surface of a cylindrical can of diameter 14 cm and height 10.5 cm. Calculate the area of the paper label.

4. The surface area of a cuboid is 586 cm^2. Given that its length and height are 12 cm and 7 cm respectively, find its breadth.
5. A closed cylinder whose height is 18 cm has a radius 3.5 cm. Calculate the total surface area of the cylinder.

(e) Surface area of a pyramid

Activity 7.8

Imagine you were hired to design a wrap for sweets in the shape of a tetrahedron.

1. Sketch a tetrahedron. Put possible dimensions on the sketch. Construct a net of the package that would be required.
2. Calculate the amount of material that would be needed to make the wrap.
3. Suggest a name for the sweet.
4. On which face would you put the name of the sweet on the package so that it attracts the eye of the customer?

Economics: Good attractive packaging tends to attract the customers into looking and finally buying a given commodity.

5. On your net, write the name of the sweet on the face you chose in (4) above.
6. Now fold the net so that the package is shown the way you want it to appear.

Activity 7.9

You are given a pyramid with a square base of sides 5 cm. The slant height of the pyramid is 6 cm. Construct an accurate net of the pyramid. Fig. 7.17.

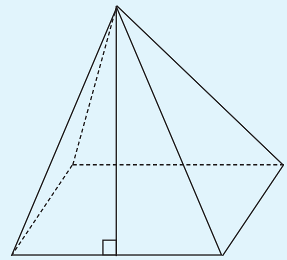


Fig. 7.17

- Record the measurements on the net.
- How many identical faces does the pyramid have?
- Calculate the area of one of the triangular faces.
- Find the total area of the net.
- What is the total surface area of the pyramid?

Learning points

- The altitude of a **slant** face of a pyramid is called a slant height of the pyramid. It is perpendicular to an edge of the base.
- Surface area of a pyramid = total area of the slant faces + the area of base.
- If the vertex of a pyramid is vertically above the centre of the base, the pyramid is called a **right pyramid**.
- The net of a pyramid is composed of 4 identical slant faces. Total surface area = 4(area of one slant face) + area of the base e.g.

  from Activity 7.9.

  Area of one slant face

  $$= (\tfrac{1}{2} \times 6 \times 5)\ cm^2$$

  Area of base = $(5 \times 5)\ cm^2$.

- Area of all slant faces

$= ((\frac{1}{2} \times 6 \times 5) \times 4)\ cm^2$

$= (4(\frac{1}{2} \times 6 \times 5) + (5 \times 5))\ cm^2$

$= 85\ cm^2$

Example 7.7

Fig 7.18 represents a tetrahedron whose dimensions are given in cm. Use the sketch to construct the net of the solid. Clearly label the net and use it to calculate the surface area of the side.

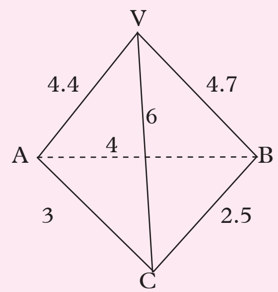


Fig. 7.18

Solution

Fig 7.19 shows the required net.

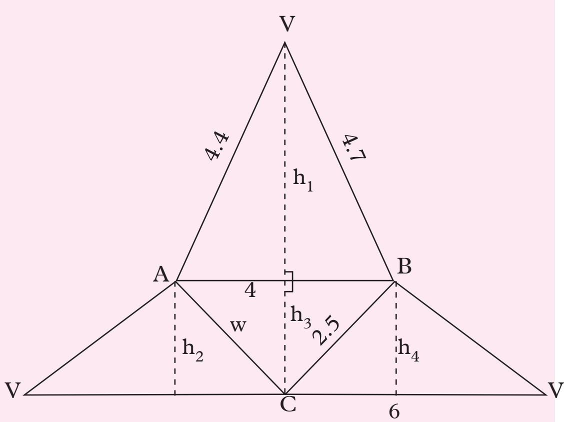


Fig. 7.19

Area of a pyramid = area $\Delta\ ABC$ + $\Delta\ ACV + \Delta\ ABV + \Delta BCV$

The altitude of the Δs are shown by broken lines labelled h_1 to h_4.

By measurement, $h_1 = 4.1 cm$, $h_2 = 2.1\ cm$, $h_3 = 2\ cm$, $h_4 = 1.8\ cm$

Area: $\Delta\ ABV = \frac{1}{2} \times 4 \times 4.1$

$= 8.2\ cm^2$

$\Delta\ ACV = \frac{1}{2} \times 6 \times 2.1$

$= 6.3\ cm^2$

$\Delta\ ABC = \frac{1}{2} \times 4 \times 2$

$= 4\ cm^2$

$\Delta\ BCV = \frac{1}{2} \times 6 \times 1.8$

$= 5.4\ cm^2$

Total surface area of solid

$= (8.2 + 6.3 + 4 + 5.4)\ cm^2$

$= 23.9\ cm^2$

Example 7.8

The base of a right pyramid is a square of sides 4 cm. The slant edges are all 6 cm long.

(a) *Draw and label a sketch of the solid.*

(b) *Draw a net of the pyramid.*

(c) *Use your net to find:*

  (i) *the slant height of the pyramid.*

  (ii) *the total surface area of the pyramid.*

Solution

(a) *Let the base of the pyramid be ABCD, and its vertex be V. Fig. 7.20 shows the sketch of the pyramid.*

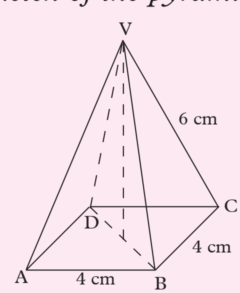


Fig. 7.20

(b) *Fig. 7.21 shows the net of the pyramid.*

(Drawn to a scale of 1 : 2).

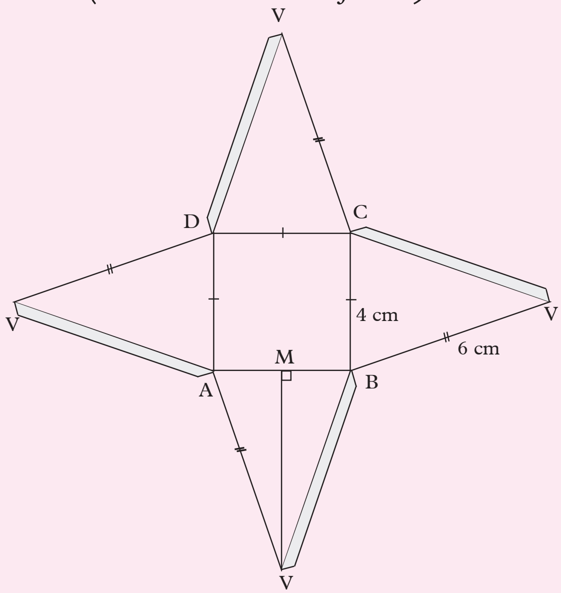


Fig. 7.21

(c) (i) The inner segment VM on the net in part (b) represents the slant height of the pyramid.

By measurement VM = 5.7 cm.

(ii) Total surface area of the pyramid

$= ((\frac{1}{2} \times 4 \times 5.7) \times 4) + (4 \times 4)$

$= 45.6 + 16$

$= 61.6\ cm^2$.

The surface area of a pyramid is obtained as the sum of the areas of the slant faces and the base.

Example 7.9

Find the surface area of the right pyramid shown in Fig. 7.22

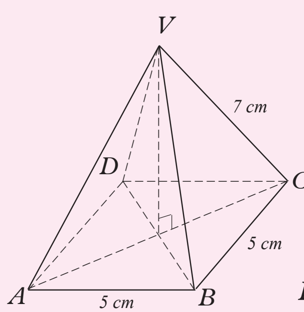


Fig. 7.22

Solution

Area of the base = 5 cm × 5 cm = 25 cm^2

Each slanting face is an isosceles triangle (Fig. 7.23).

By construction and measurement, $h = 6.5$ cm.

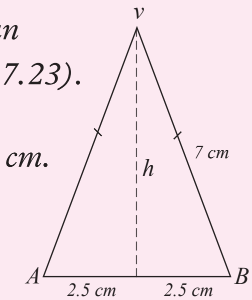


Fig. 7.23

Area of each slanting face

$= (\frac{1}{2} \times 5 \times 6.5)\ cm = \frac{32.5}{2}$

$= 16.25\ cm^2$

$\therefore$ Total area of the slanting faces

$= (4 \times 16.25)\ cm^2$

$= 65\ cm^2$

Total surface area of the pyramid

$= (25 + 65)\ cm^2$

$= 90\ cm^2$.

Exercise 7.3

In questions 1 and 2, calculate the total surface area of the given right pyramids.

1. Slant height is 5 cm, square base of sides 6 cm.
2. Slant edge 6.8 cm, rectangular base of sides 4 cm by 5 cm.
3. Fig. 7.24 shows the net of a tetrahedron, ABCDEF not drawn to scale.
   (a) Construct the net accurately.
   (b) Calculate the area of the net and hence state the surface area of the solid

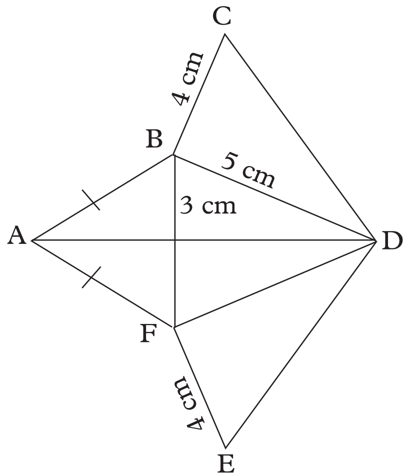


Fig. 7.24

4. ABCV in Fig. 7.25 is a tetrahedron whose base is right angled at A. The vertex, V is vertically above A. Given that AV = AC = 8 cm and AB = 6 cm, construct an accurate net for the pyramid and label it appropriately.

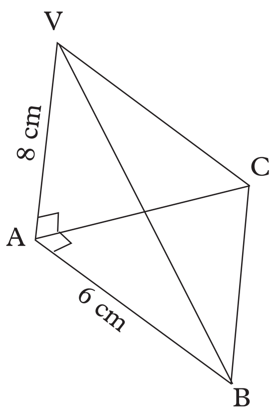


Fig. 7.25

5. A right pyramid VABCD Fig. 7.26 has a rectangular base ABCD and vertex V is vertically above the centre of the base. Given that AB = 8 cm, BC = 6 cm, and all the slant edges VA, VB, VC and VD are each 15 cm long.

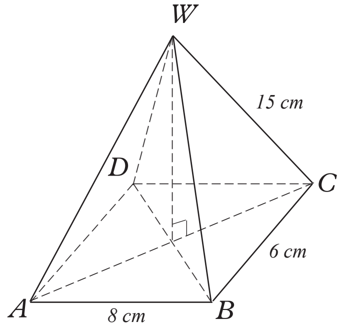


Fig. 7.26

Construct an accurate net of the pyramid and label it. Use your net to calculate the surface area of the pyramid.

In question 6, 7 and 8, calculate the total surface area for the given pyramids

6. Slant edge 12 cm; rectangular base 6 cm by 8 cm.
7. Slant edge 4cm; square base of sides 4 cm.
8. Slant height 8 cm; square base of sides 5.3 cm.
9. The base of a right pyramid is a square of sides 4 cm. The slant edges are all 6 cm long. Calculate the total surface area of the pyramid.

(f) Surface area of a cone

Activity 7.10

Carry out the following activities

1. Draw a circle, radius r (say r = 10 cm). At the centre O of the circle, measure an angle AOB (e.g. 150°) and use it to form a sector as shown in Fig. 7.27 (shaded part). Cut out the sector.

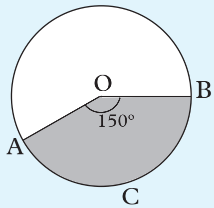


Fig. 7.27

2. Fold the sector so that OA and OB coincide. This forms the curved surface of a cone as shown in Fig. 7.28.

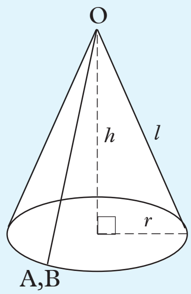


Fig. 7.28

What is the name given to such a cone?

3. What fraction of the circumference is arc ACB in Fig. 7.28? Calculate the length of the arc.

Discussion

1. What is the relationship between the length of arc ACB in Fig. 7.27 and the circumference of the base of the cone in Fig. 7.28?
2. Using the relationship in Step 1 what is the length of the radius of the base of the cone in Fig. 7.28.
3. Calculate the ratio $\frac{r}{l}$.
4. What is the circumference of the circle in Fig. 7.28 in terms of l.

   Also, find in terms of r, the circumference of the base of the cone in Fig. 7.28.

   Hence, write down, in terms of l and r, the ratio

   $$\frac{\text{circumference of base of cone}}{\text{circumference of circle}}$$

5. What fraction of the area of the circle is the shaded sector in Fig. 7.28? What is this fraction in terms of l and r?
6. What is the area of the curved surface of the cone? Give your answer in terms of l and r.

Learning points

The surface area of a cone is composed of the curved surface and the circular base.

Area of the curved surface of a cone $= \pi rl$ while the area of the circular base $= \pi r^2$. Hence, total surface area of a closed cone $= \pi r^2 + \pi rl$.

A **right circular cone** has a circular base and the vertex is vertically above the centre of the base. The word 'right' here means 'perpendicular to the base'.

The length of the arc ACB

$$= \frac{\theta}{360} \times 2\pi l$$

The length of the circumference of the base of the cone $= 2\pi l$

Since arc ACB and circumference of the base of the cone represent the same thing, therefore;

$$2\pi r = \frac{\theta}{360} \times 2\pi l$$

$$r = \frac{\theta}{360} l$$

$$\frac{r}{l} = \frac{\theta}{360}$$

Area of a sector $= \frac{\theta}{360}\pi r^2$

Area of a circle $= \pi r^2$

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\frac{\theta}{360}\pi r^2}{\pi r^2} = \frac{\theta}{360}.$$

Length of arc ACB = $\frac{\theta}{360}2\pi$

Length of circle = $2\pi r$

$$\frac{\text{Length of arc ACB}}{\text{Length of circumference of base}} = \frac{\frac{\theta}{360}2\pi r}{2\pi r} = \frac{\theta}{360}$$

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{Length of arc ACB}}{\text{Length of circle}}$$

$$\frac{\text{Area of sector}}{\pi r^2} = \frac{\frac{\theta}{360}2\pi r}{2\pi r}$$

Area of sector $= \frac{2\pi r}{2\pi} \times \pi$

$= \pi r$

Curved surface area of cone $= \pi rl$

Example 7.10

(a) *Find the curved surface area of a cone whose slant height is 5 cm. The cone was modeled using a sector of a circle whose centre angle is 216°. (Use π = 3.142).*

(b) *Calculate the radius of the base of the cone.*

Solution

Fig 7.29 (a) shows the sector of the circle used to form the cone (Fig.7.29(b)) (not drawn to scale).

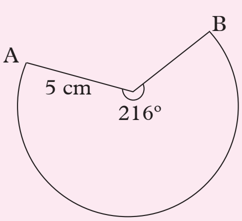


Fig. 7.29(a)

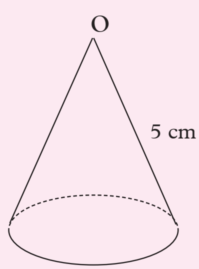


Fig. 7.29(b)

(a) *The curved area of the cone = area of the sector.*

Area of sector $= \frac{216}{360} \times \pi R^2$

$= \frac{216}{360} \times 3.142 \times 5^2$

$= \frac{16966.8}{360}$

$\therefore$ *curved surface* $= 47.13\ cm^2$

(b) *The circumference of the base of the cone = the length of the arc of the sector.*

Fig 7.30 shows the cone.

Length of the arc $= \frac{216}{360}$ *of* $2\pi r$

$= \frac{216}{360} \times 2 \times 3.142 \times 5$

$= 18.852$ *cm*

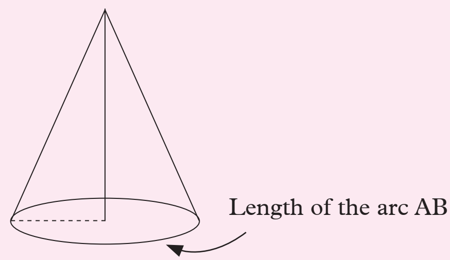


Fig 7.30

If r = radius of the cone,

circumference $= 2\pi r$

$\therefore\ 2\pi r = 18.852$

$r = \frac{18.852}{2\pi}$

$= 3$ *cm*

$\therefore$ *radius of the base of the cone = 3cm*

Example 7.11

Find the surface area of a cone whose slant height and radius are 5 cm and 3 cm respectively. (use π = 3.142).

Solution

Using Fig. 7.31

$l = 5$ *cm,*

$r = 3$ *cm*

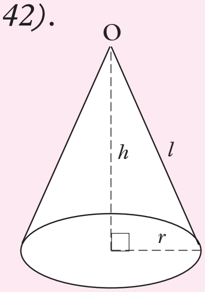


Fig. 7.31

Area of curved surface $= \pi rl.$

$= 3.142 \times 3 \times 5\ cm^2$

$= 47.13\ cm^2$

Area of circular base $= \pi r^2$

$= 3.142 \times 3^2\ cm^2$

$= 28.278\ cm^2$

Hence, total surface area $= \pi r^2 + \pi rl$

$= 28.278 + 47.13$

$= 75.408\ cm^2$

$= 75.41\ cm^2$ (2 d.p)

Exercise 7.4

In this exercise, take $\pi = 3.142$.

In questions 1 to 3, find the surface area of the given cone.

1. Slant height 8 cm; base radius 6 cm.
2. Slant height 13 cm; height 5 cm.
3. Height 8 cm; base diameter 12 cm.
4. (a) A solid cone has a radius of 8 cm and a slant height of 17cm. Calculate the total surface area of the cone.
   (b) Given that the total surface area of a cone is 319 cm^2, and the diameter of cone is 11.8 cm, find the slant height.
5. (a) The area of the curved surface area of a cone is 32.64 cm^2. If the radius of the cone is 3.7 cm, find the slant height.
   (b) The curved surface area of a cone is 184.82 cm^2 and the slant height is 0.72 cm find the radius of the cone.
6. Slant height 9 cm; perimeter of base 12 cm.
7. Height 4 cm; area of base 15 cm^2.
8. A circle has a radius of 5 cm. The length of the arc of a sector of the circle is 6 cm. Find the:
   (a) area of the sector.
   (b) surface area of the closed cone made using this sector.
9. The height of a conical tent is 3 m and the diameter of the base is 5 m. Find the area of the canvas used for making the tent. Hint: (Use an appropriate construction to determine the slant height.)

(g) Surface area of a sphere

Activity 7.11

(a) Obtain a well rounded orange or any other fruit whose shape resembles that of a ball.

(b) Carefully, cut it into halves.

(c) Using a string or a ruler, measure the diameters of each of the two halves, ensure that they are identical.

(d) Peel out each of the two halves carefully.

(e) Use the diameter obtained in C above to arrange the peels into circles with the same diameter.

Discussion

1. How many whole circles did you obtain?
2. What is the surface area of of each circle?
3. What is the total area of the four circles?
4. Compare the area of the four circles with the area of the sphere.
5. What is the name of half a sphere?

Learning points

From this activity, you should have observed that:

1. You needed approximately four circles to cover the whole sphere.
2. The surface area of the sphere is equal to 4 times the area of one circle i.e.

   Area of 1 circle = πr^2 (r being the radius of the sphere).

   Area of 4 circles = $4 \times \pi r^2 = 4\pi r^2$ (since 4 circles are needed).

Fig. 7.32 represents a solid sphere of radius *r* units.

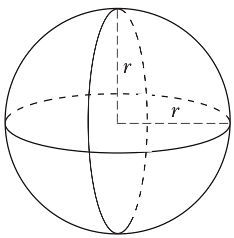


Fig. 7.32

The surface area of the sphere is given by:

Surface area = $4\pi r^2$ square units.

Footballs, tennis balls, fruits such as oranges are examples of spheres.

Half of a sphere is known as a **hemisphere**. A hemisphere has two faces, the curved surface and the circular face. Its surface area is given by:

Surface area of hemisphere

= half the area of the sphere + area of flat surface

$= \frac{1}{2} \times 4\pi r^2 + \pi r^2 = 3\pi r^2$

We also learn from that the earth is also a sphere.

Example 7.12

A solid hemisphere has a radius of 5.8 cm. Find its surface area. Take $\pi = 3.142$

Solution

Surface area of hemisphere $= 3\pi r^2$

$= (3 \times 3.142 \times 5.8 \times 5.8)\ cm^2$

$= 317.1\ cm^2$ *(4 s.f.)*

Exercise 7.5

In this exercise, use $\pi = 3.142$ or $\frac{22}{7}$ depending on the measurements given.

1. Calculate the surface area of a sphere whose radius is:
   (a) 3.2 cm (b) 1.2 cm
   (c) 4.2 cm
2. Find the surface area of Fig. 7.33 shown.
   (a) (b)

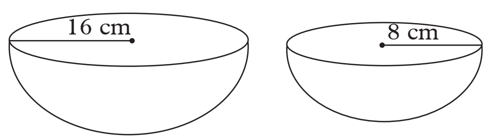


Fig. 7.33

3. Find the radius of a sphere whose surface area is:
   (a) $78.5\ cm^2$ (b) $181\ cm^2$
4. Find the total surface area of a solid hemisphere of diameter 10 cm.
5. A hollow sphere has an internal diameter of 18 cm and a thickness of 0.5 cm. Find the external surface area of the sphere.

(h) Surface area of composite solids

Activity 7.12

Carry out the following activities.

1. Obtain an orange. Cut it into two equal parts.
2. Measure the circumference of the half orange using a string.
3. Obtain a plain paper. Using the circumference you obtained in 2 above, measure an equivalent length of a rectangular paper. Use a reasonable width.
4. Cut out the rectangle. Use the cut out to form a cylinder with the length as the circumference of the cylinder. Join the two shapes by mounting the cylinder on to the half an orange as shown in Fig. 7.34 below.

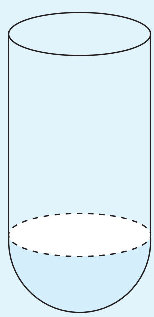


Fig. 7.34

What is the name given to such figure formed? Define it.

Discussion

1. When calculating the surface area of the figure formed, would you consider the joined surfaces (circular faces)?
2. Calculate the surface area of the figure formed.

Learning points

A composite solid is a solid formed by two or more solids. For example, a cone and a hemisphere joined together will form a composite solid.

To get the total surface area of a composite solid, we get the surface area of all the faces exposed. Faces lying on each other where the two solids are joined together is not an exposed surface and therefore we do not get the area of such a face.

Example 7.13

Fig. 7.35 shows a composite solid composed of a cuboid and a pyramid. Find its total surface area if all the slanting heights of the pyramid are equal.

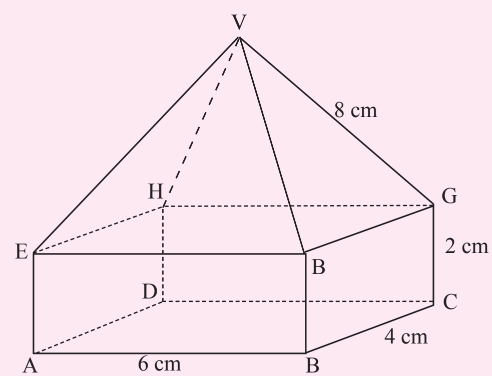


Fig. 7.35

Solution

To find the total surface area of the composite solid, we find the surface area of the four triangular faces of the pyramid and the five exposed rectangular faces of the base cuboid.

Surface area of the pyramid part = Area of the two similar triangles + area of the two similar triangles.

Area of triangle $= \frac{1}{2}\, b \times h$ *(Fig. 7.36)*

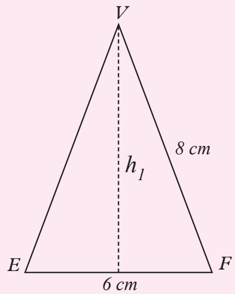


Fig. 7.36

By accurately constructing ΔVEF, and its altitude from point V, determine h_1 by measuring.

$$\therefore h_1 = 7.4 \text{ cm}$$

$$\text{Area} = (\tfrac{1}{2} \times 6 \times 7.4) \times 2$$

$$= 44.5 \text{ cm}^2$$

Area of other two similar triangles

$$= \tfrac{1}{2} \times b \times h \text{ (Fig. 7.37).}$$

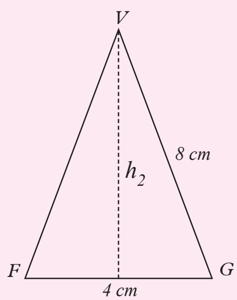


Fig. 7.37

By means of an accurate construction, determine h_2.

$$h_2 = 7.75 \text{ cm}$$

$$\text{Area} = (\tfrac{1}{2} \times 4 \times 7.75) \times 2$$

$$= 30.98 \text{ cm}^2$$

Area of pyramid part

$$= 44.5 \text{ cm}^2 + 30.98 \text{ cm}^2$$

$$= 75.48 \text{ cm}^2$$

Area of the cuboid (base)

$$= \{(2 \times 6)\ 2 + (2 \times 4)\ 2 + (6 \times 4)\}\text{cm}^2$$

$$= (24 + 16 + 24)\text{cm}^2$$

$$= 64 \text{ cm}^2$$

Total surface area of the composite solid

$$= 75.48 \text{ cm}^2 + 64 \text{ cm}^2$$

$$= 139.48 \text{ cm}^2$$

Exercise 7.6

1. Fig. 7.38 shows a composite solid made of a cylinder and a hemisphere. Find its total surface area.

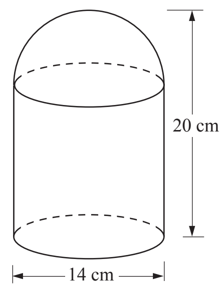


Fig. 7.38

2. A solid is made up of a hemisphere mounted on a right cone both of radius 5 cm (Fig. 7.39). Calculate the surface area of the solid.

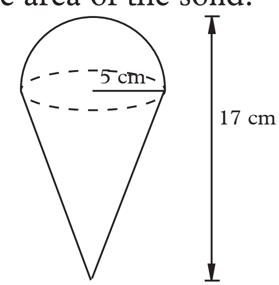


Fig 7.39

3. Fig. 7.40 shows solids which comprise of cubes surmounted with pyramids. Calculate the surface area of each solid.

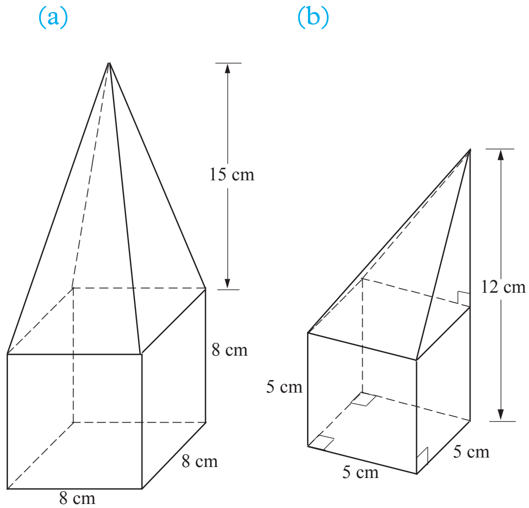


Fig. 7.40

7.3 Volume of solids

(a) Volume of cubes and cuboids

Activity 7.13

Carry out the activities described below:

(a) Using an accurate net, model a cuboid of sides 4 cm by 2 cm by 3 cm.

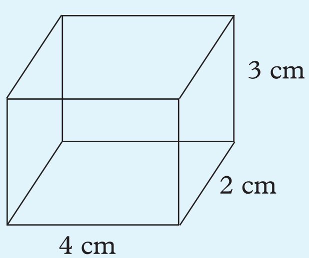


Fig. 7.41

(b) Construct small nets of a cuboid measuring 1cm. Cut out the nets and model them into cubes.

(c) How many cubes in (b) above will be needed to fill the cuboid in (a) completely?

Discussion

1. Find the product of the length, breadth and of height of the cuboid in (a) above. What do you notice?
2. Find the area of the base of the cuboid in (a) above?
3. What does the product of the area of the base and the height of the cuboid represent?

Learning points

1. From the above activities, the volume, V_1 of a solid is the number of cubic units needed to fill the solid. Thus, the volume of the cuboid in our activity has 3 layers of 1cm cubes each measuring 4 cm × 2 cm.

   Number of 1cm cubes required = product of l, b and h

   $24 = 4 \times 2 \times 3$

2. **Volume** is the amount of space occupied by an object. A unit cube (Fig. 7.42) is used as the basic unit of volume. So volume is expressed in terms of cubic units.

   The SI unit for volume is the cubic metre (m^3). Volume of cuboid = $l \times w \times h$

3. The volume of a cube of side 1 metre (i.e. a unit cube) is:

   $1\text{ m} \times 1\text{ m} \times 1\text{ m} = 1\text{ m}^3$.

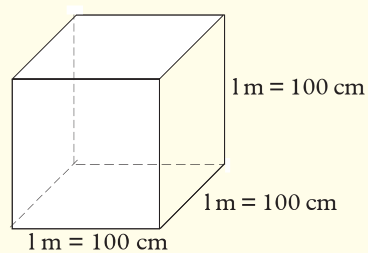


Fig. 7.42

Since 1 m^3 is too large for ordinary

use, volumes are often measured using cm^3.

$1\ m = 100\ cm$

$1\ m^3 = 100\ cm \times 100\ cm \times 100\ cm$ (Fig.7.32).

$= 1\ 000\ 000\ cm^3$.

$\frac{1}{1\ 000\ 000} m^3 = 1\ cm^3$

i.e. $1\ cm^3 = \frac{1}{1\ 000\ 000} m^3$.

Example 7.14

Fig. 7.43(a) represents a cuboid measuring 6 cm long, 5 cm wide and 3 cm high. Find its volume.

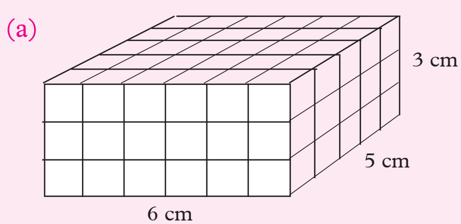


Fig. 7.43

Solution

There are 3 layers of 1 cm cubes. Each layer has 6 rows of cubes and each row has 5 cubes (Fig. 7.44(b)).

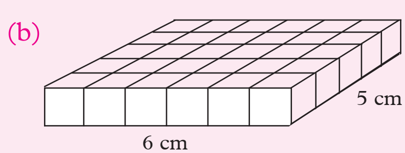


Fig. 7.44

So each layer has $6 \times 5 = 30$ *cubes.*

Number of cubes in the 3 layers

$= 30 \times 3$

$= 90$ *cubes.*

The volume of each cube

$= 1\ cm \times 1\ cm \times 1\ cm$

$= 1\ cm^3$.

$\therefore$ *Volume of the cuboid (the 90 cubes)*

$= 90 \times 1\ cm^3 = 90\ cm^3$.

We could have obtained the same result by multiplying length (6 cm) × breadth (5 cm) × height (3 cm).

$= 6cm \times 5cm \times 3\ cm = 90\ cm^3$

A cube is just a special cuboid with length = breadth = height

Thus,

Volume of a cuboid = length × breadth × height

$V = l \times b \times h$ cubic units (units^3).

Volume of a cube = $l \times l \times l$

$= l^3$ cubic units

(b) Volume of a prism

Activity 7.14

1. Use the thick foam or plasticine to model a prism as the following.

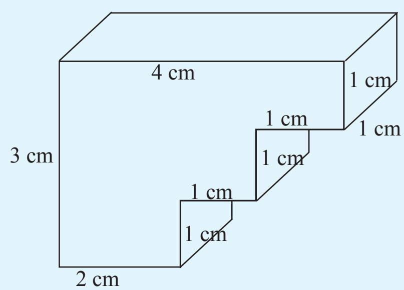


Fig. 7.45

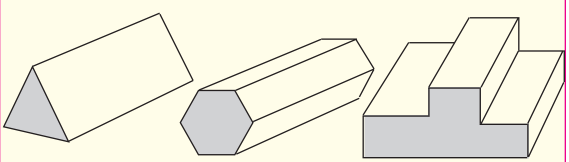
2. What is the name of a figure as Fig. 7.45 one above?
3. How many 1cm cubes can be used to make a similar identical prism?
4. Interpret your answer in (3) above.

Discussion

1. Identify the base (cross-section) of the prism.
2. Find the area of the base.
3. How can you use this area to find the volume of the prism?
4. Comment on the volume of the prism.

Learning points

From activity 7.13, you should have found that the volume = cross-section area × length of the prism.



(i) (ii) (iii)

Fig. 7.46

In Fig. 7.46 (i) we need to find the altitude of the cross-section.

In Fig 7.46 (ii) we need to divide the hexagonal face into triangles to be able to calculate its area. In Fig 7.46 (iii), we subdivide cross section into rectangles.

In general, for any prism:

Volume of a prism = area of uniform cross-section × length (or height) of the prism.

Example 7.15

A rectangular tank has 70 m^3 of water. If the tank is 5 m long and the height of water is 4 m, what is the width of the tank?

Solution

Volume of water in tank $= l \times b \times h$

$$5m \times b \times 4m = 70m^3$$

$$20b = 70$$

$$b = \frac{70}{20}$$

$$= 3.5\ m$$

$\therefore$ *The tank is 3.5 m wide.*

Example 7.16

The internal measurements of a wooden box with a lid are 10 cm by 8 cm by 7 cm. The wood is $\frac{1}{2}$ cm thick. Find the volume of the wood.

Solution

Fig. 7.47 is a sketch of the box. The external measurements are:

$$\text{Length} = 10 + \frac{1}{2} + \frac{1}{2} = 11\ cm$$

$$\text{Width} = 8 + \frac{1}{2} + \frac{1}{2} = 9\ cm$$

$$\text{Height} = 7 + \frac{1}{2} + \frac{1}{2} = 8\ cm$$

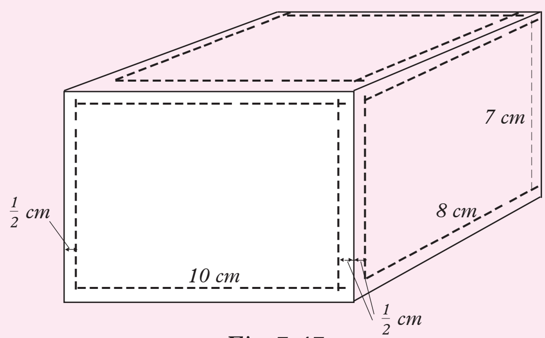


Fig. 7.47

$\therefore$ *External volume of the cuboid*

$$= (11 \times 9 \times 8)\ cm^3$$

$$= 792\ cm^3$$

Internal volume of the cuboid

$$= (10 \times 8 \times 7)\ cm^3$$

$$= 560\ cm^3$$

$\therefore$ *Volume of wood*

$= $ *external volume – internal volume*

$= 792\ cm^3 - 560\ cm^3$

$= 232\ cm^3.$

Example 7.17

A beam is shaped as in Fig. 7.48, with the measurements being in centimetres. If the length of the beam is 5 m and all angles are right angles, find its volume.

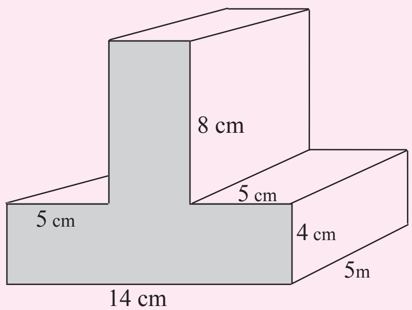


Fig. 7.48

Solution

The end face forms our uniform cross-section.

The end-face may be taken as comprising of two rectangles.

$\therefore$ *area of end-face*

$= (14 \times 4 + 8 \times 4)\ cm^2$

$= 88\ cm^2$

length of beam $= 5\ m = 500\ cm.$

$\therefore$ *volume of beam* $=$ *Base area* $\times$ *length*

$= 88\ cm^2 \times 500\ cm$

$= 44\ 000\ cm^3$

Exercise 7.7

1. Calculate the volume of the solids in Fig. 7.49. (All measurements are given in cm)

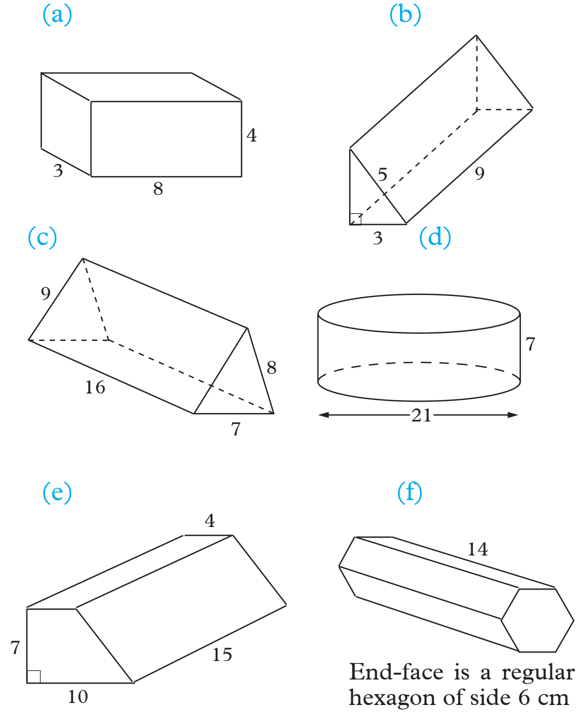


Fig. 7.49

2. A wooden beam has a rectangular cross-section measuring 21 cm by 16 cm and it is 4 m long. Calculate the volume of the beam, giving your answer in cm^3 and in m^3.

3. A block of concrete is in the shape of a wedge whose triangular end-face is such that two of its sides are 16 cm and 19 cm long and the angle between them is 50°. If the block is 1 m long, find its volume.

4. Fig. 7.50 shows the shapes of steel beams often used in construction of buildings. Calculate the volume of a 6 m length of each beam, given that all dimensions are in centimetres and that all angles are right angles.

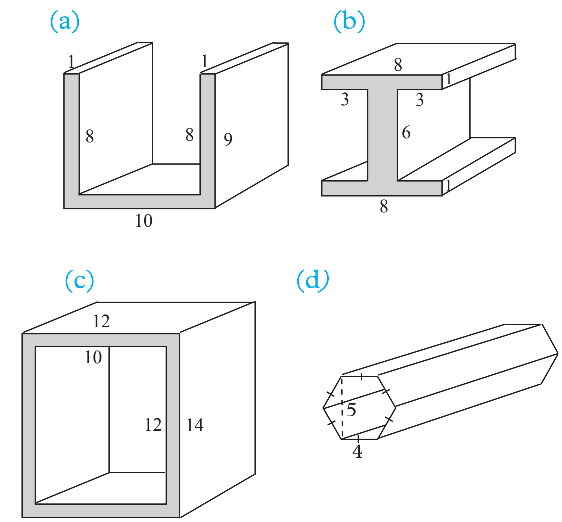


Fig. 7.50

5. A cylindrical container has a diameter of 14 cm and a height of 20 cm. Using $\pi = \frac{22}{7}$, find how many litres of liquid it holds when full.

6. 8 800 litres of diesel are poured into a cylindrical tank whose diameter is 4 m. Using $\pi = \frac{22}{7}$, find the depth of the diesel in the tank.

7. The volume of the prism in Fig. 7.51 is 1 170 cm^3. Find its length.

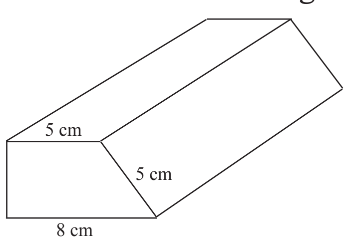


Fig. 7.51

8. The volume of a prism with a regular pentagonal base is 4 755 cm^3. If the prism is1 m long, find, in centimetres, the distance from the centre of the base to any of its vertices.

(c) Volume of a cylinder

Activity 7.15

1. Obtain 2 A4 rectangular plain papers.
2. Fold each to obtain different sizes of a cylinder by:
   (a) Joining the lengths of the papers using glue.
   (b) Joining the widths of the papers using glue as shown in Fig. 7.52 below.

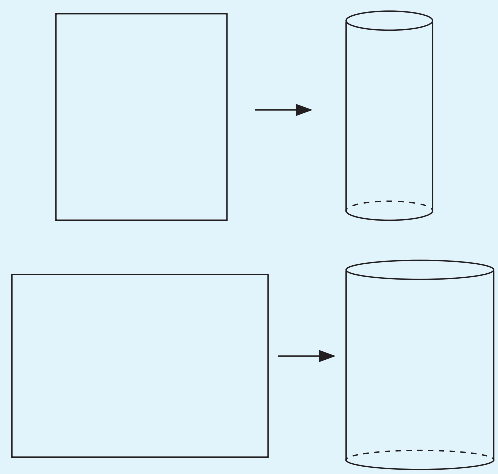


Fig. 7.52

State the measures of the height and the radius of the each of the cylinders you made.

Discussion

1. What is the measure of the circumference of each cylinder?
2. Find the cross section area of each cylinder.
3. Compare the volumes of both cylinders that you formed. Comment on your results.

Learning points

A cylinder is a prism. It has a uniform circular cross-section. Fig 7.53.

The bigger the cross-sectional area, the bigger the volume and vice versa.

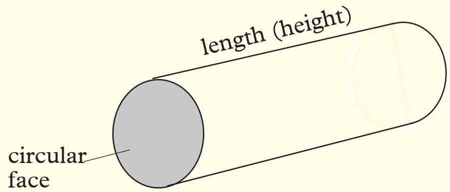


Fig. 7.53

The volume of a solid with a uniform cross-section is given by:

Volume = Area of cross-section × length.

Thus, Volume of a cylinder

= Area of the circular face × height

= $\pi r^2 h$

Example 7.18

Find the volume of a cylindrical container whose diameter is 7.9 cm if its height is 7cm. Use $\pi = 3.14$ and give your answer in 3 significant figures.

Solution

Volume of cylinder = $\pi r^2 h$

= $(3.14 \times 3.95^2 \times 7)$ cm^3

= 342.9 cm^3

= 343 cm^3

Example 7.19

A cylindrical mug has inner diameter 9.2 cm and an inner height of 12.5 cm. Taking π as 3.142, find its capacity.

Solution

Volume of cylinder = $\pi r^2 h$

$\therefore$ *Capacity of the mug*

= $(3.142 \times 4.6^2 \times 12.5)$ cm^3

= 831.1 cm^3 (to 4 s.f.)

Exercise 7.8

1. Find the volume of each of the following cylinders.
   (a) Radius 2.1 cm, height 0.3 cm.
   (b) Radius 0.7 cm, height 4.5 cm.
   (c) Radius 0.9 cm, height 14 cm.
   (d) Radius 1.5 m, height 5.6 m.
   (e) Diameter 3.5 m, height 4.9 m.

2. A cylindrical tank has a diameter of 5.0 m and contains 110 000 *l* of water. What is the height of the water in the tank?

3. Find the volume of the material of each of the following pipes.
   (a) External radius 2.5 cm, thickness 0.5 cm and length 15 cm.
   (b) External diameter 4.4 cm, thickness 0.7 cm and length 24 cm.
   (c) External diameter 15 cm, thickness 0.7 cm and length 3 m.

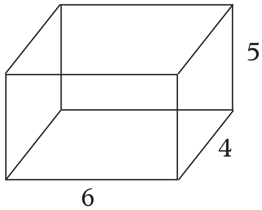
4. A cylindrical container of diameter 15 cm and depth 20 cm is full of water. If the water is poured into an empty cylindrical jar of diameter 10 cm, find the depth of the water in the jar.

5. A circular pond of diameter 40 m is surrounded by a path 2 m wide. Calculate the volume of murram required to gravel the path to a depth of 7.5 cm.

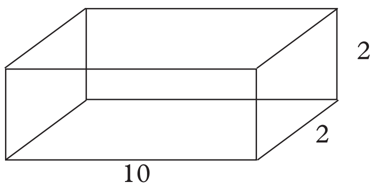
6. A cylindrical water tank has a diameter of 140 cm. To begin with, it is full of water. A leak starts at the bottom such that it loses 33 *l* of water in 1 hour. How long will it take for the water level to fall by 30 cm?

7. Find the volume of each cuboid below:

(a)



(b)



(c)

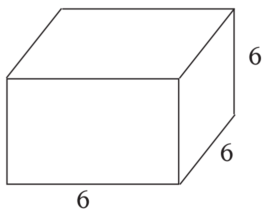


Fig. 7.54

8. In preparation to lay the foundation of a house, a hole measuring 8 m deep, 15.8 m long and 7.6 m wide was dug. What volume of earth was removed?

9. A closed wooden box has outside dimensions of 1.8 m by 0.92 m by 1.35 m.
   (a) Calculate the amount of space occupied by the box.
   (b) If the thickness of the wood that makes the box is 6cm thick, calculate the volume of the inside of the box.

10. A rectangular container is 30 cm wide and 56 cm long, and contains water to a depth of 18cm. A metal ball is immersed in the container and causes the water depth to rise 2.9 cm. Find the volume of the ball.

11. A small metal bar is 6 cm long, 2.5 cm wide and 2 cm thick. The bar is melted down and cast into a cube. Find:
   (a) The length of a side of the cube.
   (b) Assuming that no wastage occurs when melting the metal bar and that it was recast into a cylindrical bar of length 6 cm, what would be the radius of the bar?

12. A rectangular water tank is 6.4 m long, 4.5 m wide and 2.45 m deep:
   (a) Given that 1 000 cm^3 = 1 litre, find the capacity of this tank when full (in litres).
   (b) At the moment, the water level in the tank is 6 m from the top. How many litres of water are in the tank?

(d) Volume of a cone

Activity 7.16

1. Construct a cone and a cylinder so that each solid has the same radius and same height.
2. Fill the cone with dry stuff like sugar, rice, flour, beans, maize etc.
3. Pour the content of the cone into the cylinder. How many times do you need to pour the content of the cone into the cylinder to fill it up?
4. Find the base area of the cone in (1) above.

Discussion

1. Express the volume of the cone as a fraction of the volume of the cylinder. What do you obtain?

2. Express the volume of the cylinder in terms of base area, A, and the height (h).

Learning points

To fill the cylinder, you need to empty the content of the cone 3 times into it.

The volume of the cone $= \frac{1}{3}$ of the volume of the cylinder.

Volume of cone $= \frac{1}{3}$ of base area of cylinder × height

$= \text{base area} \times h$

$= Ah$

Since base of cylinder = base of cone and height of cylinder = height of cone,

Volume of a cone $= \frac{1}{3}$ base area × height of cone.

$= \frac{1}{3}\pi r^2 h$

Thus, volume of a cone

$= \frac{1}{3} \times \text{base area} \times \text{height}$

$= \frac{1}{3}\pi r^2 h$

Example 7.20

A cone has a radius of 8 cm and a slant height of 17 cm. Calculate the volume of the cone.

Solution

Let O be the centre of the base

OB = radius = 8 cm

VB = slant height

= 17 cm

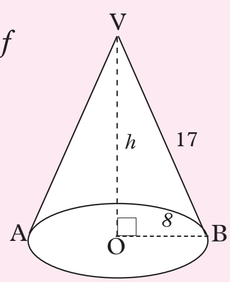
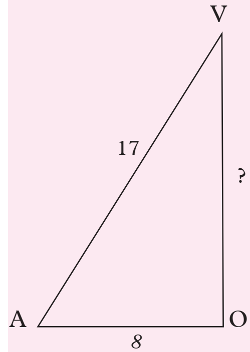


Fig. 7.55

Assuming that this is a right cone, then $\Delta VOA = \Delta VOB$, right angled at O.

By construction



$(VO)^2 + (AO)^2 = (AV)^2$

$\therefore VO = \sqrt{(AV)^2 - (AO)^2}$

$VO = \sqrt{17^2 - 8^2}$

$VO = 15$ cm

Fig. 7.56

OV = height = 15cm

Area of the base $= \pi r^2$

$= 3.142 \times 8^2$

Volume $= \frac{1}{3}$ *base area x height*

$= \frac{1}{3} \times 3.142 \times 8^2 \times 15$

$= (3.142 \times 64 \times 5)\ cm^3$

$= 1005.44\ cm^3$

Example 7.21

Find the volume of a cone whose height and radius of the base are 4 cm and 3 cm, respectively. (Take $\pi = 3.142$).

Solution

Fig 7.57 is a sketch of the cone

r = 3 cm and h = 4 cm

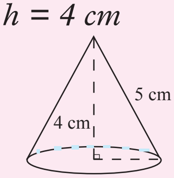


Fig. 7.57

Volume of cone $= \frac{1}{3}\pi r^2 h$

$= (\frac{1}{3} \times 3.142 \times 3 \times 3 \times 4)\ cm^3$

$= 37.7\ cm^3$

Exercise 7.9

In questions 1 to 7, find the volume of the given cone. Take $\pi = 3.142$.

1. Height 4 cm; area of base 15 cm^2.
2. Slant height 8 cm; base radius 6 cm.
3. Height 5 cm, radius 12 cm.
4. Height 8 cm; base diameter 12 cm.
5. Height 8 cm; base radius 3 cm.
6. Slant height 8.5 cm; height 6.5 cm.
7. Slant height 9 cm; perimeter of base 12 cm.
8. Find the height of a cone whose base radius is 3.72 cm and whose volume is 143 cm^3.
9. The area of a sector of a circle of radius 4 cm is 20 cm^2. What is the length of the arc of the sector? Find the radius and the volume of the cone made using this sector.
10. The height of a conical tent is 3 m and the diameter of the base is 5 m. Find the volume of the tent.
11. The value of a conical tank is 66 m^3. The area of the circular base on which it stands is 18 m^2. Find the area of the curved surface of the tank.

(e) Volume of a pyramid

Activity 7.17

Every member in the group should construct a net of a square based pyramid as shown in Fig. 7.58.
All measurements are in centimetres.

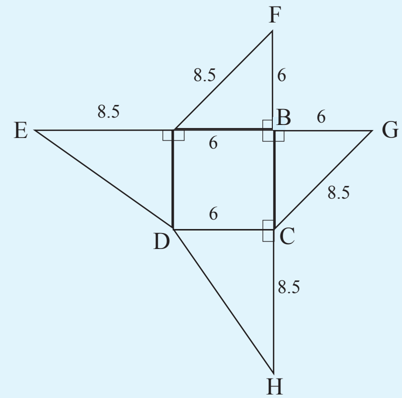


Fig. 7.58

Cut the net out and fold the triangles up to form the pyramid as shown in Fig. 7.59.

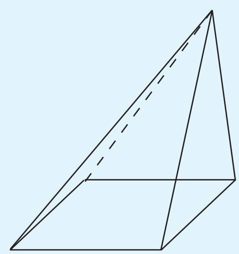


Fig. 7.59

Rotate and arrange the three pyramids (each from each group member) to make a cube as shown in Fig. 7.60.

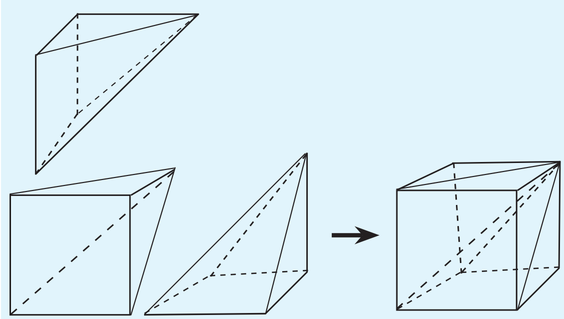


Fig. 7.60

What did you obtain?

Discussion

1. What is the volume of each pyramid?
2. What is the total volume of the three pyramids?

3. Compare the volume of the three pyramids to that of the cube. Comment on your answer.
4. Express the volume of the pyramid as a fraction of the cube.

Activity 7.18

1. Construct a cube whose base and height are equal to the base and height of the pyramids respectively.
2. Open the base of the pyramid and the top of the cube.
3. Fill the pyramid with dry substance such as sugar, rice, flour, soil etc and empty it into the cube.
4. How many times will you need to empty the contents of the pyramid into the cube in order to fill it?
5. Find the area of the base of the pyramid.

Discussion

1. Find the volume of the cube.
2. What fraction of the volume of the cube is the volume of the pyramid?
3. Express the volume of the pyramid as a fraction of the volume of the cube.
4. Express the volume of the pyramid in terms of base area and height.

Learning points

3 times the content of the pyramid = one times the content of the cube.

Volume of a pyramid = $\frac{1}{3}$ volume of cube.

Since

Volume of the cube = base Area (A) × height,

Base Area (A), of cube = base area of pyramid

Height of cube = height of pyramid.

Since

Volume of a cube = base area × height (Ah),

then the volume (V) of a pyramid is given by

$$V = \frac{1}{3}Ah$$

where A = Area of the base, and

h = the vertical height of the pyramid.

Remember

The height of a pyramid is the vertical distance between the vertex, V and the horizontal base of the pyramid.

Activity 7.19

Repeat activity 7.15 using a rectangular based pyramid. Comment on your observations comparing them with the previous activity.

Example 7.22

Fig. 7.61 shows a pyramid on a rectangular base. Find its volume. The dimensions are in centimetres.

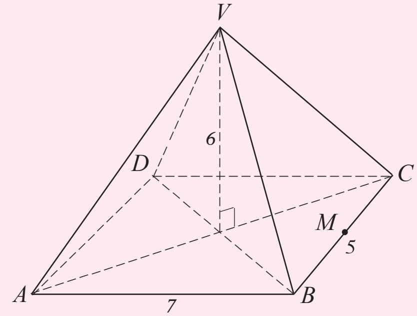


Fig. 7.61

Solution

Volume of a pyramid $= \frac{1}{3}$ *(base area) × height*

$$\therefore V = \frac{1}{3}(7\text{ cm} \times 5\text{ cm}) \times 6\text{ cm}$$

$$= 70\text{ cm}^3$$

Example 7.23

A right pyramid on a square base ABCD of sides 8 cm has slant edges, 10 cm long. Calculate the volume of the pyramid.

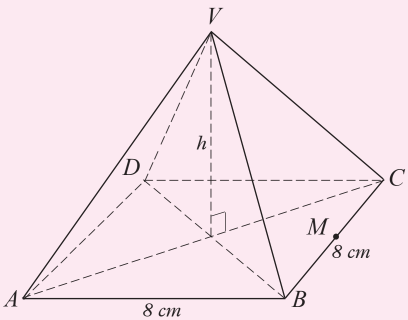


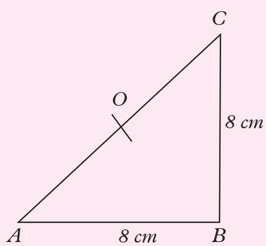
Fig. 7.62

Solution

To calculate the volume of the pyramid, we need the vertical height, h, of the pyramid. Let O be the centre of the base, the point where the diagonals AC and BD meet.

Consider ΔABC

$ABC = 90^o$, $AB = BC = 8$



$AC = \sqrt{8^2 + 8^2}$ *cm*

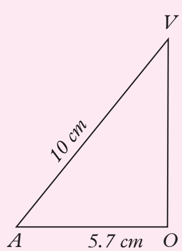
$= 11.3$ *cm*

$\frac{1}{2} AC = 5.7$ *cm*

$AO = BO = DO = CO = 5.7$ *cm*

Using AOV, $AO = 5.7$ *cm*

$VA = 10$ *cm and* $AOV = 90^o$



$OV = \sqrt{10^2 - (5.7)^2}$ *cm*

$= 8.2$ *cm*

$OV = h = 8.2$*cm*

$\text{Volume} = \frac{1}{3} \text{ base area} \times h$

$\text{Volume} = \frac{1}{3} \times 8 \text{ cm} \times 8 \text{ cm} \times 8.2 \text{ cm}$

$= 174.9 \text{ cm}^3$

Exercise 7.10

In questions 1 to 9, find the volume of each of the given right pyramid.

1. Height 4 cm; square base, side 6 cm.
2. Height 6 cm; square base of side 9 cm.
3. Height 5 cm; rectangular base, 6 cm by 4 cm.
4. Height 6 cm; rectangular base, 4 cm by 5 cm.
5. Height 16 cm; triangular base, sides 6 cm, 8 cm and 10 cm.
6. Slant edge 12 cm; rectangular base, 6 cm by 8 cm.
7. Height 10 cm; equilateral triangle base, side 6 cm.
8. Slant edge 4 cm; square base, side 4 cm.
9. Slant height 8 cm; square base, side 5.3 cm.
10. A pyramid whose height is 8 cm has a volume of 48 cm^3. What is the area of its base?
11. A pyramid has a square base of side 5 cm. What is its height if its volume is 100 cm^3?
12. A square based pyramid has a height of 6 cm and a slant edge of 8 cm. What is its volume?

(f) Volume of a sphere

Activity 7.20

Obtain an orange and a knife. Cut out a very small area (A) from the surface to the centre of the orange as shown below.

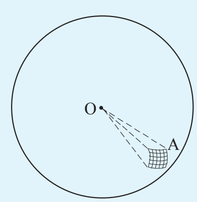


Fig. 7.63

What does A represent?

Since many of this areas (A) make up such a sphere how can we find the volume of the sphere?

Learning points

Let A represent a small square area on the surface of a sphere of radius r, centre O (Fig. 7.60).

If A is a very small area, we can look at it as almost flat.

The solid formed by joining the vertices of A to the centre O is a small 'pyramid'.

Volume of the small 'pyramid' $= \frac{1}{3} Ar$

Let there be several small 'pyramids' with base areas $A_1, A_2, A_3, \ldots$

Their volumes are $\frac{1}{3} A_1 r, \frac{1}{3} A_2 r, \frac{4}{3} A_3 r, \ldots$

Total volume $= \frac{1}{3} r(A_1 + A_2 + A_3 + \ldots)$.

For the whole surface of the sphere, the sum of all the base areas is $4\pi r^2$, (by definition).

i.e., $A_1 + A_2 + A_3 + \ldots = 4\pi r^2$.

$\therefore \frac{1}{3} r(A_1 + A_2 \ldots) = \frac{1}{3} r \cdot 4\pi r^2 = \frac{4}{3}\pi r^3$

Hence, total volume (V) of a sphere

$= \frac{1}{3} r \times 4\pi r^2$

$= \frac{4}{3}\pi r^3$

Example 7.24

Determine the radius of a sphere of volume 12 m³. Use π = 3.142.

Solution

$Volume = \frac{4}{3}\pi r^3 \Rightarrow \frac{4}{3} \times 3.142 \times r^3 = 12\ m^3$

$r^3 = \frac{12 \times 3}{4 \times 3.142}$

$r^3 = \frac{36}{12.568} = 2.8644$

$r = \sqrt[3]{2.864} = 1.42\ m$

Example 7.25

A solid hemisphere of radius 5.8 cm has density 10.5 g/cm³.

Calculate the: (a) volume,

(b) mass in kg of the solid.

Solution

(a) Volume of hemisphere

$= \frac{1}{2} \times$ volume of sphere

$= \frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$

$= \left(\frac{2}{3} \times 3.142 \times 5.8^3\right)\ cm^3$

$= 408.7\ cm^3$ (4 s.f.)

(b) Density $= \frac{Mass}{Volume}$

$\therefore$ Mass = Density × Volume

$= 10.5 g/cm^3 \times 408.7\ cm^3$

$= 4\,291.35\ g$

$= 4.291\ kg$ (4 s.f.)

Example 7.26

Find the surface area of a sphere whose volume is given as 1 000 cm³.

Solution

Let V = volume, r = radius, S = surface area

$V = \frac{4}{3}\pi r^3 \Rightarrow 1\,000 = \frac{4}{3}\pi r^3$

$r^3 = \frac{3\,000}{4\pi}$

$r^3 = 238.7$

$r = \sqrt[3]{238.7}$

$= 6.2\ cm$

$S.A = 4\pi r^2$

$\Rightarrow S.A = 4\pi \times (6.2\ cm)^2 = 483.11\ cm^2$

Exercise 7.11

In this exercise, use $\pi = 3.142$ or $\frac{22}{7}$, depending on the measurements given.

1. Calculate the volume of a sphere whose radius is:
   (a) 3.2 cm (b) 1.2 cm
   (c) 4.2 cm
2. Find the radius of a sphere whose volume is:
   (a) 73.58 cm^3 (b) 463 cm^3
3. Find the volume of a sphere whose surface area is:
   (a) 21.2 cm^2 (b) 972 cm^2
4. Find the volume of a solid hemisphere of diameter 10 cm.
5. What is the mass of a solid gold hemisphere of diameter 4 cm if the density of gold is 19.3 g/cm^3?
6. A hollow sphere has an internal diameter of 18 cm and a thickness of 0.5 cm. Find the volume of the material used in making the sphere.
7. Ten plasticine balls of diameter 2.8 cm are rolled together to form one large ball. What is the radius of the large ball?
8. Ten marbles, each of radius 1.5 cm, are placed in an empty beaker of capacity 150 ml. Water is then added to fill up the beaker. What volume of water is added?

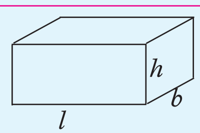
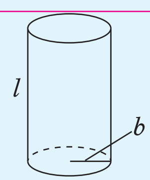
CAUTION: Conserve water by ensuring that all taps are closed after use.

9. A solid cylinder has a radius of 18 cm and height 15 cm. A conical hole of radius r is drilled in the cylinder on one of the end faces. The conical hole is 12 cm deep. If the material removed from the hole is 9% of the volume of the cylinder, find:
   (a) the surface area of the hole,
   (b) the radius of a spherical ball made out of the material.

7.3 Problem solving: Areas and volumes

Activity 7.21

Copy and complete the table 7.3 below. You may refer to your earlier work.

Shape	Name	No.of Faces	No. of Edges	No. of Vertices	Surface Area	Volume
						
						

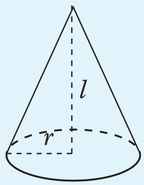
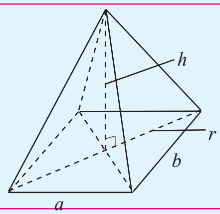
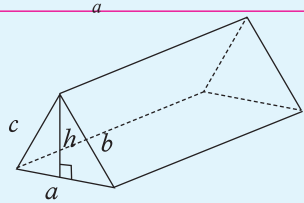
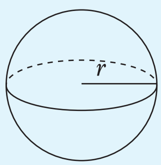
						
						
						
						

Table 7.3

Activity 7.22

1. Cut or model a cube ABCDEFGH from plasticine or thick foam.

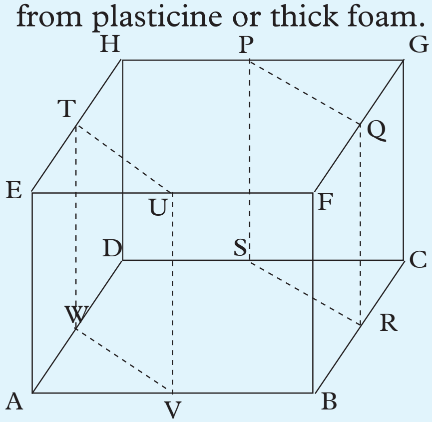


Fig. 7.64

2. Mark mid points P,Q, R, S and T, U, V, W.
3. Using a sharp cutting edge cut:
   (i) along P to Q to S to R.
   (ii) T to U to V to W.
4. Identify the resulting three solids and name them.
5. Calculate:
   (i) total surface area.
   (ii) volume of each shape.
6. Sketch the solids in your note book.
7. Re-model the cube and investigate a different way of cutting it to obtain other solids.
8. Describe the process and repeat steps 4, 5 and 6.
9. How many geometrical faces and solids can you obtain by cutting a cube?

Example 7.26

Calculate the surface area and the volume of the container shown in Fig. 7.65. Give your answers to as many figures as you think reasonable. (Take $\pi = \frac{22}{7}$)

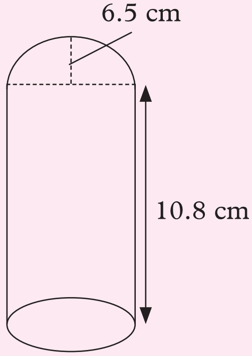


Fig. 7.65

Solution

Identify any familiar shapes in the container.
The container is made up of.

(i) *A hemisphere and*
(ii) *A cylinder*

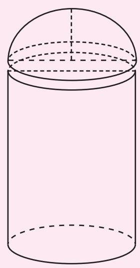


Fig. 7.66

Surface area of the hemisphere

$$= \frac{4\pi r^2}{2} = 2\pi r^2$$

$$= 2\pi \times 6.5 \text{ cm} \times 6.5 \text{ cm}$$

Curved surface area of cylinder

$= 2\pi rh$

$= 2\pi \times 6.5 \text{ cm} \times 10.8 \text{ cm}$

Surface area of the base $= \pi r^2$

$= \pi \times (6.5)^2 \text{ cm}^2$

Total surface area of container

$= (2\pi \times 6.5^2) + (2 \times 6.5 \times 10.8\pi) + (6.5^2\pi)$

$= 265.571439 + 441.2571439 + 132.785714$

$= 839.6142869$

$839.6 (1d.p)$

Measurements are given correct to 1 decimal place; it is reasonable to state the answer to the same accuracy.

Surface area of the container $= 839.6\text{cm}^2$

Volume of the cylinder $= \pi r^2 h$

$= 6.5^2 \times 10.8\pi$

$= 1434.085714 \text{ cm}^3$

Volume of the hemisphere

$$= \frac{2}{3}\pi r^3$$

$$= \frac{2}{3} \times 6.5^3\pi$$

$$= 575.4047619 \text{ cm}^3$$

Volume of container

$= 1434.085714 \text{ cm}^3 + 575.4047619 \text{ cm}^3$

$= 2008.94243 \text{ cm}^3$

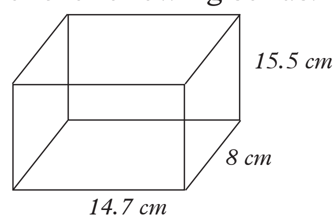
Volume of the container is 2008.9 cm^3 *(1decimal point)*

Exercise 7.12

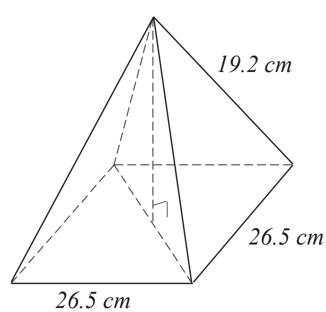
Use the appropriate formulae in this exercise.

1. Calculate the total surface area of each of the following solids. Fig 7.67

(a)



(b)



(c)

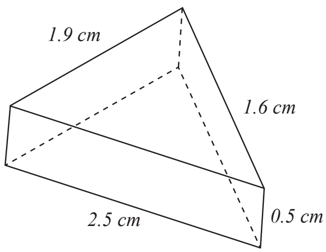


Fig. 7.67

2. Calculate the surface area of each of the solids below. Fig. 7.68.

(a)

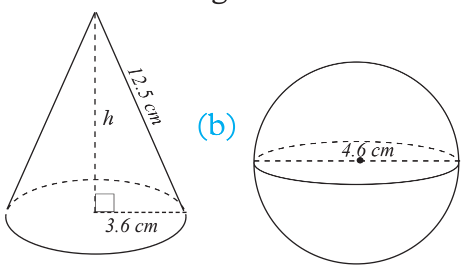
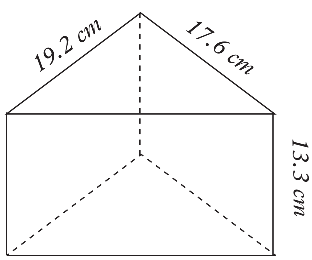


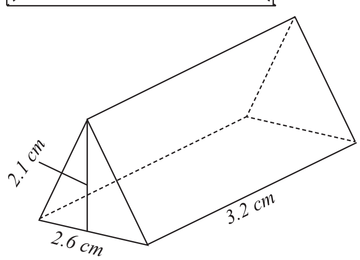
Fig. 7.68

3. For each solid in Fig. 7.69 below, calculate the volume.

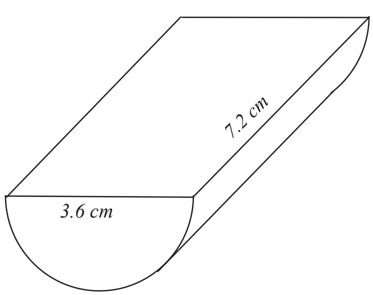
(a)



(b)



(c)



(d)

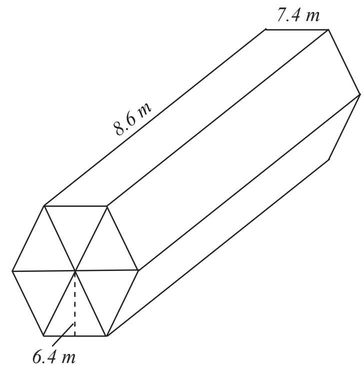


Fig. 7.69

4. A tin is 12.5 cm high and has a diameter of 8 cm. Calculate the area of the label that must be used if the overlap is 0.5 cm.

5. (a) The curved surface area of a cone is 32.64 cm^2. If the reading is 3.7 cm, find the slant height.
   (b) The curved surface area of a cone is 184.52 cm^2. If the slant height is 8.72 cm, find the radius.

6. (a) Calculate the volume of a hard ball whose radius is 3.6 cm.
   (b) How much material is needed to cover the ball?

7. A container is made by surmounting a hemisphere on a cylinder of the same radius as in Fig. 7.70.

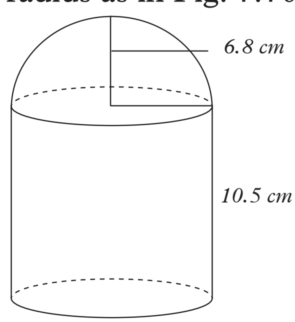


Fig. 7.70

Calculate:

(a) the surface area of the container
(b) the volume of the container.

Summary

1. Cuboid: It is a solid bounded by three pairs of identical faces which are all rectangles.
2. Cube: It is a solid bounded by six identical faces which are squares.
3. Pyramid: It is a solid figure formed with triangular slanting faces which meet at a vertex above a polygonal base.
4. Tetrahedron: It is a solid figure with four faces which are all triangular in shape.
5. Prism: It is a solid figure with identical and parallel end faces.
6. Cone: It is a solid figure which narrows to a vertex from a circular flat base.
7. Cylinder: It is a solid figure with a uniform thickness and circular ends.
8. Sphere: It is a solid figure which is entirely circular.
9. Edge: It is a line where two faces of a polyhedron meet.
10. Vertex: It is a point where edges meet.

Unit Test 7

1. A cone is surmounted on a hemisphere of radius 0.59 m. If the cone has a height of 1.23 m, calculate:

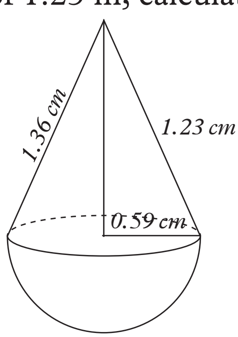


Fig. 7.71

   (a) The surface area.
   (b) The volume of the shape formed.
2. Calculate the surface area and the volume of each of the following solids:
   (a) Cylinder, radius 4.8 m, height 6.9 m
   (b) Rectangular prism, 4.3 cm by 6.8 cm by 12.8 cm
   (c) Sphere, diameter 8.25 cm
   (d) cone, diameter 12.8 cm, slant height 16.5 cm.
3. (a) The base of a rectangular tank measures 62 cm by 130 cm. If it is 40 cm high, find the amount of metal sheet required to make it.
   (b) The height of the water in the tank is 39.5 cm. If a cylindrical container of diameter 9.5 cm and a height of 20.5 cm was used to fill the tank, how many full containers were used?
4. Ice cream is sold in cylindrical containers of height 15 cm and a radius of 8 cm. A scoop of ice cream is a sphere of about 5 cm diameter. Find the number of scoops in one container.
5. A rectangular based fish aquarium has a base measuring 62 cm by 130 cm. The amount of glass needed to construct the aquarium is 23 420 cm^2. Find the height of the aquarium.
6. A classroom measures 10 m × 9 m × 4 m. Each person in the room requires 8 m^3 of air to be comfortable.
   (a) How many students should learn in the room so that everybody is comfortable?

(b) How many students should be in your school so that everyone is comfortable. Explain how you arrive at your answer.

7. Fig. 7.72 shows a conical salt-shaker which contains some salt. Calculate the volume of the salt required to fill the shaker.

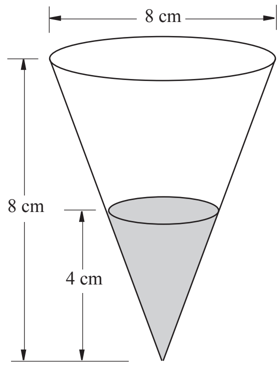


Fig. 7.72

8 STATISTICS (UNGROUPED DATA)

Key Unit Competence

By the end of this unit, I should be able to collect, to represent, and to interpret quantitative discrete data appropriate to a question or problem.

Unit outline

- Definition of data.
- Types of data.
- Collecting data.
- Frequency distribution.
- Measures of central tendency.
- Data display.
- Reading statistical graphs.
- Converting statistical graphs into frequency tables.

Introduction

In Primary 4, 5 and 6, you were introduced to the study of statistics. Everyday, we are confronted with information in the newspapers, on the television, in school, at work etc. It is often necessary to interpret that information intelligently and make decisions based on our understanding.

8.1 What is statistics?

Activity 8.1

- Discuss the word statistics and come up with a simple but comprehensive definition. You may use a mathematics dictionary or any other reference.
- What does the study of statistics involve?
- What use is it in our lives?
- Let your group secretary record and report your findings.
- Give examples of information about your class that can be classified as statistics.

From the group activity above, you should have established the following:-

1. Statistics is the study of numbers.
2. It involves collection, organization, representation, display and interpretation of numerical data.
3. It is used in making inferences, predictions and decisions based on the data available.
4. Information such as age, mass, height of the students are examples of statistics.

8.2 Types of data

Activity 8.2

- Use an English dictionary, a mathematical dictionary or search from the internet the meaning of the following terms:

  Data, qualitative data, quantitative data, continuous data and discrete data.
- As you work, ensure that you appoint a secretary for your group.
- Let the secretary record and report your findings.

- Which of the three sources of information did you find most useful?

Compare your findings with those of other groups.

From Activity 8.2, you found that:

1. Data is a set of values or observations that give raw information in an organized form. Such information is collected together for reference or analysis.
2. Qualitative data is type of data whose numerical value cannot be measured. This kind of data include attributes such as the sex of a person, race, place of birth, favourite food, smoking habit etc.
3. Quantitative data is either continuous or discrete. It is a type of data which can be measured numerically. Such data include age, mass, height, salary etc.
4. Continuous data is the type of data that may take all values within a given range. For example if x: x < 5, x is a continuous variate since it can take any value that is less than 5.
5. Discrete data is a variate or a group of items which has a finite and distinct number of possible values. Suppose our set of data is the set of even numbers between 10 and 100. This distribution has a finite number of items i.e. the set has members 12, 14, 16, 18 …………… 98.

Activity 8.3

List at least five more examples of :

(i) Qualitative data.
(ii) Quantitative data.

Activity 8.4

Use table 8.1 that shows marks scored by 40 students in a Biology test to answer the questions that follow.

78	46	55	47	77	63	52	52	62
46	77	47	40	35	67	61	58	52
42	40	48	57	66	54	75	78	75
59	75	47	59	35	62	53	72	57
51	69	55	57					

Table 8.1

(a) What is the highest score?
(b) What is the lowest score?
(c) What is the difference between the highest and the lowest scores?
(d) What mark was scored by most students?
(e) How many students scored above 70?
(f) If the pass mark was 45, how many students failed the test?
(g) How many students scored between:
 (i) 30 to 39, (ii) 40 to 49,
 (iii) 50 to 59, (iv) 60 to 69,
 (v) 70 to 79 marks?
(h) What was the general performance of the students in that test?

Points to note

1. Information such as the one in Table 8.1 is called **raw data** or simply **data**.
2. The data in Table 8.1 is both quantitative and discrete.

As you can see, there is a lot of information that is hidden in this set of numbers. Answering the provided questions reveals some of the hidden information.

Activity 8.5

1. Working in pairs, use the data in the distribution table 8.1 to discuss and answer the following questions:

   Whom do you think would be interested in such a data?
2. Working in groups, and basing your activity on the personal information available in your class about your class-mates, name five possible sets of :-
   (a) Qualitative data.
   (b) Quantitative data.

   Where possible, explain who could be interested in such data.
3. Name and discuss examples of:-
   (a) Continuous data.
   (b) Discrete data.

8.3 Collecting and organising data

Activity 8.6

Carry out the following activities:

1. Go outside your school compound to the nearest busy road. Stand by the road side.
2. Count the number of vehicles of each type that passby.
3. With a paper, a pen and a table such as the following, fill in the number of vehicles that passed by for each type in a span of two hours.

Vehicle	**Number**
Tractor	
Motorbike	
Car	
Minibus	
Lorry	
Bus	

Table 8.2

(i) What is the total number of vehicles that passed by the school gate in 2 hours?

(ii) What would you call such an activity?

(iii) Name other methods of collecting data.

Discussion

From Activity 8.6, you realise that you obtained a list of vehicles that passed by your school in two hours. This is called **data collection**.

The data was collected through **direct observation**.

Information could also be collected using an existing table that shows types of vehicles which passed at a given point over a given time interval.

This method of collecting data is reliable and accurate.

Activity 8.7

1. A detergent manufacturer wishes to introduce a new detergent in the market. To predict the expected sales, people are interviewed through a questionnaire. Four approaches are proposed for selecting people to participate in the questionnaire. Discuss the suitability of each method and choose one, explaining why you think it is the best.

i. Pick every 20th name on the electoral register of the town.

ii. Choose people as they enter a supermarket ensuring a balance in gender, age, social class, etc.

iii. Select houses at random from the town plan and interview one person per house.

iv. Choose at random, one name from each page of the telephone directory and call them up.

2. Name other three possible projects that would need to use methods similar to the ones suggested above to gather information. How would such information be used in your project of choice?

Activity 8.8

1. Identify and describe some other methods of collecting data.
2. Describe the reliability and accuracy of each method.
3. Make a list of these methods and against each method list some advantages and disadvantages.

Other methods of collecting data include interviews, prepared questionnaires and so on.

Frequency distribution table

Activity 8.9

Carry out the following activities:

1. In your exercise book, draw a table such as the one below.

Vehicle	**Tally marks**	**Totals (Frequency)**
Tractor		
Motor bike		
Car		
Minibus		
Lorry		
Bus		

Table 8.3

2. Use the list you obtained in Activity 8.6 and proceed as follows.

   (a) For every vehicle that passed, put stroke (/) in the 'Tally' column in the row corresponding to that type of vehicle.

   (b) Make every fifth stroke against a given type of vehicle cross the other four, so that instead of having five strokes as ///// we have ~~////~~. The next stroke will be next to the group of five strokes, i.e. ~~////~~, / etc. Do this for all the vehicles in your list. Record the total number of strokes against each type of vehicle in the 'Totals' column.

3. (a) Did you find the tally method easier? Explain?

   (b) State how bundling the strokes together made your work easier.

   (c) What name would you give to such a table?

Learning points

1. The tallying method of recording makes counting easier and more accurate especially when big numbers are involved.

2. Bundling strokes into fives (~~////~~) makes it easier to count them.

Activity 8.10

A group of statistics students carried out a survey in 72 major towns in a region. They wished to establish how serious the spread of HIV and AIDS was. Table 8.4 shows the number of people infected, giving their figures to the nearest 10 persons.

150	150	150	120	150	160	140	150	150
140	140	170	150	170	140	120	130	170
120	170	120	160	150	120	120	140	150
150	160	170	170	160	140	150	170	170
130	150	140	150	150	140	140	170	150
170	140	150	120	170	150	140	120	140
140	120	170	150	150	120	160	150	150
120	150	150	170	160	140	130	160	160

Table 8.4

Use the data in the table to construct a frequency distribution table.

A table such as Table 8.3 or 8.4 is called a **frequency distribution table** or simply a **frequency table**. **Frequency** means the number of times an item or value occurs. It shows the frequency of various items in a set of data. Each item in the table corresponds to a frequency of the occurrences of the item within a particular set. The table summarizes the distribution of values in the sample under discussion.

Example 8.1

Table 8.5 shows the grades scored by a class of 30 students in a Mathematics examination.

C	B	C	A	C	B	B	D	A	A
B	B	C	C	B	D	B	C	B	D
A	B	A	C	B	A	C	C	B	C

Table 8.5

Make a frequency table for the information in this table.

Solution

Table 8.6 is the required frequency table.

Grade	*Tally*	*Frequency*
A	~~////~~ /	*6*
B	~~////~~ ~~////~~ /	*11*
C	~~////~~ ~~////~~	*10*
D	///	*3*
Total		*30*

Table 8.6

Exercise 8.1

1. Collect the following data from each member of your group: height, mass, size of shoes worn, favourite subject, and favourite sport. Each group leader should record information about his/her group as shown in Table 8.7.

Name	Height (cm)	Mass (kg)	Size of shoes	Favourite subject	Favourite sport

Table 8.7

Each leader should then collect the data from other groups so that each group has data for the whole class.

2. Describe how you might collect data to fund the following. You may use more than one method.
   a. The top five soccer clubs of the year.
   b. The most popular make of car in a town.
   c. Most popular soft drink.
3. Suppose you are a businessman. You wish to set up a guest house in town.
   a. How can you collect information to help you make a decision for or against the project?
   b. What other factors would you take into consideration for the success of your project?

   Make a frequency table for each statistic except the name.
4. Record the temperature outside the classroom at the following times: 7.30 a.m, 10.30 a.m, 1.30 p.m and 4.30 p.m. This should be done everyday for a whole week (Different groups could be assigned different days).
5. Find out from the school library how many books are there in each of the following subject categories.
   (a) English
   (b) Other languages (French, German, etc.)
   (c) Mathematics
   (d) Science
   (e) Religion
   (f) Others

   What use is this information?

   Who would be interested in it?
6. On this page that you are reading, count all the words that have one letter, two letters, three letters, four letters, and five letters. Make a frequency table for this data.
7. Table 8.8 shows the amount of milk in litres produced by 36 cows in Mugisha's farm in one day.

8	10	11	9	9	4	8	18	13
16	15	12	13	4	6	11	15	4
8	7	9	13	7	10	11	8	5
14	12	9	14	9	12	5	14	14

Table 8.8

   Make a frequency distribution table for this data.

   Imagine this was your farm, is there any decision you might make based on the performance of your cows? Explain.
8. Table 8.9 shows the sizes of shoes worn by 40 students in a Senior 2 class in a local High school.
   (a) Make a frequency table for the data.

6	10	7	6	7	8	7	9	11	11
10	7	8	6	8	7	9	11	6	7
9	9	8	7	10	10	8	8	7	8
8	7	8	8	7	9	8	7	10	9

Table 8.9

   (b) Whom do you think would be interested in this information?
9. Table 8.10 shows the grades scored by 40 pupils in a mathematics examination. Make a frequency table for the data.

E	A	B	D	C	C	B	C
B	C	C	B	D	C	B	D
A	C	B	C	D	A	B	B
C	B	D	A	C	B	B	C
D	C	C	B	C	B	D	C

Table 8.10

8.4 Measures of central tendency

Activity 8.11

1. Use your mathematics dictionary to find the meaning of the following terms.
   (i) Arithmetic mean
   (ii) The median
   (iii) The mode
2. Research from the internet the meaning of terms mean, median and mode. Find also the meaning of the expression "measures of central tendency."

From the above activity, we can now summarize the methods used to determine measures of central tendency.

The most common types of averages are the mean, the mode and the median.

Averages are also called measures of **central tendency** because they show how the values in a data distribution tend to cluster around a central value.

8.4.1. The arithmetic mean

Activity 8.12

(a) Choose a group leader, collect the following data from the group; age, mass and height.

(b) Your group leader should also obtain the same data from group leaders of three the other groups.

(c) Each group should compile three data sets for the whole class.

(d) Using this data, calculate:-
- The mean age.
- The mean mass.
- The mean height.

The **mean** of a numerical set of observations (data) is defined as "the sum of the observations divided by the number of observations".

$$\text{Mean} = \frac{\text{sum of all the values in the group}}{\text{the number of all data item in the group}}$$

Thus, mean $= \dfrac{x_1 + x_2 + x_3 + x_4 + .. x_n}{n}$

this can be written as

mean $= \dfrac{\sum x}{n}$ (**where** $\sum$ means *"sum of"*)

Example 8.2

Calculate the mean of the pocket money of some 5 students who get 2 500 FRW, 4 000 FRW, 5 500 FRW, 7 500 FRW and 3 000 FRW.

Solution

$$Mean = \frac{\sum x}{n}$$

$$Mean = \frac{(2500 + 4000 + 5500 + 7500 + 3000)}{5}$$

$$= \frac{22\,500}{5}\ FRW = 4\,500\ FRW$$

Example 8.3

Table 8.11 shows the masses, in grams, of some 40 mangoes sold in a grocery shop. Calculate the mean mass of the mangoes.

70	100	90	120	110	80	110	80	100	90
120	110	120	80	110	100	110	90	110	100
80	70	100	80	90	110	110	90	100	90
100	100	90	100	90	90	80	100	100	80

Table 8.11

Solution

Method 1

Mean mass $= \dfrac{\text{sum of masses of all mangoes}}{\text{Total number of mangoes}}$

Add all the values in Table 8.11. The total mass is 3 850 g.

$\therefore$ *mean mass* $= \dfrac{3\,850}{40}\text{ g} = 96.25\text{ g}$

Method 2

Make a frequency table as shown in Table 8.12.

Mass x (g)	*Tally*	*Frequency, f*	*fx*
70	//	2	140
80	~~////~~ //	7	560
90	~~////~~ ////	9	810
100	~~////~~ ~~////~~ /	11	1 100
110	~~////~~ ///	8	880
120	///	3	360
		$\sum f = 40$	$\sum fx = 3\,850$

Table 8.12

The column 'fx' represents the total mass of mangoes. For example, if there are 2 mangoes each of mass 70 g, their total mass is (70 × 2) g = 140 g.

Similarly, there are 7 mangoes, each of mass 80 g, having a total mass of (80 × 7) g = 560 g, etc.

The symbol $\sum$ *(Greek letter 'sigma'), stands for 'sum of'.*

Thus, $\sum f$ *means 'sum of frequencies' and* $\sum fx$ *means 'sum of products of f and x'.*

The mean, denoted by $\overline{x}$*, is given by:*

$$\text{mean, } \overline{x} = \frac{\sum fx}{\sum f}$$

In the example, $\sum f = 40$, $\sum fx = 3\,850$

$$\therefore \quad \overline{x} = \frac{\sum fx}{\sum f} = \frac{3\,850}{40} = 96.25\text{ g}$$

Exercise 8.2

1. Calculate the mean of the following:

(a) 1, 3, 5, 7
(b) 2, 4, 6, 8
(c) 1, 2, 3, 4, 5
(d) 6, 7, 8, 9, 10
(e) 2, 2, 3, 4, 4, 5
(f) 3, 4, 4, 7, 8, 9
(g) 3, 9, 4, 7, 2, 8, 7, 9
(h) 10, 4, 11, 13, 15, 19, 21, 5
(i) 8, 0, 3, 3, 1, 7, 4, 1, 4

2. Calculate the mean of the following:

(a) 2.1, 1.4, 3.5, 2.7
(b) 4.8, 4.5, 3.2, 1.8, 2.2
(c) 0.7, 0.9, 0.3, 0.8, 0.7, 0.9, 0.8, 0.6, 0.5, 0.2
(d) $3\frac{1}{2}$, $4\frac{1}{4}$, $2\frac{1}{8}$, $3\frac{3}{4}$

3. Eight ladies had massses of as follows: 51 kg, 44 kg, 57 kg, 63 kg, 48 kg, 49 kg, 45 kg, 53 kg. Find the mean of their masses.

4. The mean mark scored by 5 students in a mathematics test was 19. Four students had the following scores 15, 18, 17, and 16. What was the score for 5^{th} student?

5. Table 8.13 shows the lengths, in centimetres, of a sample of 50 seedlings found in a certain seedbed.

2	7	4	3	2	6	5	5	4	6
5	7	3	5	4	7	3	4	3	2
4	6	4	7	5	3	5	5	6	3
6	5	7	6	6	4	6	6	5	7
5	4	2	7	4	3	6	5	5	7

Table 8.13

Construct a frequency table and use it to calculate the mean length of the seedlings.

Trees are very important in our environment. Let us join hands to plant as many trees as possible.

6. Table 8.14 shows the number of goals scored in a series of football matches.

Number of goals	1	2	3
Number of matches	8	8	x

Table 8.14

If the mean number of goals is 2, find the value of x?

7. In this question, letter grades are assigned the values shown in Table 8.15.

A	A$^-$	B$^+$	B	B$^-$	C$^+$	C	C$^-$	D$^+$	D
12	11	10	9	8	7	6	5	4	3

Table 8.15

Use the values in Table 8.16 to determine the mean grade for each subject as obtained in an examination by students in the school.

	A	A$^-$	B$^+$	B	B$^-$	C$^+$	C	C$^-$	D$^+$	D
Biology	20	15	26	12	16	7	4	1	–	–
English	14	9	16	11	9	3	–	–	–	–
German	1	2	6	7	10	1	3	3	–	–
French	1	1	6	9	5	5	1	7	3	5
Art	–	–	–	3	2	5	3	4	2	1
Maths	–	–	–	5	4	13	22	35	6	10

Table 8.16

8.4.2 Mode

Activity 8.13

1. Using the data in Table 8.17 below make a frequency table.

3	8	5	4	3	7	6	6	5	7
6	8	4	6	5	8	4	5	4	3
5	7	5	8	6	4	6	6	7	4
7	6	8	7	7	5	7	7	6	8
5	3	8	5	4	7	6	6	8	

Table 8.17

2. Does any of the items have a higher frequency than all the others?
3. If your answer in 2 above is yes, state the frequency, and note the corresponding entry from your activity.

From Activity 8.13, you should have observed that;

i. 6 has the highest frequency i.e. 12.

ii. The item with the highest frequency in a given set of data is called the mode.

Therefore, in this set the mode = 6

In a given data distribution, the value or the item that has the highest frequency is called the **mode** (from the French 'a la mode' meaning 'fashionable').

Example 8.4

What is the mode of: 71, 71, 72, 75, 73, 75, 76, 76, 75, 72, 78, 79, 75, 78, 79, 75, 71, 73, 75, 76?

Solution

Table 8.18 is the frequency table for the data.

Number	Frequency
71	3
72	2
73	2
75	6
76	3
78	2
7	2
	20

Table 8.18

In Table 8.18, 75 has the highest frequency (i.e. 6). Thus, 75 is the mode and 6 is the ***modal frequency****.*

Think! Is it possible to represent the mode of a set of data by means of a graph?

8. 4.3 The median

Activity 8.14

Work in the same group as before.

Use the class data you compiled earlier i.e. age, height, mass to;

(i) Arrange age in rank order – lowest to highest.

(ii) Identify the age that stands in the middle of the set.

(iii) Call the value in (ii) middle value or median value.

Repeat the process for the heights and mass.

From Activity 8.14, you notice that to get the median value, we choose a value which is centrally placed, which divides the set into two i.e. as many items below the chosen one as there are above it.

The data item in the middle position when all the data values are arranged in increasing order is acalled the **median**.

Let n be the number of data values or number of observations

(a) Median for odd number of data values

When the number of data values (n) is odd, the median is the data value in the $\frac{1}{2}(n + 1)^{th}$ position in the rank order.

(b) Median for even number of data values

When the number of data values (n) is even there are two middle data values i.e. one in the $\frac{1}{2}n^{th}$ position and the one in the $(\frac{1}{2}n + 1)^{th}$ position.

The median is half way between the two middle numbers.

The *average (mean) of these two data values give the median.*

Thus, we determine the two values, add them up then divide their sum by 2

Example 8.5

Find the median of the following numbers:

(a) 15, 12, 13, 13, 9, 10, 8, 11, 10, 8, 7, 9, 10, 10, 11

(b) 12, 8, 21, 11, 4, 12, 13, 18, 20, 19, 21, 11

Solution

Arrange the values in order from the smallest to the largest.

(a) 7, 8, 8, 9, 9, 10, 10, $\boxed{10}$, 10, 11, 11, 12, 13, 13, 15

This is the middle value
i.e. median = 10

(b) *4, 8, 11, 11, 12, 12, 13, 18, 19, 20, 21, 21*

In this case, there is no one value that is in the middle. In such a case the median is the mean of the two middle values.

Thus, median $= \frac{12 + 13}{2} = 12.5$

Note: When the number of values is **odd**, the median is the **middle value** but when the number is **even**, the median is the mean of the **two middle** values.

For odd number of values,

$$\text{median} = \left(\frac{n+1}{2}\right)^{th} \text{value or } \left(X_{\frac{n+1}{2}}\right)$$

For even number of values,

$$\text{median} = \frac{1}{2}\left(X_{\frac{n}{2}} + X_{\frac{n+1}{2}}\right)$$

Example 8.6

Table 8.19 shows the masses of some tomatoes bought from a farmer.

Mass (in g)	58	59	60	61	62	63
Frequency	2	6	12	9	8	3

Table 8.19

Find: (a) the mean, (b) the median, (c) the mode of the masses of the tomatoes.

Solution

(a) *Mean mass =*

$$\frac{(58 \times 2) + (59 \times 6) + (60 \times 12) + (61 \times 9) + (62 \times 8) + (63 \times 3}{(2 + 6 + 12 + 9 + 8 + 3)}$$

$$= \frac{2\ 424}{40} = 60.6 \text{ g.}$$

(b) *Since there are 40 tomatoes, the median mass is the mass between those of the 20*th *and 21*st *tomatoes.*

From Table 8.21, we see that there are 20 tomatoes with mass 60 g and less. The 21st *tomato must, therefore, have a mass of 61 g.*

Mass (g)	58	59	60	61	62	63
Frequency	2	6	12	9	8	3
Cumulative frequency	2	8	20	29	37	40

Table 8.20

Cumulative frequency is a running total of frequencies showing what the total frequency is at the end of each class.

The mass of the 20th *tomato = 60g, and the mass of the 21*st *tomato = 61 g.*

$\therefore$ *median* $= \frac{60 + 61}{2} = 60.5$ g.

(c) *The mode is 60 g.*

Exercise 8.3

1. Find the mean, median and mode of each of the following groups of numbers:
   (a) 4, 5, 8, 6, 4, 12, 3
   (b) 9, 9, 7, 7, 4, 13, 3, 17, 2, 21, 7
   (c) 7, 9, 4, 9, 3, 9, 10, 1, 1, 12
   (d) 5, 4, 5, 5, 4, 2, 3, 2, 1, 3, 1, 4, 0, 2, 2, 0, 1
2. Write down 5 numbers such that the mean is 6, the median is 5 and the mode is 4.
3. Seven pieces of luggage have masses of 48 kg, 45 kg, 49 kg, 63 kg, 57 kg, 44 kg and 51 kg.
   (a) Find the mean mass of the seven pieces.
   (b) If the lightest and the heaviest pieces are taken away, what is the mean mass of the remaining ones?
   (c) What is the median mass if the lightest piece is removed?

4. Table 8.21 shows lengths of some nails picked at random.

Length of nail (cm)	2	3	4	5	6	7
Frequency (f)	3	6	7	11	8	5

Table 8.21

Find: (a) the mean
(b) the mode
(c) the median length of the nails.

5. A student carried out a survey on the number of people in the cars passing at a certain point. Table 8.22 shows the data he collected.

No. of occupants	1	2	3	4
No. of cars	7	11	7	x

Table 8.22

(a) Find x if the mean number of occupants is $2\frac{1}{3}$.
(b) What is the largest possible value of x if the mode is 2?
(c) What is the largest possible value of x if the median is 2?

8.4.4 Quartiles

Activity 8.15

Begin this activity by using your dictionary to find out the meaning of the word quartile.

Consider the following marks scored by 11 students arranged in order of rank in Table 8.23.

Rank order	1	2	3	4	5	6	7	8	9	10	11
No. of marks	45	46	48	52	54	56	63	64	77	77	78

Table 8.23

i. In this set, identify the median mark.
ii. How many marks are there below the median mark?
iii. How many marks are above the median mark?
iv. Find the middle value of the lower half of the set.
v. Find the middle value of the upper half of the set.
vi. Together with the median mark, what do the middle values in (iv) and (v) do to the given data? Discuss.

From Activity 8.15 above you should have observed that:-

i. The median, M, lies at the 6^{th} position and divides the set into two halves, thus M = 56.
ii. There are 5 marks below M and 5 above M.
iii. The middle value of the lower half lies in the third position, and it is 48.
iv. The middle value of the upper half lies in the 9^{th} position and it is 77.
v. The two middle values, together with the median divide the set into four quarters.

In short, the lower middle value is called the **first (lower) Quartile** denoted by Q_1.

The middle value of the upper half is called the **third (upper) Quartile** denoted by Q_3.

Thus: M = 56

$Q_1 = 48$

$Q_3 = 77$

The difference between the upper Quartile and the lower Quartile is called the **interquartile range**.

i.e. $Q_3 - Q_1 = 77 - 48$
$= 29$

$Q_3 - Q_1$ represents a measure of spread in the ranked distribution. It describes the spread or variability of a given set of data.

Q2 and the median are the same.

Activity 8.16

Use the class data you compiled earlier in this chapter on age, height and mass to determine the following for each set of data:

i. The quartiles.

ii. The median.

iii. The interquartile range.

Compare your findings with those of other groups in your class.

Example 8.7

The heights in cm of 13 boys are:-

163, 162, 170, 161, 165, 163, 162, 163, 164, 160, 158, 153, 165.

Determine:-

(a) The median.

(b) The interquartile range.

Solution

We begin by arranging the heights in ascending rank or order.

Rank order	1	2	3	4	5	6
Height in cm	153	158	160	161	162	162

Q_1

7	8	9	10	11	12	13
163	163	163	164	165	165	170

Q_2 Q_3

Table 8.24

(a) The middle height lies in the 7^{th} position.

median = 163 cm

(b) The lower quartile lies between the third and the fourth height.

3^{rd} height = 160 cm

4^{th} height = 161 cm

∴ The lower quartile, Q_1

$$= \frac{1}{2}(160 + 161)$$

$$= \frac{1}{2} \times 321$$

$$= 160.5 \text{ cm}$$

The upper quartile, Q_3 lies between the 10^{th} and the 11^{th} height.

The 10^{th} height = 164 cm

The 11^{th} height = 165 cm

∴ The upper quartile,

$$Q_3 = \frac{1}{2}(164 + 165)$$

$$= \frac{1}{2}(329)$$

$$= 164.5 \text{ cm}$$

The interquartile range = $(Q_3 - Q_1)$

$$= 164.5 - 160.5$$

$$= 4 \text{ cm}$$

Learning points

In our example, there are 13 items.

1. To obtain median, we take $\frac{13+1}{2} = 7$. This tells us that the median is in the 7^{th} position.
2. To obtain the lower quartile, we take $\frac{13+1}{4} = 3.5$. This means the lower quartile, Q_1 is between the 3^{rd} and the 4^{th} position i.e. 3.5^{th} position.
3. To obtain Q_3, we take $(\frac{13+1}{4})3 = 10.5$. This means Q_3 lies between the 10^{th} and the 11^{th} items i.e. 10.5^{th} position.

- In general, the lower quartile, Q_1 takes the $[\frac{1}{4}(n+1)]^{th}$ position from the lower end on the rank order.
- The upper quartile, Q_3, takes the $[\frac{3}{4}(n+1)]^{th}$ position on the rank order.
- For large population, it is enough to use $(\frac{1}{4}n)^{th}$ and $(\frac{3}{4}n)^{th}$ positions for the lower and upper quartiles respectively.

Exercise 8.4

1. In a certain weather station, the annual rainfall in cm for 15 consecutive years were recorded as follows:-

   24 38 26 37 41 33 26 25 23 22 32 31 34 34 43 29

   Use the rank order to determine:-
   (a) The median.
   (b) The quartiles.
   (c) Interquartile range.
2. Table 8.25 shows marks scored by a group of students in a test.

Marks	38	39	40	41	42	43	44
f	1	2	3	1	3	3	2

Table 8.25

   Use the information to determine:-
   (a) The number of students in the group.
   (b) The median.
   (c) The upper and the lower quartiles.
   (d) The interquartile range.
3. In a given set of numbers, the mode is 5. If the numbers are 2, 5, 4, 8, 3, 6, x, 9, 5, 4, 7

   Find:-
   (a) The value of x.
   (b) The median.
   (c) The lower and the upper quartiles.
   (d) The interquartile range.

8.5 Presentation and reading of statistical graphs

Once data has been collected, they may be presented or displayed in various ways. Such displays make it easier to interpret and compare the data. The following are some of the ways.

8.5.1 Rank order list

Activity 8.17

Read and carry out the following activity.

Five pupils had the following scores in a Mathematics test: 15, 12, 21, 13, 18.

1. Rank the scores from;
   (a) the largest to the smallest.
   (b) the lowest to the biggest.
2. Define a rank order list.
3. How helpful is a rank order list?

Learning point

1. A **rank order list** is a list showing items that have been arranged in order from the highest to the lowest or from the lowest to the highest.
2. The rank order list helps us to find the:
   (a) highest value.
   (b) lowest value.
   (c) most common value.
   (d) value which is in the middle.

(e) number of those above or below a given value, etc.

(f) helps compare values easily.

8.5.2 Pictogram (or pictograph)

Activity 8.18

Carry out the following activities:

- Look at the following list which show the number of computers allocated to each class:
  1. Senior class 1 – 22 computers
  2. Senior class 2 – 16 computers
  3. Senior class 3 – 14 computers
  4. Senior class 4 – 24 computers
  5. Senior class 5 – 20 computers
  6. Senior class 6 – 12 computers
- On a piece of paper, draw 1 computer to represent four computers for each class until all classes are represented.
- Where there is a remainder of two computers, draw half of the computer.
- Each class and the computer coins drawn should be on one row. You should then have 6 rows, one for each class.
- Write a scale for your presentation i.e  represent 4 computers.

In groups,

1. What is the name given to such a presentation.
2. To present information in a pictograph;

   (a) What key components are needed?

   (b) What information can you see at a glance from the pictograph?

   (c) In which situation would you prefer this method?

   (d) List; (i) Advantages (ii) Disadvantages of using this method.

Learning point

1. The presentation above is called a **pictograph**.

   A **pictogram** (pictograph) is a diagram that represents statistical data in a pictorial form. Each picture or drawing represents a certain number or value from the data.
2. The picture to be used is chosen to represent the data subject as closely as possible, e.g. a computer to represent the number of computers, a car to represent the number of cars etc.
3. A fraction of 2 is represented by a fraction of the drawing and so on.
4. A pictogram is used to display information on posters, magazines and newspapers to attract attention. They are means of advertisements.

Example 8.8

Represent the data in Table 8.26 in a pictogram.

Size	6	7	8	9	10	11
No. of shoes	4	11	11	8	7	3

Table 8.26

Solution

Let  *represent 2 pairs of shoes. Since 4 pupils wore size 6 shoes, then 4 will be represented by*  *, etc. So the data in Table 8.25 would be represented as in Fig. 8.1.*

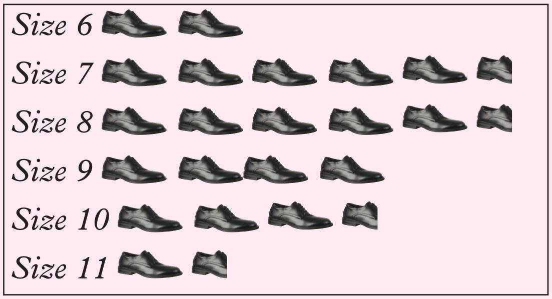
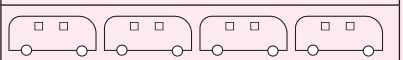
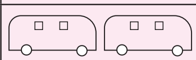
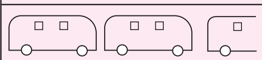
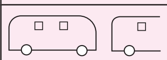


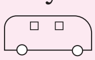
Fig. 8.1

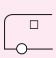
Example 8.9

In a national safari rally, there were several contestants. They all had different types of vehicles namely, Nissan, Toyota, Isuzu and Daihatsu. The following pictograph shows how many vehicles of each type were used during the safari rally.

Type of vehicle	**Icon**
Nissan	
Toyota	
Isuzu	
Daihatsu	

Key

 represents 6 vehicles.

 represents 3 vehicles.

(a) *How many Toyota participated in the rally?*

(b) *Which type of vehicle participated most in the rally?*

(c) *How many vehicles participated in the rally altogether?*

(d) *Draw a frequency table to display the information above.*

Solution

From the pictograph, we can get/obtain several information as follows:

(a) *2 full icons shows the number of Toyota vehicles that participated in the safari rally.*

1 icon = 6 vehicles.

2 icons = 2×6

= 12 Toyota vehicles participated.

(b) *Nissan had the most icons i.e 4.*

1 icon = 6 vehicles.

4 icons = $4 \times 6 = 24$

Nissan were the most type of vehicles participated in the rally.

(c) *Total number of vehicles that participated in the rally are as follows:*

Nissan = 4 icons × 6 = 24 vehicles

Toyota = 2 icons × 6 = 12 vehicles

Isuzu = $2\frac{1}{2}$ icons × 6 = $\frac{5}{2} \times 6$

= 15 vehicles

Daihatsu = $1\frac{1}{2}$ icons × 6 = $\frac{3}{2} \times 6$

= 9 vehicles

Total number of vehicles

= 24 + 12 + 15 + 9 = 60 vehicles

60 vehicles took part in the safari rally

(d) *The frequency table is shown below.*

Type of vehicle	Tally mark	Frequency
Nissani	~~////~~ ~~////~~ ~~////~~ ~~////~~ ////	24
Toyota	~~////~~ ~~////~~ //	12
Isuzu	~~////~~ ~~////~~ ~~////~~	15
Daihatsu	~~////~~ ////	9
	Total	60

Exercise 8.5

1. Table 8.27 represents the number of deaths recorded at an accident black sport over a period of 4 succesive years.

Year	1st	2nd	3rd	4th
No. of deaths	27	10	13	10

Table 8.27

(a) Choose a suitable diagram and use it to represent the data in a pitograph.

(b) What message is the pictograph trying to convey?

(c) How effective do you think the poster is?

2. The picture graph below shows the number of fruits a green grocery sold in one week.

Scale: One  represents 10 fruits

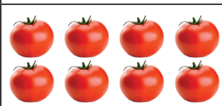
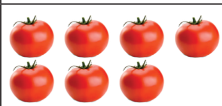
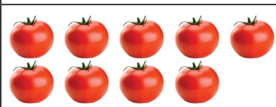
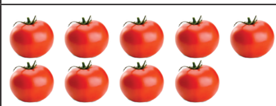
Day	Number of fruits sold
Monday	
Tuesday	
Wednesday	
Thursday	
Friday	
Saturday	
Sunday	

Table 8.28

(a) Which day(s) did the green grocery sell most fruits?

(b) Which day(s) did the green grocery sell the least fruits?

(c) Which day(s) did the green grocery sell the same number of fruits?

(d) Draw a frequency table from the given pictogram?

3. In an inter-houses ball games competition the following goals were scored as shown in the picture graph below. One ball represents one goal.

Scale:  represents 1 goal

House	**Number of goals scored**
Green House	
Red House	
Blue House	
Yellow House	
Orange House	
Purple House	
Brown House	
Violet House	
White House	

Table 8.29

(a) Which house scored most goals?

(b) Which house scored the least goals?

(c) Which house scored the same number of goals?

(d) Find the total number of goals scored.

(e) Use the pictograph to draw a frequency table.

4. Eighty five students were asked how they travelled to school everyday. The following information was recorded.

Means of Transport	Car	Bus	Walking	Bicycle
No. of people	30	40	10	5

Table 8.30

Draw a pictograph to represent this data. Use the scale of 1 person to represent 5 people.

5. 30 pupils were asked what they were writing with. Their responses were recorded in a table as follows

Writing instrument	Black pen	Blue pen	Pencil
No. of people	12	9	9

Table 8.31

(a) Choose a suitable diagram and use it to represent the data in a pictograph.

(b) What does one diagram represent?

8.5.3 Pie-chart

Activity 8.19

Discuss and answer the following questions.

1. What are the key components needed when drawing a pie chart?
2. What is the procedure of getting the right sectors in a pie-chart?
3. Describe the procedure of constructing the chart.
4. What information can you identify on the chart at a glance?
5. In which situation would you prefer to display information by this method?
6. Are there any advantages or disadvantages in using this method?
7. Illustrate this method using a simple example.

Learning point

A pie-chart is a graph or a diagram in which different proportions of a given data distribution are represented by sectors of a circle.

The diagram is looked at as a circular 'pie', hence the name pie chart.

Example 8.10

Table 8.32 shows grades scored by 15 candidates who sat for a certain test.

Grade	*A*	*B*	*C*	*D*	*E*
No. of candidates	*2*	*5*	*4*	*1*	*3*

Table 8.32

Draw a pie chart for this data.

Solution

Work out the fractions of numbers of candidates who scored each grade. For example, for grade A we have $\frac{2}{15}$.

Since the angle at the centre of a circle is 360°, we calculate the angle to represent grade A as $\frac{2}{15}$ *of 360°*

i.e. $\frac{2}{15} \times 360° = 48°$.

A is represented by an angle of 48°.

Table 8.33 shows all the angles.

Grade	No. of candidates	Fraction of total	Angle at the centre of the circle	%
A	2	$\frac{2}{15}$	$\frac{2}{15} \times 360° = 48°$	13.3%
B	5	$\frac{5}{15}$	$\frac{5}{15} \times 360° = 120°$	33.30%
C	4	$\frac{4}{15}$	$\frac{4}{15} \times 360° = 96°$	26.7
D	1	$\frac{1}{15}$	$\frac{1}{15} \times 360° = 24°$	6.7%
E	3	$\frac{3}{15}$	$\frac{3}{15} \times 360° = 72$	20%

Table 8.33

Fig. 8.2 shows the required pie chart.

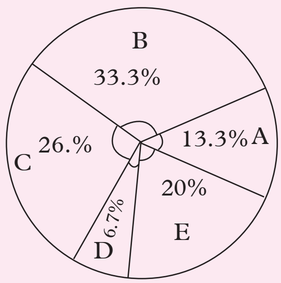


Fig. 8.2

Example 8.11

After selling fruits in a market, Asha had a total of 144 fruits remaining. The pie chart below shows each type of fruit that remained.

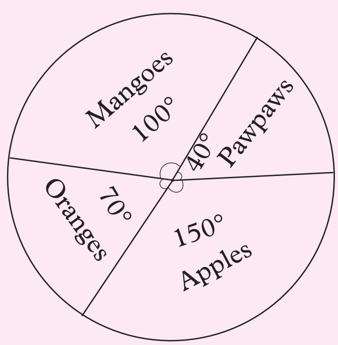


Fig. 8.3

(a) *Find the total cost of mangoes and pawpaws remaining if a mango sells at 30 FRW and a pawpaw at 160 FRW each.*

(b) *Which type of fruit remained the most?*

(c) *What was the median number of fruit that remained?*

(d) *Draw a frequency table to display the information on the pie chart.*

Solution

(a) Numbers of mangoes $= \frac{100}{360} \times 144$

$= 40$ mangoes

Number of pawpaws $= \frac{40}{360} \times 144$

$= 16$ pawpaws

Cost of mangoes $= 40 \times 30$

$= 1\ 200$ FRW

Cost of paw paws $= 16 \times 160$

$= 2\ 560$ FRW

Total cost $= 1\ 200 + 2\ 560$

$= 3\ 760$ FRW

(b) Apples remained the most unsold i.e

$\frac{150}{360} \times 144 = 60$ apples

(c) Median number of remaining fruit

Mangoes $= 40$

Pawpaws $= 16$

Apples $= 60$

Oranges $= \frac{70}{360} \times 144 = 28$

Median $= 16, 28, 40, 60$

$= \frac{28 + 40}{2}$

$= \frac{68}{2}$

$= 34$ fruits

(d)

Type of fruit	Frequency (number remaining)
Mangoes	40
Oranges	28
Pawpaws	16
Apples	60
Total	144

Table 8.34

Learning points

1. Usually there are no numbers on a pie chart beside the angles sizes.
2. The sizes of the sectors give a comparison between the quantities represented.
3. The order in which the sectors are presented does not matter.
4. Sectors may be shaded with different patterns (or colours) to give a better visual impression.

8.5.4 Bar chart/Bar graph

A **bar chart** (or bar graph) is a graph consisting of rectangular bars whose lengths are proportional to the frequencies in a data distribution.

Activity 8.20

Answer the following questions.

1. Name the key components needed to draw a bar-graph.
2. What information can the length of each bar display at a glance?
3. What information stands out on the graph at a glance?
4. In which situation would you use this method of display?
5. List the advantages and the disadvantages of this method.
6. Create a simple example to illustrate this method.

Steps in drawing a bar chart

To draw a bar chart

- First identify what quantities to represent of the on the horizontal and vertical axis.
- Determine the most suitable scale to use for frequency quantity.
- Determine the suitable width and uniform gap between the bars. The bars should have the same width and uniform gap between them.
- Draw the bar chart and shade them with a visible color.

Example 8.12

Table 8.35 shows the sizes of sweaters worn by 30 Senior 1 students in a certain school.

Size	Small (S)	Medium (M)	Large (L)	Extra large (XL)
No. of pupils	5	13	8	4

Table 8.35

Represent the data on:

(a) a horizontal bar chart

(b) a vertical bar chart.

Solution

(a) Horizontal bar chart Fig. 8.4(a).

In a horizontal bar chart, frequency is represented on the horizontal axis.

Bars are drawn with spaces between them (as in Fig, 8.4(a)) and they may be shaded or not.

Fig. 8.4 shows the required diagram

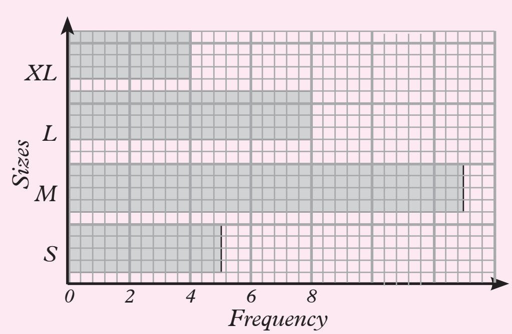


Fig. 8.4(a)

(b) *Vertical bar chart Fig. 8.4(b).*

In a vertical bar graph, frequency is represented on the vertical axis.

Fig. 8.3 shows the required diagram

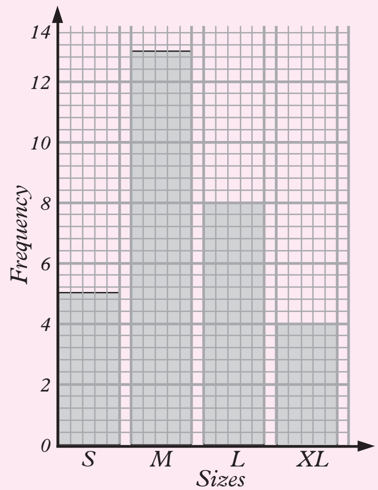


Fig. 8.4 (b)

Example 8.13

The bar graph shows the number of athletes who represented five African countries in an international championship.

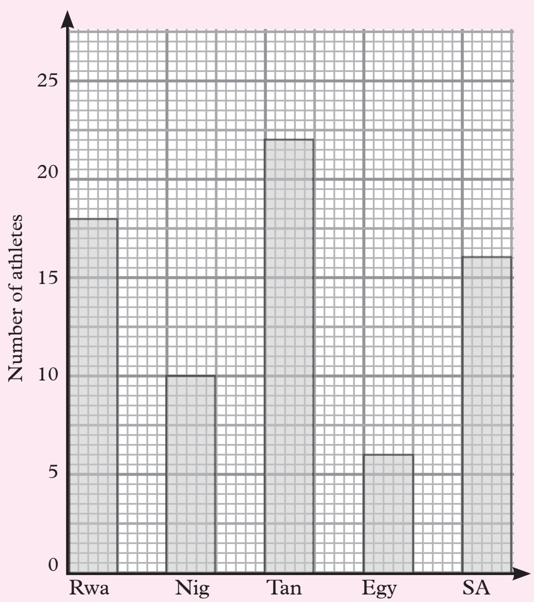


Fig. 8.5

Key

Rwa – Rwanda

Nig – Nigeria

Tan – Tanzania

Egy – Egypt

SA – South Africa

(a) *What was the total number of athletes representing the five countries?*

(b) *What was the smallest number of athletes representing one country?*

(c) *What was the most number of athletes representing a country?*

(d) *Represent the information on the graph on a frequency table.*

Solution

Total number of athletes are:

(a) *18 + 10 + 22 + 6 + 16 = 72 athletes*

(b) *6 athletes*

(c) *22 athletes*

(d)

Country	*Number of athletes*
Rwanda	*18*
Nigeria	*10*
Tanzania	*22*
Egypt	*6*
South Africa	*16*
Total	*72*

Table 8.36

Learning points

In a bar chart:

1. The widths of the bars are the same.
2. The height of a bar is proportional to the corresponding frequency.

3. The bars can be placed vertically or horizontally provided the bars do not touch. A bar chart drawn vertically is also called column bar chart.

Exercise 8.6

1. In a survey on soft-drinks, 180 people were asked to state the brand they preferred. 35 chose brand A, 30 chose brand B, 100 chose brand C and 15 chose brand D. Draw a pie-chart to display this information.
2. At the semi-final stage of a football competition, 72 neutral observers were asked to predict which team they thought would win. Table 8.37 shows their predictions.

Team	No. of predictions
Team A	9
Team B	40
Team C	22
Team D	1

Table 8.37

Draw a pie-chart to display the predictions.

3. A farmer has 48 hectares of land. The circle graph below shows how the farmer made use of her land.

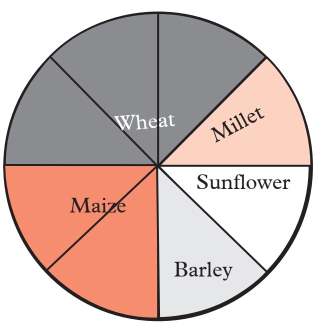


Fig. 8.6

(a) What fraction of land is under:-
   (i) wheat (ii) sunflower
   (iii) millet (iv) barley
   (v) maize
(b) How many hectares of land are under:-
   (i) wheat (ii) sunflower
   (iii) millet (iv) barley
   (v) maize
(c) Which crop has made most use of the land?
(d) Display the information on a frequency table.

4. Mr. Onani has a monthly income of 12 000 FRW. The pie-chart in Fig. 8.7 shows how he spends the money.

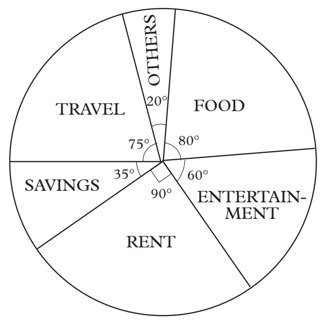


Fig. 8.7

How much does he spend on:
(a) Food (b) Rent (c) Savings
(d) Entertainment (e) Travel?
What do you think the sector marked others represent.

5. Daliso scored the following marks in her end of term examination.
English 72, Biology 78, Mathematics 94, Physics 84 and History 86.
(a) Draw the table to illustrate this information.
(b) Draw a bar graph to represent the given information.

(c) (ii) What is the difference between the highest and the lowest marks?

(ii) What is the total marks Daliso scored in the examination?

6. During a certain month, a wholesale trader sold 80 bags of maize, 40 bags of beans, 55 bags of millet, 35 bags of sorghum, 55 bags of wheat and 70 bags of rice.
   (a) Draw a table to represent the information above.
   (b) Use the table drawn in (a) above to draw a bar graph representing the given information.
   (c) Which commodity sold the
   (i) most number of bags?
   (ii) least number of bags?
   (d) Which two commodities sold the same number of bags?
   (e) What was the total number of bags sold that month?
   (f) How many more bags of maize were sold than those of millet?

7. The pie chart below represents how John uses his money every month. He earns 96 000 FRW.

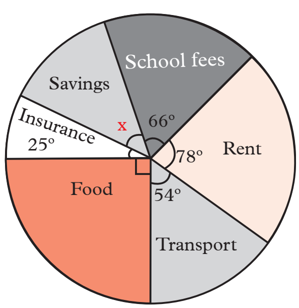


Fig. 8.8

   (a) Find the value of x.
   (b) Find the fraction of each item of spending.
   (c) Calculate the money spent on each item.
   (d) On which item did he spend most?
   (e) On which item did he spend least?

8. The bar graph below represents the number of eggs collected in five days. Use it to answer the questions that follow.

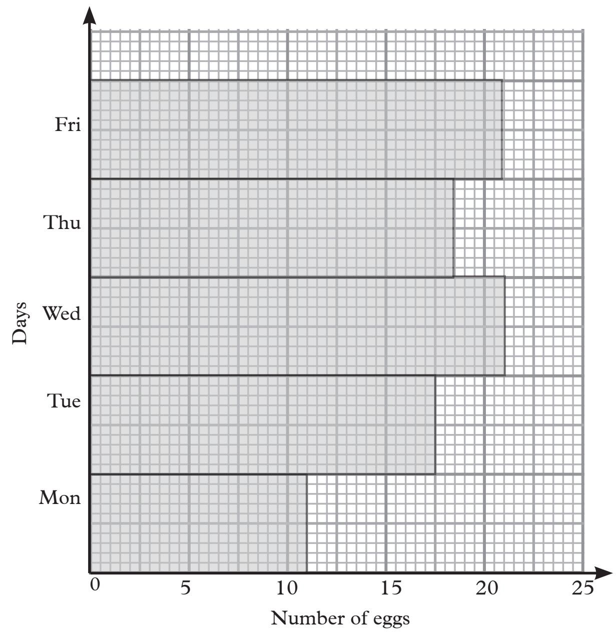


Fig. 8.9

   (a) What scale has been used on the horizontal axis?
   (b) Which day was:
   (i) The highest number of eggs collected?
   (ii) The least number of eggs collected?
   (c) Find the mean daily collection of eggs for that week.
   (d) If one egg was sold for 60 FRW, what was the total amount of money collected that week?
   (e) Represent the data in the bar chart on a frequency table.

9. The table below shows the rainfall in millimetres recorded in a weather station for one year.

Month	J	F	M	A	M	J
Rainfall (mm)	20	30	70	125	110	100
	J	A	S	O	N	D
	75	60	40	75	105	90

(a) Using a vertical scale of 1 cm to represent 20 mm and a horizontal scale of 0.5 cm to represent one month, draw a bar graph to represent this information.

(b) What is the difference between the sum of the three wettest months and the sum of the three driest months?

(c) What is the mean rainfall in millimetres?

(d) In which two months was the same amount of rainfall recorded?

10. The bar graph below shows how far it is by road from Gicumbi to nine other towns in Rwanda. Study the bar graph and answer the questions that follow.

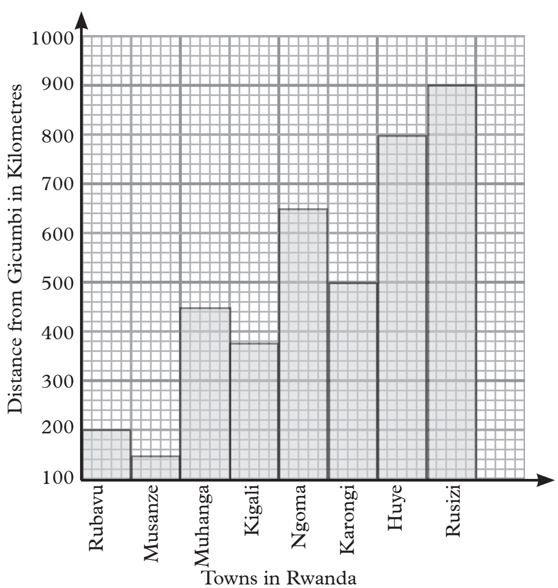


Fig. 8.10

(a) On which axis of the bar graph is distance shown?

(b) On which axis of the graph are towns shown?

(c) How far is it from Gicumbi to Rusizi?

(d) Write down the names of the towns on the bar graph which are over 500 km from Gicumbi?

(e) If someone makes a return journey from Gicumbi to Karongi, what is the total distance the person will have to cover?

(f) Which is the farthest town on the bar graph from Gicumbi?

(g) Draw a frequency table to display the information on the chart?

11. Of the animals on Sigele's farm, 35% are cows, 20% are goats, 15% are sheep, 2% are donkeys and 28% are pigs. Find the angle representing each type of animal. Draw a pie chart to illustrate this information.

8.5.5 Line graph

Activity 8.21

Read, discuss and answer the following questions.

When data relates two measures, each set of data can be shown by a point on a graph. The points are then joined to form a broken-line graph.

1. What are the key components needed in order to draw a line graph?
2. Is any information obvious on the graph?
3. What benefit would you get by using this method rather than any other?

4. List some advantages and disadvantages of using this method.
5. Create a simple example and use it to illustrate this method.

Example 8.14

Table 8.36 shows the temperature in degrees Celsius, observed at 2-hourly intervals of a patient who was admitted at a hospital.

Time	6 a.m	8 a.m	10 a.m	12 noon	2 p.m	4 p.m
Temp °C	38.2	38.6	38.9	38.8	38.8	38.5

Time	6 p.m	8 p.m	10 p.m	mid-night	2 a.m	4 a.m
Temp °C	38.2	37.8	37.1	37.0	36.8	36.8

Table 8.38

(a) Plot a graph of time against temperature.

(b) Estimate the patient's temperature at:

(i) 9 a.m (ii) 3 p.m

(iii) 11 p.m (iv) 3 a.m

(c) What can you say about the patient's temperature;

(i) during the day?

(ii) during the night?

Solution

(a) When a graph of time against temperature is plotted using the data in Table 8.38 and the points joined by dotted line segments, Fig. 8.11 is obtained.

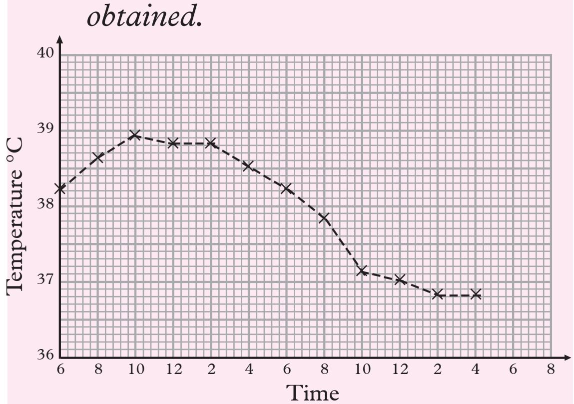


Fig. 8.11

(b) (i) 38.4°C (ii) 38.6°C

(iii) 37.1° C (iv) 36.8° C

(c) (i) The temperature rises during the day in the morning hours and in the afternoon it starts to fall down.

(ii) During the night, the temperature falls back to normal temeperature.

A graph, such as Fig. 8.11 which is formed by line segments joining the points representing given data is known as a **line graph**.

Why are points joined by dotted lines? When could such points be joined by continuous line segments?

Example 8.15

The following line graph shows the relationship between the price of a kilogram of meat and the quantity of meat. Use it to answer the questions that follow.

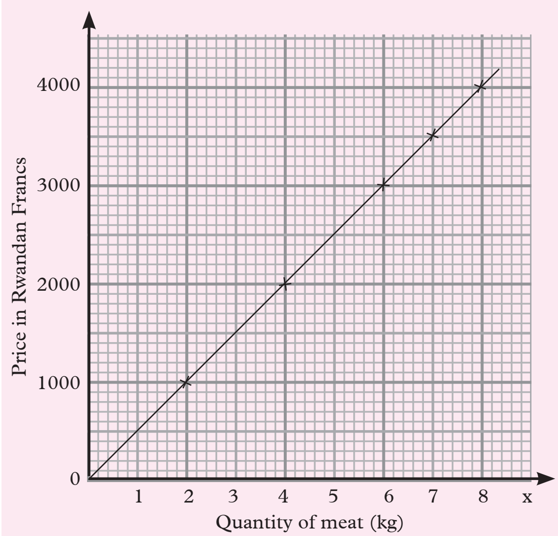


Fig. 8.12

From the graph:

(i) What is the price of:

(a) 1 kg of meat?

(b) 5 kg of meat?

(ii) Draw a frequency table to show the quantity of meat and price per quantity.

Solution

From the graph:

(i) (a) 1kg of meat costs 500 FRW.

(b) 5kg of meat costs 2 500 FRW.

(ii)

Quantity (kg)	1	2	3	4	5	6	7	8
Price (RWF)	500	1000	1500	2000	2500	3000	3500	4000

NOTE:
A line graph helps us appreciate the pattern or trend of a given variable i.e. how the variable changes with time.

Activity 8.22

In a certain season, Jane was the top scorer in the National Women's Basket Ball Team. Her scores for the season are tabulated in the table below.

Game	Number of points
1	9
2	12
3	18
4	20
5	16
6	7
7	10
8	24
9	15
10	19

Table 8.39

(a) Represent the information using 4 different methods you have learned.

(b) Which graph is most appropriate? Explain.

(c) In which game did Jane perform;
(i) her best (ii) her worst?

(d) Which graph is most helpful in answering (c)? Why?

(e) What percentage of points were scored in the first half of the season?

(f) Did she do better in first or second half of the season?

(g) Which graph is most useful in helping you answer (e) and why?

(h) Calculate the average number of points scored per game.

This activity emphasizes the need to use a method that is appropriate for the given data. This helps to ease interpretation of the graph and make appropriate decisions and inferences.

Activity 8.23

A survey was done in spending habits of some 56 tourists. The average amount spent per day by each is tabulated below:

98	72	52	91	48	51	64	83	69
61	61	55	56	87	82	86	77	58
65	64	73	92	92	59	96	77	88
67	86	77	67	73	66	85	71	61
42	91	68	69	56	63	81	57	74
76	73	66	84	77	76	73	58	62
61	58							

(a) What percentage of the tourists spent more than (i) \$ 60, (ii) \$ 80 per day?

(b) If you are planning for a holiday to the same destination, what would you make your budget?

What factors have you taken into consideration?

What assumptions have you made?

(c) Which method is appropriate to represent information in the table above? Explain why or why not.

Exercise 8.7

1. The average masses of pupils of different ages in a certain school were obtained and recorded as in Table 8.40.

Age (years)	11	12	13	14	15	16	17
Average mass (kg)	36.7	38.6	42.4	46.8	51.9	60.4	65.6

Table 8.40

(a) Represent this data on a line graph.

(b) What is the estimate for the average mass of pupils who are:

(i) $11\frac{1}{2}$ years, (ii) $13\frac{1}{2}$ years,

(iii) $15\frac{1}{2}$ years old?

2. Table 8.41 shows the population of a certain country, in thousands, between the year 1901 to 1966.

Year	1901	1911	1921	1931	1941	1951	1961	1966
Pop.	5 400	7 200	8 800	10 400	11 500	14 000	18 200	20 000

Table 8.41

(a) Draw a line graph for the data.

(b) Estimate the population in:

(i) 1926 (ii) 1971.

3. The line graph below shows bags of cement produced by cement factory in a minute.

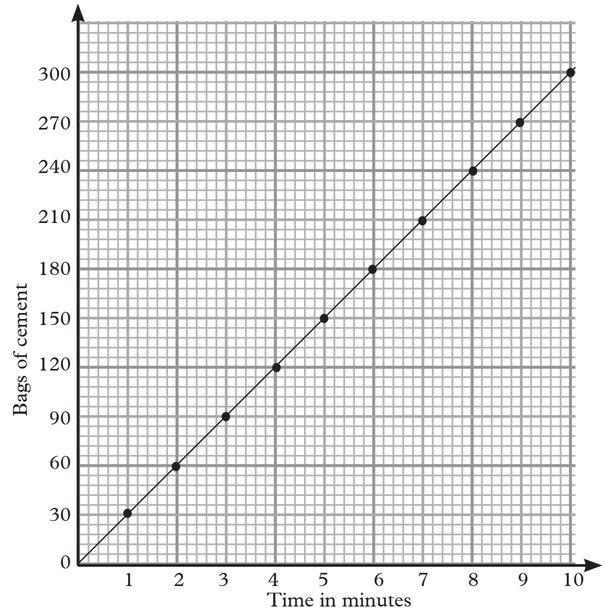


Fig. 8.13

(a) Find how many bags of cement will be produced in:-

(i) 8 minutes

(ii) 3 minutes 12 seconds

(iii) 5 minutes

(iv) 7 minutes 24 seconds

(v) 10 minutes

(b) Find how long will it take to produce:-

(i) 78 bags of cement

(ii) 120 bags of cement

(iii) 300 bags of cement

(iv) 54 bags of cement

(v) 246 bags of cement

(c) Draw a frequency table to show the number of bags produced and the time taken.

4. Table 8.42 shows, the maximum and minimum temperatures in degrees, recorded for the first 12 days of September in a certain town.

Day	1	2	3	4	5	6	7	8	9	10	11	12
Max (°C)	26	21	17	23	26	21	20	21	19	19	19	17
Min (°C)	15	13	12	13	13	14	16	11	11	10	10	12

Table 8.42

(a) Using the same axes, draw line graphs to represent these temperatures.

(b) Which of the two sets of temperatures show the greater variation?

(c) If the temperatures for 5th and 11th September had been omitted, could you have estimated them?

(d) Using the trends shown by the graph, make a forecast for 13th September.

5. Table 8.43 shows deaths, in thousands, from two diseases during a period of 10 years.

Year	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956
TB deaths	23.9	22.9	17.5	14.1	12	9.3	7.9	7.1	5.8	4.9
Pneu-monia deaths	26.7	20.7	23.6	20.3	25.6	21	23	20.4	23.8	24.8

Table 8.43

(a) On the same axis, draw line graphs to represent this data.

(b) What can you say about deaths from each disease?

6. The travel graph below shows the journey made by a traveller between 6 am and 12 noon.

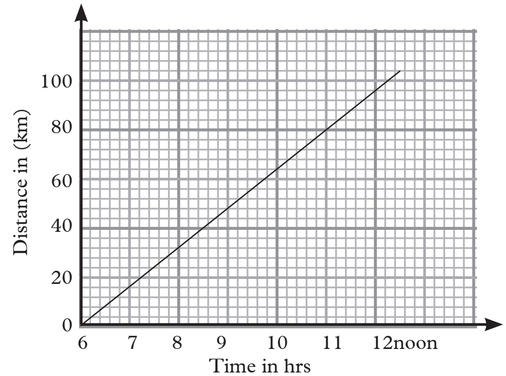


Fig 8.14

Use the graph to answer the following questions.

(a) Find:

(i) The vertical scale.

(ii) The horizontal scale.

(b) How far had the traveller moved by 8 am?

(c) At what time had she covered 92 km?

7. Table 8.44 shows the time taken by a certain car test to accelerate from 0 to the given speeds.

Acceleration	Time taken (s)
0 to 50 km/h	4.8
0 to 65 km/h	6.4
0 to 80 km/h	9.0
0 to 95 km/h	12.3
0 to 110 km/h	16.1
0 to 125 km/h	21.4

Table 8.44

(a) Represent the data on a line graph with speed on the horizontal axis.

(b) How long would the car take to accelerate from:

(i) 0 to 70 km/h,

(ii) 0 to 120 km/h?

(c) What speed would the car gain from rest in: (i) 8 s, (ii) 14 s?

8. Table 8.45 shows data from students who participated in blood transfusion at Maera Secondary School in a particular year.

	Month 1	Month 2	Month 3	Month 4
Boys	4	8	9	10
Girls	6	5	11	12

Table 8.45

(a) What type of graph would you draw using this data?

(b) Draw the graph for the data.

(c) Use the graph to determine.

(i) The least number of students who participated in the blood transfusion.

(ii) The largest number of students who participated in the blood transfusion.

(iii) The mean number of students who participated in the blood transfusion.

8.5.6 Histogram and frequency polygon

Activity 8.24

From a mathematics dictionary, find the meaning of the terms histogram and frequency polygon.

Search from the internet the meaning of the two terms and compare your findings.

Can you think of any other source where you might find this information?

Distinguish between a histogram and a bar graph.

Learning point

A **histogram** is a diagram used to represent frequency distribution in an ungrouped data. A histogram resembles a bar graph or bar chart with the bars touching one another.

A **frequency polygon** is a line graph drawn by joining all the midpoints of the top of the bars of a histogram.

Example 8.16

Table 8.46 – shows how a group of 50 students preformed in a maths quiz marked out of ten.

Mark	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
f	*2*	*2*	*11*	*11*	*12*	*7*	*4*	*1*	*0*	*0*

Table 8.46

(a) Using a suitable scale, say vertical scale 2cm: 4 units and horizontal scale. 1cm: 1unit, represent this information on a histogram.

(b) On the same graph, draw a frequency polygon.

Solution

Since this set represents discrete elements which are ungrouped, all the rectangles will be of the same width, and the frequencies will be represented by the heights of the rectangles.

Figure 8.15 shows the histogram and the frequency polygon for the frequency distribution in Table 8.46.

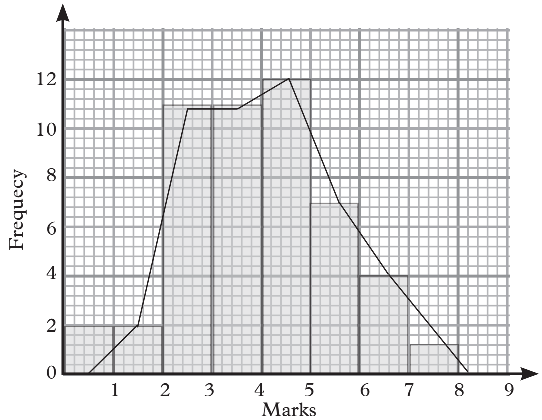


Fig. 8.15

Learning point

The graph obtained by joining the midpoints of the top of the consecutive bars is called a frequency polygon.

To complete the polygon, join the midpoint of the first bar to the bottom left hand corner of the bar and the midpoints of the last bar to the bottom right hand corner of that bar.

Example 8.17

The masses of a group of students are measured to the nearest kilogram and masses recorded as in table 8.47 below

Mass (kg)	57	58	59	60	61	62	63	64	65	66	67
Frequency	20	36	44	46	39	30	22	17	16	4	2

Table 8.47

(a) *Construct a histogram to represent the data.*

(b) *Use your graph to estimate the mode.*

(c) *State the range of the distribution.*

(d) *Draw the frequency polygon represented by the histograph.*

Solution

Using a scale vertical scale 2cm:10 students

Horizontal scale 2cm:10

(a) *Fig. 8.16 shows the required histogram.*

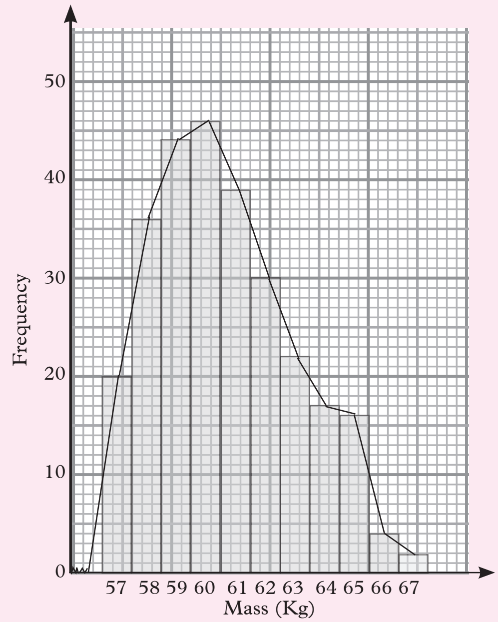


Fig. 8.16

(b) *To estimate the mode graphically, we identify the bar that represents the highest frequency.*

The mass with the highest frequency is 60 kg. It represents the mode.

(c) *The highest mass = 67 kg*

The lowest mass = 57 kg

$\therefore$ *The range = 10 kg*

(d) *The graph joining the midpoints of top of the bars is the* ***frequency polygon****. i.e Fig. (See fig 8.16 above)*

Example 8.18

The following histogram (fig. 8.17) shows the monthly wages (in FRW) of workers in a certain factory.

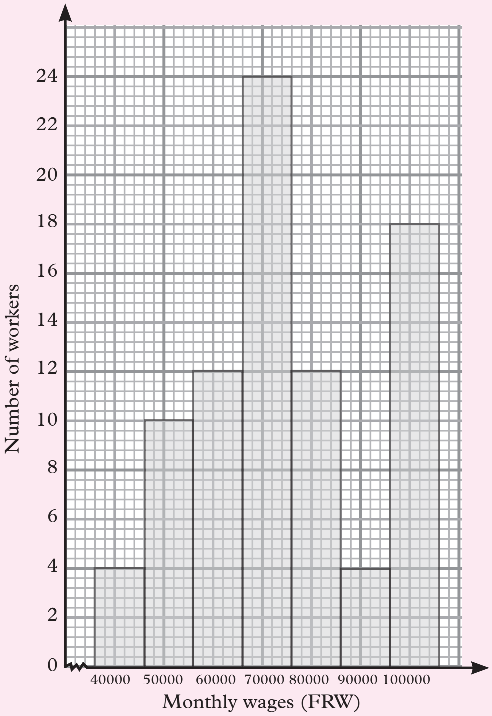


Fig. 8.17

(a) *Find the largest number of workers getting a similar wage.*

(b) *Find the least wage and the highest wage with the number of workers earning in each.*

(c) *How many workers get a monthly wage of 85 000 FRW or less?*

(d) *Draw a frequency table from the histogram showing the monthly wages and the number of workers.*

Solution

(a) *The largest number of workers getting a similar wage is 24. They get wages between 65 000 – 75 000 FRW.*

(b) *The least wage is between 35 000 FRW and 45 000 FRW with 4 workers getting that. The corresponding figures for highest wage is between 95 000 FRW and 105 000 FRW and 18 workers get that.*

(c) *60 workers get a wage of 80 000 FRW or less as*

(35 000 - 45 000) FRW - 4 workers

(45 000 - 55 000) FRW -10 workers

(55 000 - 65 000) FRW -12 workers

(65 000 - 75 000) FRW -24 workers

(75 000 - 85 000) FRW -12 workers

Total = 62.

Thus, 62 workers earn 85 000 FRW and below.

(d)

Monthly wage (FRW)	Number of workers
40 000	4
50 000	10
60 000	12
70 000	24
80 000	12
90 000	4
100 000	18
Total	84

Exercise 8.8

Use the table to answer the questions 1 - 4.

Table 8.42 below shows marks scored out of 100 by a group of students.

26	64	76	87	35	76	35	64	55	76
55	76	46	92	64	76	55	46	64	40
26	87	64	64	46	55	64	26	55	87

Table 8. 48

1. (a) Make a frequency distribution table.

   (b) Construct a histogram for the data.

   (c) State the mode of the data.

(d) Construct a frequency polygon for the data.

2. Use the histogram in question 1 to find:

   (a) The number of students who scored 80% and above.

   (b) The number of student who failed if the highest failing mark was 50%?

3. Estimate:

   (a) the mean.

   (b) the median of the data.

   Which of the two values best represents the data? Give reasons for your answer.

4. Use a graphical method to estimate the mode.

5. Interpret the data represented by the histogram in fig. 8.10 by answering the following questions:

   (a) Which shirt size was sold the least?

   (b) Which shirt size was sold the highest?

   (c) How many shirts were sold up to the shirt size 43?

   (d) How many shirts of size 43-46 were sold?

   (e) Draw a frequency table to show the sizes of the shirts sold and the number of each shirt size sold.

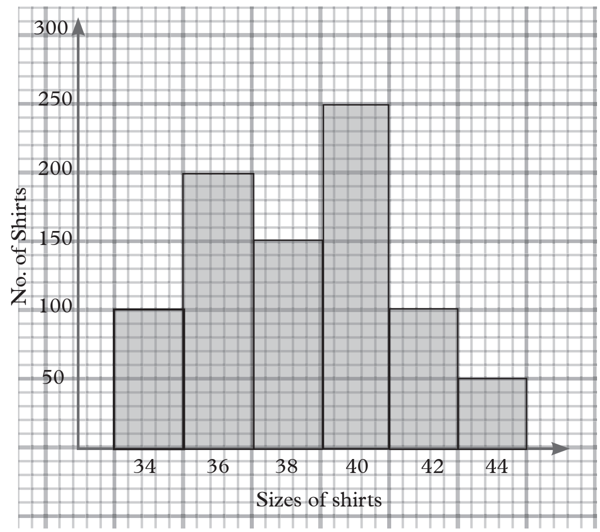


Fig. 8.18

6. The histogram below shows the daily earnings of 100 shopkeepers. Use it to answer the questions that follow.

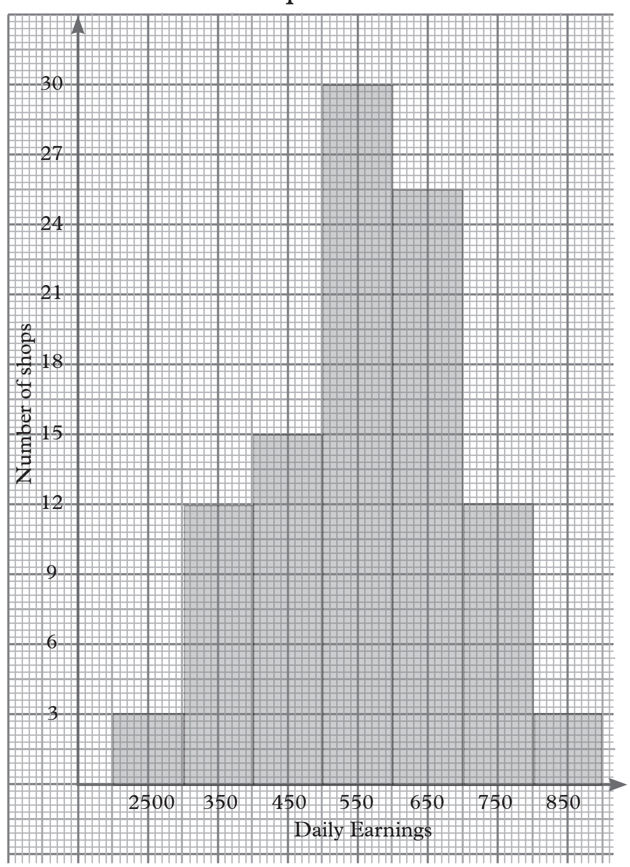


Fig. 8.19

(a) Find the least number of shopkeepers earning a similar amount.

(b) Find the highest and the least earning with numbers of shopkeepers earning each week.

(c) Find the number of shopkeepers earning a similar amount.

(d) Draw a frequency distribution table to show the daily earning and the number of shops in each case.

(e) Find their average earning for all the shopkeepers.

8.5.7 Cumulative frequency diagram

Activity 8.25

1. The distribution Table 8.49 below shows a group of children whose ages are recorded to the nearest year.

Age (yrs)	7	8	9	10	11	12	13
f	0	10	24	13	6	4	3

Table 8.49

2. Add another column in the table which shows running totals of the frequencies i.e.

   For the age 7, total = 0

   For the age 8, total = 0 + 10,

   For the age 9, total = 0 + 10 + 24

   and so on.

3. What name would you call the running totals?

4. Draw a graph by plotting age against the corresponding running totals i.e. (7,0) (8,10), (9,34), (10, 47), (11, 53), (12, 57) and (13, 60).

5. What is the name of such a graph as drawn in step 4?

Learning points

- The running totals are known as the cumulative frequencies (cf).
- From Activity 8.25, you should have obtained a cumulative frequency table as the following:

Age (yrs)	frequency (f)	Cumulative frequency	Age range represented by cf
7	0	0	up to 8 years
8	10	10	up to 9 years
9	24	34	up to 10 years
10	13	47	up to 11 years
11	6	53	up to 12 years
12	4	57	up to 13 years
13	3	60	up to 14 years

Table 8.50

- The graph you plotted is called a cummulative frequency graph and should be as follows:

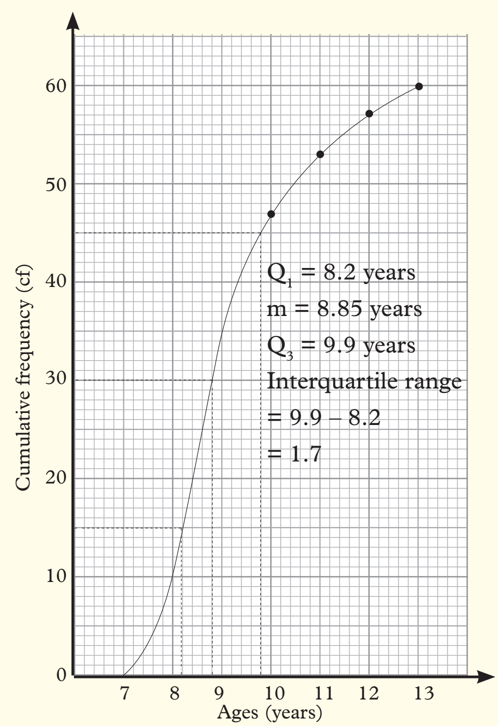


Fig. 8.20

Activity 8.26

Use the graph you drew in Actvity 8.25 to find age when cf = 30.5; cf when age = 8.2 ; and age when cf = 45.6.

From your graph we can locate the following:-

The median is in position 30.5, and corresponds to age 8.85. Thus median ≈ 8 years 10 months, lower quartile is in 15.5^{th} position.

When cf is 15.3, age = 8.2 years.

∴ Lower quartile ≈ 8 years 2 months

Upper quartile is in 45.5^{th} position

When cf is ≈ 45.5, age = 9.9 years

∴ upper quartile = 8 years 11 months

Example 8.19

Table 8.51 shows the length of a metal bar according to the estimates by a group of 32 students.

Length (cm)	35	36	37	38	39	40	41	42
f	1	3	4	8	6	5	3	2

Table 8.51

Find:

(a) The range of these estimates.

(b) Draw a cumulative frequency table and plot a cumulative frequency curve from this information.

(c) From the curve, estimate the median and the interquartile range.

(d) Use the information in table 8.51 to construct a histogram.

(i) Use the histogram to estimate the mode.

(ii) On the histogram, construct the frequency polygon.

Solution

(a) The longest metal bar is 42 cm.

The shortest metal bar is 35 cm.

∴ The range = 42 – 35

= 7cm

(b) Table 8.52 and Fig. 8.10 are the required cumulative frequency table and cumulative frequency curve.

Length (cm)	f	cf
35	1	1
36	3	4
37	4	8
38	8	16
39	6	22
40	5	27
41	3	30
42	2	32

Table 8.52

Cumulative frequency curve Fig. 8.21

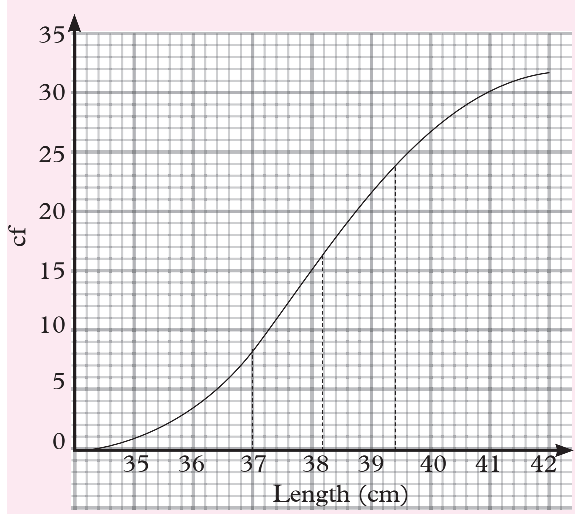


Fig. 8.21

(c) Median Length = 38 cm

LQ =Q_1 = 37

UQ = Q_3 = 39.4

Interquartile range

= 39.4 – 37

= 2.4

Histogram Fig. 8.22

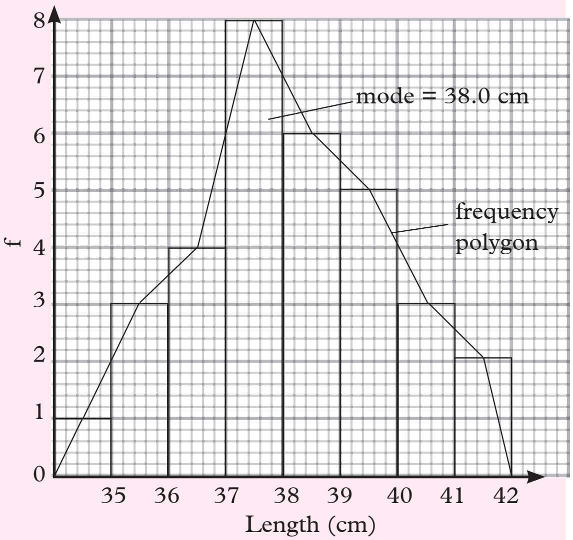


Fig. 8.22

The bars represent the histogram.

Exercise 8.9

1. A tour company recorded the number of people travelling to a holiday resort each day for 30 consecutive days as in Table 8.53 below

No. of people	125	175	225	275	325	375	425
f	0	6	7	8	5	3	1

Table 8.53

(a) Make a cumulative frequency table for the data.

(b) Draw a cumulative frequency curve and use it to estimate:-

(i) The median.

(ii) The lower quartile.

(iii) The upper quartile.

(iv) The interquartile range.

2. Table 8.54 shows the heights, in centimetres, and their frequencies of a group of people.

Height (cm)	153	158	160	163	165	168	170	173	175	178	180
f	2	3	5	7	12	11	8	5	4	2	1

Table 8.54

Use a graphical method to estimate:

(a) The median.

(b) The quartile.

(c) The interquartile range.

Summary

1. Statistics is the study of collecting, organizing, representing, displaying and analysing numerical data. Examples of data include age, mass, heights of students etc.

2. Data is a set of values and observations that gives raw information in a more organized form. Data are of two types:

(a) Qualitative data – is data whose numerical values cannot be measured e.g. race, smoking habit, sex etc.

(b) Quantitative data – is data that can be measured numerically e.g. weight, mass, height, salary etc.

3. Quantitative data is of two types:

(a) Continuous data – is data that takes all values within a given range e.g. x < 5

(b) Discrete data – is a group of items with a finite and distinct number of possible values e.g. 12, 14, 16, 18 and so on.

4. Methods of data collection include observation, interview, prepared questionnaires etc.
5. Frequency – is the number of times an item or a value occurs.
6. Methods used in determining measures of central tendencies include:

(a) Mean: This is the average of numbers

$$= \frac{\text{sum of observation}}{\text{total number of all data item in a group}}$$

$$= \frac{\sum fx}{\sum f}$$

(b) Mode: This is the item with the highest frequency in a given set of data.

(c) Median: This is the middle value in a set of data that divides a set into two.

For an odd set Median = $[\frac{1}{2}(n + 1)]^{th}$ position

For even set Median = $\frac{1}{2}(\frac{X_n}{2} + \frac{X_{n+1}}{2})$

7. When the number of values is odd, the median is the middle value. When the value is even, the median is the mean of the two middle values.
8. Cumulative frequency is the running total of frequencies showing the end total frequency of each class.
9. Histogram is a graphical display of data using bars of various heights.
10. Frequency polygon is a line graph drawn by joining all the midpoints of the top of the bars of a histogram.
11. Interquartile range is the difference between the lower quartile (Q_1) and the upper quartile (Q_3)

$Q_3 - Q_1$

Lower quartile Q_1 = $[\frac{1}{4}(n + 1)]^{th}$ position

Upper quartile Q_3 = $[\frac{3}{4}(n + 1)]^{th}$ position

Unit Test 8

1. Find the mode and the median of each of the following sets of the numbers.
   (a) 5, 4, 1, 1, 7, 6, 4, 1, 5, 4
   (b) 1, 0, 2, 2, 1, 3, 0, 1
   (c) 16, 22, 15, 20, 28
   (d) 2.3, 3.4, 2.3, 2.3, 5.4, 3.4, 2.3, 5.3, 3.4, 4.4, 2.3
2. Find the mode, mean and median of the following sets.
   (a) 3, 4, 4, 9, 11, 4
   (b) 1, 3, 7, 7, 8, 7, 6, 6, 9
   (c) 14, 15, 12, 13, 15, 16, 12, 12, 10, 12, 12
3. The mass of 30 new-born babies were recorded in kg as follows:

   1.8, 1.7, 1.6, 2.1, 1.8, 1.9, 2.5, 1.7, 1.8, 1.6, 1.5, 1.4, 2.0, 2.1, 1.8, 1.6, 1.7, 2.1, 1.9, 1.8, 1.2, 1.9, 1.8, 1.8, 1.9, 1.7, 1.8, 2.0, 1.8, 1.6

   (a) State the mode.
   (b) Make a frequency distribution table and work out the mean using the values.
4. The following are marks obtained by

40 University students in a Physics exam:

30, 50, 65, 40, 70, 43, 65, 41, 60, 35, 30, 40, 55, 50, 65, 40, 60, 40, 43, 30, 65, 56, 30, 30, 50, 30, 40, 44, 30, 43, 30, 40, 55, 40, 50, 65, 50, 70, 30 and 30.

Make a frequency distribution table and find the mean mark

5. 12 million bags of maize were distributed in four Towns, A, B, C and D in Rwanda as shown in the pie- chart below.

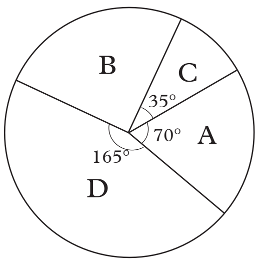


Fig. 8.23

Find the number of bags distributed in every province.

6. The distribution below shows marks of a test of 50 students that was marked out of 50 marks. Make a frequency distribution table using classes of width 5 starting from 10–14.

35	28	21	16	30
44	32	36	23	24
28	28	30	44	17
38	41	30	17	14
12	26	48	13	14
23	43	47	25	26
33	27	33	36	32
24	22	26	44	28
34	25	23	44	35
36	37	47	32	41

Table 8.55

Use the frequency distribution table to draw a frequency polygon for the data in table 8.55.

7. Make a frequency distribution table for the following data.

15, 18, 11, 15, 11, 18, 17, 11, 20, 10, 11, 25, 24, 18, 10, 24, 14, 15, 20, 15, 15, 18, 20, 19, 13, 11, 17, 17, 12, 19, 17, 13, 11, 20.

Table 8.56

Use the table to find the mean and the mode of the data.

8. The data show the number of children born to 25 families.

3, 5, 4, 2, 2, 4, 6, 8, 10, 4, 3, 5, 4, 8, 4, 7, 6, 6, 4, 6, 2, 3, 6, 5, 8.

Make a frequency distribution table and find:

(a) the mean number of children.

(b) the mode.

9. In a given year, rainfall in a certain district was recorded in mm for six months as follows.

Month	Jan	Feb	Mar	Apr	May	Jun
Rainfall in mm	63	65	176	150	204	74

Table 8.57

Represent this information in a bar chart.

10. Mr Bindyo has a monthly income of 40 000 FRW. He spends 10 000 FRW on house rent, 8 000 FRW on food, 4 000 FRW on transport, 9 000 FRW on entertainment and saves the remaining amount. Draw a pie chart to represent this information.

11. Table 8.58 below shows the distribution of marks scored by 60 candidates in an examination:

Marks	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90
Frequency	2	5	6	10	14	11	9	3

Table 8.58

Use Table 8.58 to draw a histogram and frequency polygon on the same grid.

12. A die was thrown 25 times and the face that appeared at the top was recorded. The scores were as shown in Table 8.59.

Face	1	2	3	4	5	6
No. of times	3	6	3	2	4	7

Table 8.59

Draw a bar chart to represent the information above.

13. The mean of the numbers 3, 4, a, 5, 7, 9, 5, 8, 5 and 9 is equal to the mode. Find the value of a and hence the median of the data.

14. Forty-five pupils in a mixed class sat for a Mathematics examination. The mean mark for 15 boys was 40.5 marks while the mean mark for the whole class was 52.3 marks. Find the mean mark for the girls in the class.

15. The number of patients who attended a clinic by age was grouped as shown in Table 8.60.

Age (years)	26–30	31–35	36–40	41–45	46–50	51–55
No. of patients	9	13	20	15	6	2

Table 8.60

(a) Calculate the mean and the median age of attendance.

(b) State the modal class.

(c) Calculate the mode of the distribution.

(d) On the same axes draw a histogram to represent the information.

16. Table 8.61 shows high altitude wind speeds in knots recorded at a weather station in a period of 100 days.

Wind speed	0–19	20–39	40–59	60–79	80–99	100–119	120–139	140–159	160–179
Frequency	9	19	22	18	13	11	5	2	1

Table 8.61

(a) Draw a cumulative frequency graph for the data.

(b) Use the graph to estimate:

(i) The interquartile range.

(ii) The number of days when the wind exceeded 125 knots.

9 PROBABILITY

Key Unit Competence

By the end of this unit, I should be able to determine the probability of an event happening using equally likely events or experiment.

Unit outline

- Definition of an event and outcome.
- Examples of random events.
- Probability of equally likely outcomes.
- Estimation of probabilities where experimental data is required.

9.1 Introduction

Activity 9.1

Discuss the following scenarios:

1. The doctor told S1 students that three out of 5 cases of stomach upset cases are caused by not washing hands before eating. One day, a student visited the toilet then chewed some piece of sugarcane without washing his hands. What do you think happened to the student? Discuss amongst the three of you. Which of the following is the most accurate prediction ?
   (a) He definitely had a stomach upset.
   (b) It is not likely that he got stomach upset
   (c) It is most likely he got stomach upset.
3. You have been sent to the market to get vegetables by your mother. In order to get fresh vegetables, you have to go early to the market. What is the likelihood of getting fresh vegetables if you go to the market in the evening?
4. How likely are the following events:
   (a) In the last two weeks it has been raining everyday. It will rain tomorrow.
   (b) The national football team has won 5 of their last 6 matches. The team will win the next match.
   (c) This medicine has helped 8 out of 10 patients who had a problem similar to yours. Try it, you will be helped.

In Activity 9.1, you were trying to tell the likelihood of an event occurring.In our day to day life, we come across many events and issues that require our judgment on their presumed occurrence or happening.

For instance, when planning for outdoor events such as a wedding, you evaluate the likelihood of raining on the proposed date of the wedding.

Similarly, when farmers are preparing their land for planting, they do these hoping to have rains during a given time period. For instance, the likelihood of having long rains in March is higher compared to January.

The likelihood or the possibility of an event occurring is known as probabilty.

9.2 Definition of terms used to describe probability

Activity 9.2

Carry out the following activities:

1. Obtain a Rwanda coin. Let the side with the crown be the head (H) and the other side the tail (T).

Throw the coin to the ground at random. Let your partner record the side that comes up as the result, a head or a tail. Let your partner repeat this activity as you record. Each one of you to do twice.

How many heads did you obtain? How many tails did you obtain? What name would you give to these results?

2. There are 52 playing cards in a packet with spades, diamonds, clubs and hearts. Pick 5 cards from the packet. What did you obtain? Write down all the possible results if a card is picked from the pack.

Playing cards

3. Obtain a die. Observe it and record the number represented on by the dots on each side. Throw it twice. What number did you obtain on top? Write down all the possible results when a die is thrown once.

Die

Lets us use your results in Activity 9.2 to define some terms that are used to describe probability.

(i) An experiment is any activity or process through which data is obtained and analysed.

(ii) All likely results of an experiment are called possible outcomes.

(iii) A set of all possible outcomes that may occur in a particular experiment is called a sample space, usually denoted by S. For example,

(a) When a coin is tossed: S = {H,T}.

(b) When two coins are tossed:
S = {HT, TH, HH, TT}.

(c) When three coins are tossed:
S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}
Note that H= Head and T=Tail then S=Sample space/outcomes.

(d) When a die is thrown:
S = {1, 2, 3, 4, 5, 6}, e.t.c.

(iv) An event is a set consisting of possible outcomes of an experiment with the desired qualities. It is a subset of a sample space.

Examples of events are:

1. Getting a tail when tossing a coin.
2. Rolling and scoring a "5" in a die game.

3. Getting a card from a well shuffled pack of 52 playing cards.
4. Winning, drawing or losing a football match in a pool of 32 competitive matches.

Types of events

Activity 9.3

Discuss whether the following events are possible or not. Estimate the likelyhood in each case

1. When a stone is thrown upwards into the air, it will finally drop to the ground.
2. When a Rwanda coin is thrown at a random, it will show a head (H) Shows two sides at the same time.
3. When a dice is thrown it will give a 10.

An event that will definitely take place is called a sure or certain event.The following are examples of sure events.

(a) If you put a shirt in water, it will get wet.

(b) If you put your exercise book in fire, it will burn.

An event that cannot take place is known as an impossible event. The following are examples of impossible events.

(a) A dice cannot give a 9.

(b) A cassava plant will produce potatoes.

An event that may or may not take place is known as an uncertain event. The following are examples of uncertain events.

a. It will rain tomorrow.

b. The national football team will win their next match.

c. Mala will win the elections.

9.3 Numerical expression of probability

We use words like **probable**, **likely**, **unlikely, chance** to describe the likely hood of an event happening.

We usually assign numerical values to our predictions. Thus in the case of the national football team winnig (Actvity 9.1), we would say that the chance of the team winning is $\frac{5}{6}$. Similarly, in the case of the medicine helping a patient, we would say that the chance of it helping the patient is $\frac{8}{10}$.

The numerical values $\frac{5}{6}$ and $\frac{8}{10}$ are called probabilities.

Learning point

Thus, probability is the branch of Mathematics in which appropriate *numerical values* are assigned as measures of the chances of *uncertain events* occuring or not occuring.

The numerical values assigned are called probabilities (singular: probability), derived from the word "probable".

Thus we can say: "The probability of the national football team winning is $\frac{5}{6}$."
"The probability of the medicine helping a patient is $\frac{8}{10}$."

9.4 Experimental probability

Probabilities which are determined from experiments or from practical events are called **experimental probabilities**.

Learning point

In general, the experimental probability P(E) of an outcome

$$P(E) = \frac{\text{the number of outcomes}}{\text{the total number of trials}}$$

Thus, $P(H) = \frac{\text{Number of heads}}{\text{Total number of tosses}}$

$P(T) = \frac{\text{Number of tails}}{\text{Total number of tosses}}$

Table 14.1 shows part of the results of a football competition.

P = Number of matches played

W = Number of matches won

D = Number of matches drawn

L = Number of matches lost

Teams	P	W	D	L
Lions	18	9	7	2
Tigers	18	9	4	5
Cheaters	18	5	9	4
Leopards	18	2	5	11

Table 14.1

In this case, each match is a **trial**. The outcome of each trial is either a win (W), a draw (D) or a loss (L). Thus the experimental probability of

1. Lions winning $= \frac{9}{18}$
2. Lions losing $= \frac{2}{18}$
3. Leopards winning $= \frac{2}{18}$

What is the probability of

(a) Mumiad drawing?

(b) Brewers winning?

(c) Tigers losing?

In a pack of playing cards, there are 52 cards of which 13 are spades, 13 clubs, 13 diamonds and 13 hearts. So when a card is picked, it can be a spade, a club, a diamond or a heart. This is the list of possible outcomes.

Since there are 4 possible outcomes, the probability of picking a heart is 1 out of 4, that is P (heart) $= \frac{1}{4}$

The spades and clubs are black in colour while diamonds and hearts are red. What is the probability of picking a red card?

The list of possible outcomes is red, black.

Hence there are two possible outcomes.

Hence the probability of picking a red card is one out of two, that is

Probability (red card) $= \frac{1}{2}$

Basic rules in probability

Activity 9.4

Carry out the following activities:

1. Toss a coin is toss once. Determine all the possible outcomes and their probalities. Sum up the all the probabilities of all possible outcomes. What did you obtain? Repeat the same for a dice. Draw a conclusion on the range of probability
2. Determine the probability of getting a 9 on throwing a dice. What value did you obtain? Draw a conclusion on the value of probability of an impossible event.
3. Jump upwards in the air five times. Determine the probability of droping to the ground when you jump the sixth time. Draw a conclusion on the value of probability of a certain event.

From Activity 9.3, you have realised the following:

1. Probabilities of any possible events is in the range of 0 to 1. i.e.

   In general, for any event A,

   $0 \leq P(A) \leq 1$

   In addition

If the only possible events are A and B, and when A takes place B does not, and vice versa, then

$$P(A) + P(B) = 1.$$

A' or ~A is used to denote that event A has not taken place (has not occured). Thus, $P(A) + P(A') = 1$.

No event or a combination of events can have a probability of more than one.

2. If an event E is certain its probability of occuring $P(E) = 1$
3. If an event is impossible , its probability of occuring is 0: $P(E) = 0$ (zero).

Example 9.1

Consider an event of throwing a die once.

(i) Write down the sample space.

(ii) What is the probability of each element?

Solution:

(i) The sample space is $S = \{1, 2, 3, 4, 5, 6\}$

(ii) The probability of each element is $\frac{1}{6}$ since each element is likely to occur once in one throw.

Example 9.2

In an experiment of drawing a card from a deck of 52 cards, what is the probability of drawing an ace?

Solution

There are 4 aces expected. So the probability of getting an ace is P(A). This can be obtained as $P(A) = \frac{4}{52} = \frac{1}{13}$

Example 9.3

A spinner has 4 equal sectors colored yellow, blue, green and red. After spinning the spinner, what is the probability of landing on each colour?

Solution

Outcomes: The possible outcomes of this experiment are yellow, blue, green and red.

Probabilities:

P(yellow)

$$= \frac{\text{number of ways to land on yellow}}{\text{total number of colour}} = \frac{1}{4}$$

P(blue)

$$= \frac{\text{number of ways to land on blue}}{\text{total number of colour}} = \frac{1}{4}$$

P(green)

$$= \frac{\text{number of ways to land on green}}{\text{total number of colour}} = \frac{1}{4}$$

P(red)

$$= \frac{\text{number of ways to land on red}}{\text{total number of colour}} = \frac{1}{4}$$

Example 9.4

A single 6-sided die is rolled. What is the probability of each outcome? What is the probability of :

(a) rolling an even number?

(b) rolling an odd number?

Solution

Outcomes: The possible outcomes of this experiment are 1, 2, 3, 4, 5 and 6.

Probabilities:

$$P(1) = \frac{\text{number of ways to roll a 1}}{\text{total number of sides}} = \frac{1}{6}$$

$$P(2) = \frac{\text{number of ways to roll a 2}}{\text{total number of sides}} = \frac{1}{6}$$

$$P(3) = \frac{\text{number of ways to roll a 3}}{\text{total number of sides}} = \frac{1}{6}$$

$$P(4) = \frac{\textit{number of ways to roll a 4}}{\textit{total number of sides}} = \frac{1}{6}$$

$$P(5) = \frac{\textit{number of ways to roll a 5}}{\textit{total number of sides}} = \frac{1}{6}$$

$$P(6) = \frac{\textit{number of ways to roll a 6}}{\textit{total number of sides}} = \frac{1}{6}$$

$$P(even) = \frac{\textit{number of ways to roll an even number}}{\textit{total number of sides}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

$$P(odd) = \frac{\textit{number of ways to roll an odd number}}{\textit{total number of sides}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

Example 9.5

It has been found that the probability that Keza arrives at work on time is 0.2. How many times would you expect her to be on time in the next 20 days?

Solution

Let the number of days Keza arrives on time in the 20 days be x. Then, the experimental probability of her being on time

$$= \frac{\textit{Number of heads}}{\textit{Total number of tosses}} = \frac{x}{20}$$

Since the probability that she is on time is 0.2, then $\frac{x}{20} = 0.2$

$\Rightarrow \quad x = 0.2 \times 20 \text{ days} = 4 \text{ days}.$

Risk matters and probabilty!!!

Did you know that we can use probability in finance? We bear risks when transacting business activities but at the end of it all is having profits and losses.

"A person, who risks nothing, is nothing and has nothing!!

Exercise 9.1

1. A die has 6 faces labeled 1 – 6.
   (a) List all the possible outcomes when the die is rolled.
   (b) Find the probability that a face marked with an even number shows up.
   (c) Find the probability that a face marked with a multiple of 3 shows up.
2. In a pack of 52 playing cards, 13 are spades, 13 clubs, 13 diamonds and 13 hearts. The spades and clubs are black in colour, the diamonds and hearts are red. What is the probability of picking.
   (a) A spade
   (b) A black card
   (c) A yellow card
3. Open a textbook. Write down the last digit of the right hand page. Do this 40 times. For example, if the book opens at pages 120 and 121, write down 1.
   (a) What is the probability of getting
       (i) 0 (ii) 1
       (iii) 2 (iv) 4?
   (b) What is the probability of getting a number less than 5?
   (c) What is the probability of getting an odd number?
4. Toss a bottle top 50 times. Each time record whether the bottle top faces up or down.
   (a) What is the probability of the bottle top facing up?
   (b) What is the probability of the bottle top facing down?

5. Toss two bottle tops together. Record whether both face down, both face up or one faces down while the other faces up. Do this 40 times. What is the probability that
   (a) both face down?
   (b) both face up?
   (c) one faces down and one faces up?
6. Toss two coins together. Record whether they show 2 heads (HH), 2 tails (TT) or 1 head and 1 tail (HT). Do this 40 times. What is the probability of obtaining
   (a) HH (b) TT (c) HT?
7. Open a textbook. Record the number of letters of the first and last words on a page. Do this 30 times. What is the probability that
   (a) a word has more than 5 letters?
   (b) a word has 4 letters?
   (c) a word has less than 4 letters?
8. In the past twenty days that Zikomo has been going to work, he has had a lift seven times, used a bus nine times and walked four times. What is the probability that
   (a) he gets a lift?
   (b) he does not pay to get to work?
9. In the last season, it was found that the probability of Victory Football Club winning a match was $\frac{1}{8}$. How many matches would you expect the team to win if they play 24 matches in the next season?
10. In 30 fights, a boxer has won 24 fights, drawn 2 and lost 4. What is the probability that
    (a) he wins in the next fight?
    (b) he does not win the next fight?
11. The probability that it rains on Christmas day in town X is 0.3. What is the probability that it will not rain on Christmas day in that town?
12. If the probability of the school's volleyball team winning is 0.4, how many games has the team possibly played if it has won 16 games?
13. (a) In Question 2, what do you get when you add the answers to (a) and (b)?
    (b) In Question 3, what do you get when you add the answers to (a), (b) and (c)?
    (c) In Question 4, what do you get when you add your answers to (a), (b) and (c)?
    (d) In Question 8, what do you get when you add your answers to (a) and (b)?
14. Toss a coin 20 times. Each time, record whether heads or tails show up. Find the experimental probability.
    (a) P(H) (b) P(T)
    (c) P(H) + P(T)

    What do you notice in each case?

    Repeat the process when the coin is tossed:

    (i) 40 times (ii) 60 times
    (iii) 80 times and (iv) 100 times

    Copy and complete Table 12.2.

No. of tosses	20	40	60	80	100
P(H)					
P(T)					
P(H) + P(T)					

Table 12.2

What do you notice about the values of P(H) and P(T) as the number of tosses increases? What do you notice about P(H) + P(T) in each case?

15. (a) Place 10 bottle tops (3 of drink A, 5 of drink B, 2 of drink C) in a bag. Mix them, then pick one at random and note which brand it is and then return it. Do this 20 times. Find the experimental probability

(i) P(A) (ii) P(B)

(iii) P(C)

(b) Repeat part (a) using 10 bottle tops, all of drink A.

9.5 Theoretical probability

In this section, we see how certain probabilities can be found without experimenting.

In Question 14 of Exercise 9.1, you should have noticed that as the number of tosses increased, the values of P(H) and P(T) were each closer and closer to 0.5 (or $\frac{1}{2}$). If the number of tosses is very large, for all practical purposes we have P(H) = P(T) = $\frac{1}{2}$(or 0.5).

When a fair coin is tossed, there are only two possible outcomes H or T. The symmetry of the coin tells us that "heads" and "tails" have equal chances of occuring. We say that they are **equally likely**. It follows that P(H) = P(T).

Since P(H) + P(T) = 1 (from the earlier section), then P(H) = P (T) = $\frac{1}{2}$.

Alternatively,

All the outcomes are H, T.

Number of all outcomes	=	2
Number of times H appears	=	1
Number of times T appears	=	1

$$\therefore \text{P(H)} = \frac{\text{Number of times H appears}}{\text{Total number of outcomes}} = \frac{1}{2}$$

$$\text{and P(T)} = \frac{\text{Number of times T appears}}{\text{Total number of outcomes}} = \frac{1}{2}$$

Since these values can be obtained without tossing a coin, they are called **theoretical probabilities**.

Learning point

In general, if N is number of all possible outcomes of A and n is number of times a particular outcome appears, then

$$\text{P(A)} = \frac{n}{N}$$

Key words in probabilty

In any probability question, when the following terms are used, great care must be taken. To understand the terms, let us take an example a family of 4 children, where B represents a boy and G represents a girl.

(i) **at least**

'At least two are boys, B ≥ 2. Therefore, B can take values 2, 3 or 4.

ii) **'at most'**

'At most two are boys', i.e. B ≤ 2. Therefore, B can take values 2, 1 or 0.

(iii) **Not more than**

'Not more than two boys' and 'at most two boys' mean the same.

(iv) **Not less than**

'Not less than two boys 'and 'at least two boys' mean the same.

(v) **More than**

'More than two boys', i.e. B > 2. Therefore B can take values 3 or 4.

(vi) **Less than**

'Less than two boys', i.e. B < 2. Therefore, B can take values 0 or 1.

(vii) Equally likely

Have equal probabilities.

Example 9.6

Numbers 1 to 20 are each written on a card. The 20 cards are mixed together. One card is chosen at random from the pack. Find the probability that the number on the card is:

(a) *Even* *(b)* *A factor of 24*

(c) *Prime.*

Solution

We will use 'P(x)' to mean 'the probability of x'. Let S be the sample space such that S= {1, 2, 3, 4, 5, 6, 7, 8… 20}.

(a) $P\ (\text{even}) = \dfrac{\text{number of even numbers}}{\text{total number of cards in the pack}}$

$= \dfrac{10}{20} = \dfrac{1}{2}$

(b) $P\ (\text{a factor of } 24) = \dfrac{\text{number of factors of } 24}{\text{total number of cards in the pack}}$

The factors of 24 are: $F_{24} = \{1, 2, 3, 4, 6, \text{and } 8\ 12, 24\}$.

$n(F_{24}) = 8$

Therefore, $P\ (\text{a factor of } 24)$

$= \dfrac{8}{20} = \dfrac{2}{5}$

(c) *Prime numbers in the pack =* $\{2, 3, 5, 7, 11, 13, 17, 19\}$

$P\ (\text{prime}) = \dfrac{\text{number of prime numbers in the pack}}{\text{total number of cards in the pack}}$

$= \dfrac{8}{20} = \dfrac{2}{5}$

Example 9.7

A black die and a white die are thrown at the same time. Display all the possible outcomes. Find the probability of obtaining:

(a) *A total of 5,*

(b) *A total of 11,*

(c) *A 'two' on the black die and a 'six' on the white die.*

Solution

It is convenient to display all the possible outcomes on a grid. There are 36 possible outcomes, as shown below.

	Number on white die					
Number on black die	**1**	**2**	**3**	**4**	**5**	**6**
1	(1,1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

(a) *There are four ways of obtaining a total of 5 on the two dice. These are:* $(1, 4), (4, 1), (2, 3)$ *and* $(3, 2)$

Probability of obtaining a total of 5 $= \dfrac{4}{36} = \dfrac{1}{9}$

(b) *There are two ways of obtaining a total of 11. These are* $(5, 6)$, *and* $(6, 5)$.

$P\ (\text{total of } 11) = \dfrac{2}{36} = \dfrac{1}{18}$

(c) *There is only one way of obtaining a 'two' on the black die and a 'six' on the white die.*

Therefore, P *(a two on black and a 6 on white* $= \dfrac{1}{36}$.

Example 9.8

A fair coin is tossed and a fair die is rolled. Find the probability of obtaining a 'head' and a 'six'.

Solution

When a coin is tossed once, the sample space is: $S = \{H, T\}$

Where H denotes a 'head' and T a 'tail'.

So, $P(H) = P(T) = \frac{1}{2}$.

Similarly, the sample space when a die is tossed once is: $S = \{1, 2, 3, 4, 5, \text{and } 6\}$

$P(\text{six}) = \frac{1}{6}$

Example 9.9

Two coins are tossed at once. Find the probability of obtaining

(a) 2 heads (b) 1 head, 1 tail

(c) 2 tails

Solution

If the coin shows heads we indicate the outcome as HH.

If one coin shows head and the second coin shows tail, we indicate the outcome as HT, etc.

So, all the possible outcomes are HH, HT, TH, TT.

Number of all possible outcomes = 4

Number of outcomes with two heads = 1

Number of outcomes with 1 head and 1 tail = 2

Number of outcomes with two tails = 1

Hence

(a) $P(2 \text{ heads}) = P(HH) = \frac{1}{4}$.

(b) $P(1 \text{ head}, 1 \text{ tail}) = P(HT) = \frac{2}{4} = \frac{1}{2}$.

(c) $P(2 \text{ tails}) = P(TT) = \frac{1}{4}$.

Example 9.10

A bag contains 10 bottle tops of which 3 are of drink A, 5 are of drink B and 2 are of drink C. If a bottle top is picked from the bag at random, what is the probability that it is

(a) a drink A (b) a drink B

(c) a drink C?

Solution

Let A represent drink A, B represent drink B and C represent drink C.

All the possible outcomes are

A, A, A, B, B, B, B, B, C, C.

Thus, the number of possible outcomes is 10.

Each bottle top is equally likely to be picked.

Number of possible outcomes for drink A is 3.

Number of possible outcomes for drink B is 5.

Number of possible outcomes for drink C is 2.

(a) $P(A)$

$$= \frac{\text{No. of possible outcome for drink A}}{\text{No. of all possible outcomes}}$$

i.e. $P(A) = \frac{\text{No. of drink A bottle tops}}{\text{No. of all bottle tops}} = \frac{3}{10}$

(b) $P(B) = \frac{\text{No. of drink B bottle tops}}{\text{No. of all bottle tops}} = \frac{5}{10} = \frac{1}{2}$

(c) $P(C) = \frac{\text{No. of drink C bottle tops}}{\text{No. of all bottle tops}} = \frac{2}{10} = \frac{1}{5}$

(Compare these results with your answers for Question 15(a) of Exercise 9.1).

BEWARE!!!!

The probability of contracting Sexually Transmitted Diseases which includes HIV and AIDS is high among the youth of today. In that case, avoid all the unnecessary ways of contracting such diseases so as to have a brighter future!! Abstain from engaging in sexual acts before marriage and remain faithful to one partner once married.

Exercise 9.2

1. A box contains 12 boiled eggs and 25 raw ones. An egg is taken at random. What is the probability that the egg is raw?
2. A card is drawn from a pack of 52 playing cards. Find the probability that it is
   (a) a club.
   (b) not a diamond.
3. If I choose a number at random from 11, 13, 15, 17, ... 41, what is the probability that the number is either prime or a multiple of three?
4. A die is tossed once. What is the probability that the number appearing on top is prime?
5. Two dice are thrown. Find the probability of getting:
   (a) at least one 4.
   (b) a total of at most 4
   (c) At most one 6
   (c) A five and a five in both dice
6 In a room there are three gentlemen and four ladies. One person is picked at random. What is the probability that the person is a lady?
7. A vendor has fifteen 5 FRW coins and nine 10 FRW coins in a bag. He picked one coin from the bag at random. What is the probability that the coin was a 10 FRW coin?
8. In a class of 20, there were 8 boys. If one student was picked at random, what is the probability that a girl was picked?
9. A set of cards are numbered from 1, 2, ...30. One card is picked at random. What is the probability that the number of the card is a
   (a) multiple of 3?
   (b) factor of 28?
10. In 2011, a national soccer team played against four other national teams. If the team won once and drew three times. What is the probability that the team will win in 2014?
11. A bag contains 10 red balls, 5 blue balls and 7 green balls. Find the probability of selecting at random:
   (a) A red ball,
   (b) A green ball.
12. One ball is selected at random from a bag containing 12 balls of which x balls are white.
   (a) What is the probability of selecting a white ball?
   (b) When a further 6 white balls are added, the probability of selecting a white ball is doubled. Find x.
13. Choose a number at random from 1 to 5. What is the probability of each outcome? What is the probability that

the number chosen is even? What is the probability that the number chosen is 0.

Summary

1. Probability is the likelihood/possibility of an occurrence of an event at a given period of time.
2. Sample space/probability space – is the list of possible outcomes in a trial.
3. An event – is any possible outcome from the sample space OR an event is a sub set of a sample space.
4. Probability that an event occurs $= \dfrac{\text{number of favourable outcome}}{\text{Total number of outcomes}}$.
5. Probability values are real numbers that range from 0 to 1, $0 \leq P(E) \leq 1$.
6. The probability of certainty is 1 (one) while the probability of impossibility is 0 (zero).

Unit Test 9

1. It has been observed that in a certain region, five out of every 100 children die at an age below five years. What is the probability that a child born in the region will live for more than five years?
2. A box contains 12 boiled eggs and 25 raw ones. An egg is taken at random. What is the probability that the egg is raw?
3. Two dice are thrown. Find the probability of getting:
   (a) a total of more than 8.
   (b) at least one 6.
   (c) Atmost one 5
4. If I choose a number at random from 11, 13, 15, 17,....41, what is the probability that the number is either prime or a multiple of three?
5. A cards is drawn, with replacement, from a well shuffled pack of 52 playing cards. What is the probability that it is:
   (a) A king?
   (b) An ace
6. A bag contains 6 red balls and 4 green balls.
   (a) Find the probability of selecting at random:
      (i) A red ball.
      (ii) A green ball.
   (b) If one red ball is removed from the bag, find the new probability of selecting at random:
      (i) A red ball.
      (ii) a green ball.
7. A fair die is thrown once. Find the probability of obtaining:
   (a) A six,
   (b) An even number,
   (c) A number greater than 3,
   (d) A three or a five.
8. One letter is selected at random from the word 'UNNECESSARY'. Find the probability of selecting letter:
   (a) R (b) E (c) O

Glossary

- Set is a group of items with a common feature.
- Relation is a connection between two or more numbers or things.
- Composite function is a function formed by combining two or more functions to form a function.
- Percentage is fraction whose denominator is 100.
- Discount is the amount reduced from the marked price of a commodity.
- Commission is the money paid to sales agents or representatives for sales made.
- Profit is the extra amount gained after selling a commodity at a price higher than the buying price.
- Loss is the amount of money lost when a commodity is sold below the actual buying price.
- Interest is the fee paid for borrowing or depositing money for a specific period of time. It is a fixed percentage charged on unpaid loans.
- Ratio is a mathematical statement of how two or more quantities or numbers compare.
- Sharing is to share a quantity into two parts in the ratio a : b is where the quantity is split into a + b equal parts and the required parts become $\frac{a}{a+b}$ and $\frac{b}{a+b}$ of the quantity.
- Direct proportion is where the quantities are such that, when one quantity increases or decreases in the ratio $\frac{a}{b}$, the other quantity decrease or (increases) in the ratio $\frac{b}{a}$.
- Inverse proportion is where two quantities are such that, when one quantity increases or decreases in the ration $\frac{a}{b}$, the other quantity decreases or increases in the ratio $\frac{b}{a}$.
- A ray is a straight line which starts from a fixed point and moves towards one direction without an end.
- A line segment is a part of a straight line which has two fixed points, the starting and the ending points.
- A transversal line is a straight line which cuts through two lines in the same plane at two distinct points.
- Cuboid is a solid bounded by three pairs of identical faces which are all rectangles.
- Cube is a solid bounded by six identical faces which are squares.
- Pyramid is a solid figure formed with triangular slanting faces which meet at a vertex above a polygonal base.
- Statistics is the study of collecting, organising, representing and displaying of numerical data. Examples of data include age, mass, heights of students etc.
- Data is a set of values and observations that gives raw information in a more organized form.
- Frequency is the number of times an item or a value occurs.
- Cumulative frequency is the running

total of frequencies showing the end total frequency of each class.

- Histogram is a graphical display of data using bars of various heights.
- Interquartile range is the difference between the lower quartile (Q_1) and the upper quartile (Q_3)
- Probability is the likelihood/possibility of an occurrence of an event at a given period of time.
- Sample space/probability space is the list of possible outcomes in a trial.
- An event is any possible outcome from the sample space OR an event is a sub set of a sample space.

REFERENCES

1. Rwanda Education Board (2015). *Mathematics Syllabus for Ordinary Level Senior 1-3*, Kigali.

2. Allen R (2004). *Intermediate Algebra for College Students*, Pearson Education, Inc, New Jersey.

3. Richard E (1997). *Algebra I,* Addison –Wesley Publishing Company, Inc, Phillipines.

4. Engelsohn H., Feit J (1980). *Basic Mathematics*, Moriah Publishing, New York.

5. Mugo K., Maina L., Ondera J (2003). *Excel and Succed Mathematics Form 1*, Longhorn Publishers, Nairobi.

6. Mugo K., Maina L., Ondera J (2003). *Excel and Succed Mathematics Form 2*, Longhorn Publishers, Nairobi.

7. Penguin Books Ltd (1998). The Penguin Dictionary of Mathematics, England.