

MATHEMATICS FOR TTCs

STUDENT'S BOOK

YEAR

3

OPTION:

Early Childhood and Lower Primary Education
(ECLPE)

© 2020 Rwanda Education Board

All rights reserved

This book is property of the Government of Rwanda. Credit must be given to REB when the content is quoted.

FOREWORD

Dear Student,

Rwanda Education Board (REB) is honored to present Year Three Mathematics book for Early Childhood and Lower Primary Education (ECLPE) Student Teachers. This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics content. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the student-teacher where concepts are mainly introduced by an activity, situation or scenario that helps the student-teacher to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other student-teachers through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by student-teachers in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, secondary school teachers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year Three student teachers in Early Childhood and Lower Primary Education (ECLPE). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different Universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers, secondary school teachers and TTC tutors whose efforts during writing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook elaboration.

Joan MURUNGI

Head of CTRLR Department

Table of Contents

FOREWORD..... iii

ACKNOWLEDGEMENTV

UNIT 1: POLYNOMIAL, RATIONAL AND IRRATIONAL FUNCTIONS.... 1

1.1. Types of functions.....3

1.2. Injective, surjective and bijective functions.....5

1.3. Domain and range of a numerical function.....13

1.4. Composition and inverse of numerical functions.....31

1.5. Even functions and odd functions.....35

1.6 Introduction to limits of a function.....42

1.7. End of unit assessment.....84

UNIT 2: DIFFERENTIATION 85

2.1 Concepts of derivative of a function.....86

2.2 Rules of differentiation.....91

2.3 Differentiation of trigonometric functions98

2.4 Derivative and the variation of a function102

2.5 Derivative and Concavity of a function.....107

2.6 Derivative and the table of variation for a function.....110

2.7 Derivative and limit with indeterminate cases: Hospital's rule115

2.8 Derivative, tangent line equation and Normal line equation.....116

2.9 Applications of differentiation in Economics and finance120

2.10 Applications of differentiation: rate of change problems,
optimization problems130

2.11 End of unit assessment.....136

UNIT 3: MATRICES AND SYSTEM OF LINER EQUATIONS 137

3.1. Definition, size and types of matrices.....137

3.2. Operations on matrices.....143

3.3. Transpose of Matrix.....155

3.4 Determinants and inverse of matrices of order two and three.....	157
3.5 Applications of matrices and determinants	173
3.6 End of unit assessment.....	185
UNIT 4: LOGARITHM AND EXPONENTIAL FUNCTION	187
4.1. Logarithmic functions	188
4.2. Exponential function	216
4.3. Modeling and solving problems involving logarithms and exponential functions	238
4.4 End unit assesment.....	247
REFERENCES	248

UNIT 1

POLYNOMIAL, RATIONAL AND IRRATIONAL FUNCTIONS

Key Unit Competence:

Evaluate correctly domain of definition and limit of a function and determine asymptote to related function.



1.0. Introductory activity

- Consider the following sentences:
 - The **function** of the heart is to pump blood.
 - Last Saturday, my sister got married; the arrangement of chairs in the main hall was in **function** of the number of guests.
 - The area of a square is **function** of the length of its side.

Explain what is meant by the word “**function**” in each of the three sentences above.

- Any function involves at least two variables. Identify the “**independent variable**” and the “**dependent variable**” in the following functions:

i) $y = \frac{4x-4}{(x-1)^2}$

ii) $A = \pi r^2$

iii) $S = \sqrt{A}$

- Classify the following functions as “**polynomial**”, “**rational**” or “**irrational**”

a. $f(x) = \sqrt{\frac{x^2+1}{x-2}}$

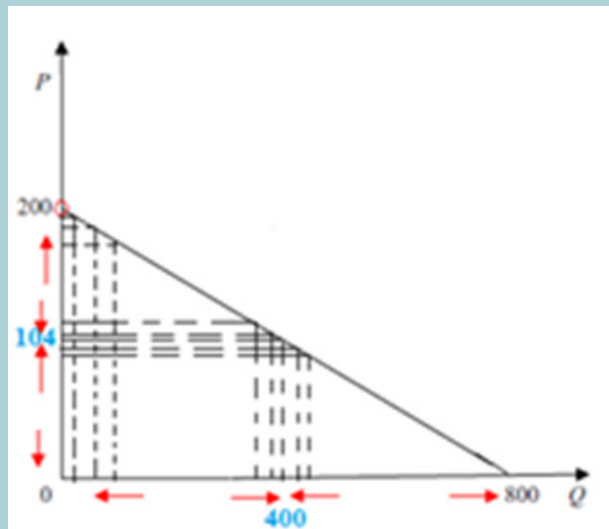
b. $f(x) = \frac{x+1}{x-5}$

c. $f(x) = \sqrt{x^2 - 1}$

d. $f(x) = 2x - 7$

e. $f(x) = \frac{x^3 + 2x - 4}{5x}$

4. If we agree that the set of all possible values, the independent variable can assume is called the “**Domain**” of the function and the set of all possible values, the dependent value can assume is called the “**Range**” of the function, determine the range and the domain of each of the functions in question (2).
5. For each of the following functions $f(x) = \frac{1}{x}$; $g(x) = \sqrt{x}$, $h(x) = x^2$, write]down the set of real numbers that are not in the:
- domain of the function
 - the range of the function
6. Consider and observe the graph of the price function of quantity Q given by $P = 200 - 0.24Q$



- Complete the table and approximate the value of P when Q approaches 0.
- When Q approaches 20, from left and from right, the price gets “closer and closer” to, this can be written mathematically as $\lim_{x \rightarrow \dots} P = \dots$

1.1. Types of functions

Activity 1.1



Differentiate rational from irrational numbers. Guess which of the following functions is a polynomial, rational or irrational function

1. $f(x) = (x+1)^2$

2. $h(x) = \frac{x^3 + 2x + 1}{x - 4}$

3. $f(x) = \sqrt{x^2 + x - 2}$

Content summary

Definition: a function is a relationship between the members of two sets (not necessarily distinct), such that to each value of the independent variable there corresponds at most one value of the dependent variable.

The familiar notation of a function is $y = f(x)$ which we read as “ y equals f of x ”. Here x is the independent variable and y is the dependent variable.

a. Polynomial

A function that is expressible as the sum of finitely many monomials in x is called **polynomial in x** .

Example:

$x^3 + 4x + 7$; $17 - \frac{2}{3}x$; y and x^5 are polynomials. Also $(x-2)^3$ is a polynomial in x because it is expressible as a sum of monomials.

In general, f is a polynomial in x if it is expressible in the form $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ where n is a positive integer, and a_0, a_1, \dots, a_n are real constants. The greatest positive integer n , for which $a_n \neq 0$, is called the **degree** of the polynomial, and in this case, a_n is called the **leading coefficient** of the polynomial.

A polynomial is called **linear** if it is of degree 1, that is, if it can be expressed in

the form $a_0 + a_1x$, where $a_1 \neq 0$. The polynomial is said to be **quadratic** if it is of degree 2, which means it can be expressed in the form $a_0 + a_1x + a_2x^2$, $a_2 \neq 0$,

The polynomial is **cubic**, or of degree 3, if it is possible to rewrite it as $a_0 + a_1x + a_2x^2 + a_3x^3$, $a_3 \neq 0$,

An n^{th} **degree polynomial** has the form $a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots + a_nx^n$; $a_n \neq 0$

b. Rational function

A function that is expressible as ratio of two polynomials is called **rational**

function. It has the form $f(x) = \frac{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_mx^m}$

Example:

$$f(x) = \frac{x^2 + 4}{x - 1}, \quad g(x) = \frac{1}{3x - 5} \text{ are rational functions}$$

c. Irrational function

A function that can be expressed as $\sqrt[n]{f(x)}$, where $f(x)$ contains the variable x .

Example:

$$f(x) = \frac{\sqrt{x^2 + 4}}{\sqrt[3]{x - 1}}, \quad g(x) = \sqrt{\frac{x}{x - 5}} \text{ are irrational functions}$$



Application activity 1.1

Observe the given functions and categorize them into polynomial, rational or irrational functions.

$$f(x) = x^3 + 2x^2 - 2$$

$$g(x) = \frac{x^3 + 2x^2 - 2}{x - 5}$$

$$h(x) = \sqrt{x^3 + 2x^2 - 2}$$

1.2. Injective, surjective and bijective functions

Activity 1.2



a. Consider the function f defined by $f : \mathbb{R} \rightarrow \mathbb{R}$

$$x \mapsto f(x) = x^2$$

- i. Is there any real number missing an image?
- ii. State whether f is a mapping or not
- iii. Is there any real number that is image of more than one real number? If yes, give an example. If not, explain.
- iv. State whether f is *one to one* or not.
- v. Is there any real number that is not image under function f ? If yes, give an example. If not, explain.
- vi. State whether f is *onto* or not.

b. Consider now the function f defined by $f : \mathbb{R} \rightarrow \mathbb{R}^+$

$$x \mapsto f(x) = x^2$$

Repeat questions iii, iv, v and vi of (a) with $f : \mathbb{R} \rightarrow \mathbb{R}^+$ given.

c. Finally, consider the function f defined by $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$x \mapsto f(x) = x^2$$

Repeat questions iii, iv, v and vi of (a)

d. Consider the function f defined by $f : \mathbb{R} \rightarrow \mathbb{R}$

$$x \mapsto f(x) = -x^2 + 4x$$

Determine the greatest subsets A and B of \mathbb{R} such that function

by $f : A \rightarrow B : x \mapsto f(x) = -x^2 + 4x$ is bijective (that is, both one to one and onto)

Content summary

Given sets A and B , a **function** f from A to B is a correspondence, or a rule that associates to any element of A either one image in B , or no image in B .

A function such that no element of A is missing an image is called a **mapping**, thus, under a mapping any element of A has exactly one image in B (not less than one, and not more than one)

A mapping such that any element of B is image of either one element of A , or of no element of A , is called a **one- to- one mapping**, or an **injective mapping** or simply an **injection**; under a one-to-one mapping no two elements of A share the common image in B .

Mathematically, $(\forall x_1 \in A)(\forall x_2 \in A); f(x_2) = f(x_1) \Rightarrow x_2 = x_1$

Or $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$.

A mapping such that any element of B is image of at least one element of A (image of one element of A , or image of more than one element of A), is called an **onto mapping**, or a **surjective mapping** or simply a **surjection**

Mathematically, $(\forall y \in B)(\exists x \in A); f(x) = y$

A mapping both one-to-one and onto is said to be a bijective mapping, or simply, a bijection

In particular, linear function $f: \mathbb{R} \rightarrow \mathbb{R}$

$x \mapsto f(x) = ax + b$, where $a \neq 0$, is bijective, there is no restrictions on the variables (independent or dependent)

Quadratic function $f: \mathbb{R} \rightarrow \mathbb{R}$

$x \mapsto f(x) = ax^2 + bx + c$, where $a \neq 0$, is not bijective, since some real numbers share images, or some real numbers are not images under function f .

The function f is bijective under the following restrictions:

i. $f: \left[\frac{-b}{2a}, +\infty[\rightarrow \left[-\frac{\Delta}{4a}, +\infty[$

$x \mapsto f(x) = ax^2 + bx + c, a > 0;$

ii. $f:]-\infty, -\frac{b}{2a}] \rightarrow \left[-\frac{\Delta}{4a}, +\infty[$

$$x \mapsto f(x) = ax^2 + bx + c, a > 0;$$

$$\text{iii. } f: \left[\frac{-b}{2a}, +\infty[\rightarrow]-\infty, -\frac{\Delta}{4a} \right]$$

$$x \mapsto f(x) = ax^2 + bx + c, a < 0 \text{ and}$$

$$\text{iv. } f:]-\infty, -\frac{b}{2a}] \rightarrow]-\infty, \frac{\Delta}{4a}]$$

$$x \mapsto f(x) = ax^2 + bx + c, a < 0 \text{ are bijective}$$

The homographic function $f: \mathbb{R} \rightarrow \mathbb{R}$

$$x \mapsto f(x) = \frac{ax + b}{cx + d} \text{ is not bijective, since it is not a mapping } (x = -\frac{d}{c} \text{ has no image}$$

under function f , or $y = \frac{a}{c}$ is not image under function f)

But the restriction $f: \mathbb{R} - \{-\frac{d}{c}\} \rightarrow \mathbb{R} - \{\frac{a}{c}\}$

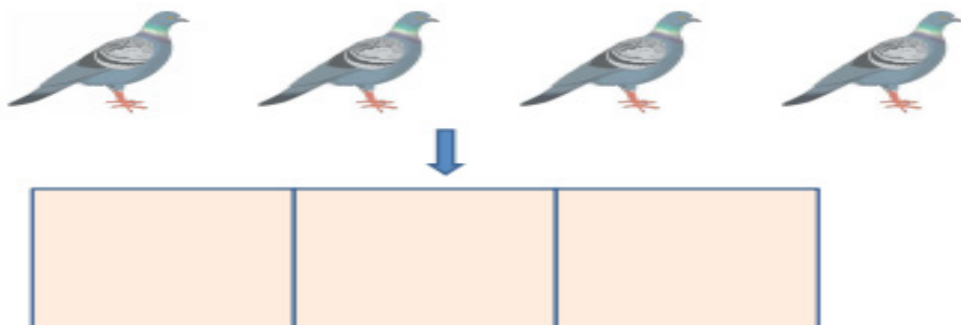
$$x \mapsto f(x) = \frac{ax + b}{cx + d} \text{ is bijective}$$

Example 1

Consider the set of pigeons and the set of pigeonholes on the diagram below to answer the questions:

Determine whether it can be established or not between the two sets:

- A mapping,
- A one-to-one mapping,
- An onto mapping,
- A bijective mapping:



Solution:

Let the pigeons be numbered a, b, c, d and the pigeonholes be numbered $1, 2, 3$.

- It is possible to establish a mapping between the two sets. For example, $\{(a, 1); (b, 2); (c, 3); (d, 3)\}$. This function is a mapping since each pigeon is accommodated in exactly one pigeonhole, though pigeons c and d are in the same pigeonhole.
- It is not possible to establish a one-to-one mapping, since sharing images is not allowed. A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements that have the same image.
- The example given in part (a) illustrates a mapping that is onto: no pigeonhole is empty.
- It is impossible to define a bijection, since it is already impossible to establish a one-to-one mapping.

Example 2

Determine whether function $f: \mathbb{R} \rightarrow \mathbb{R}$

$$x \mapsto f(x) = 3x - 5 \text{ is (or is not)}$$

- One-to-one
- Onto
- Bijjective.

Solution:

- Let x_1 and x_2 be real numbers such that $f(x_1) = f(x_2)$. Then $3x_1 - 5 = 3x_2 - 5$

This is equivalent, successively to $3x_1 = 3x_2$ (by adding 5 on both sides);

$$x_1 = x_2 \text{ (Dividing both sides by 3)}$$

Since the equality $f(x_1) = f(x_2)$ implies $x_1 = x_2$, the function is one-to-one.

- Suppose y a real number. Let us look for real number x , if possible, such that $f(x) = y$.

Then $3x - 5 = y$. It follows that $x = \frac{y+5}{3}$; such x exists for any value of y ;

$$f\left(\frac{y+5}{3}\right) = 3\left(\frac{y+5}{3}\right) - 5 = y$$

Therefore, function f is onto.

c. Since, from points (a) and (b), f is one-to-one and onto, function f is bijective.

Example 3

Show that function f defined by $f: \mathbb{R} \rightarrow \mathbb{R}$

$$x \mapsto f(x) = x^2 + 2x - 3 \text{ is neither one-to-one,} \\ \text{nor onto}$$

Solution

$$f(-2) = (-2)^2 + 2(-2) - 3 = -3 \text{ and } f(0) = (0)^2 + 2(0) - 3 = -3$$

Since $f(-2) = f(0)$ and $-2 \neq 0$, the function is not one-to-one.

On the other side, there is no x such that $f(x) = -5$;

$$\text{in fact, } f(x) = -5 \Leftrightarrow x^2 + 2x - 3 = -5$$

$$\Leftrightarrow x^2 + 2x + 2 = 0$$

$$\text{No such } x \text{ since } \Delta = 2^2 - 4(1)(2) = -4 < 0$$

Therefore, the function is not onto

Example 4

Consider the function f defined by $f: \mathbb{R} \rightarrow \mathbb{R}$

$$x \mapsto f(x) = -x^2 + 4x$$

Determine the greatest subsets A and B of \mathbb{R} such that function

$$f: A \rightarrow B$$

$$x \mapsto f(x) = -x^2 + 4x \text{ is bijective}$$

Solution:

The maximum value of the function occurs for $x = 2$ and $f(2) = 4$. Therefore,

$$A = [2, +\infty[\text{ and } B =]-\infty, 4], \text{ or } A =]-\infty, 2] \text{ and } B =]-\infty, 4]$$

Example 5

Function $f: \mathbb{R} - \{x_0\} \rightarrow \mathbb{R} - \{x_1\}$

$$x \mapsto f(x) = \frac{2x-5}{3-x} \text{ is bijective}$$

- Find the values of x_0 and x_1
- Show that f is one-to-one
- Find the real number whose image is 2

Solution:

- a. f is bijective if $x_0 = 3$ and $x_1 = -2$, (from the restriction $f: \mathbb{R} - \{-\frac{d}{c}\} \rightarrow \mathbb{R} - \{\frac{a}{c}\}$)

$$x \mapsto f(x) = \frac{ax+b}{cx+d} \text{ is bijective, for our case}$$

$$x_0 = -\frac{d}{c} \text{ and } x_1 = \frac{a}{c}; \text{ where } a = 2, b = -5, c = -1, d = 3,$$

- b. Let $x_1 \neq 3$ and $x_2 \neq 3$ be such that $f(x_1) = f(x_2)$, that is $\frac{2x_1-5}{3-x_1} = \frac{2x_2-5}{3-x_2}$

$$\text{Then } 6x_1 - 2x_1x_2 - 15 + 5x_2 = 6x_2 - 2x_1x_2 - 15 + 5x_1,$$

$$\text{which is equivalent to } 6(x_1 - x_2) - 5(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2$$

Therefore, f is one-to-one

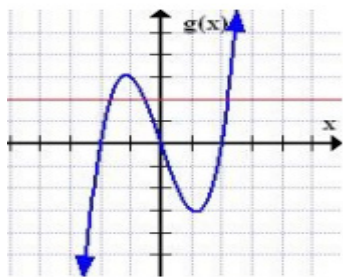
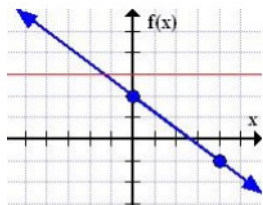
- c. Let x be the number. Then $f(x) = 2 \Leftrightarrow \frac{2x-5}{3-x} = 2$

Solving this equation, we get $x = 2$

Horizontal Line Test

Horizontal Line Test states that a function is a one to one (injective) function if

there is no horizontal line that intersects the graph of the function at more than one point.

Graph representation	Interpretation	Conclusion
	<p>You can see that for this graph, there are horizontal lines that intersect the graph more than once.</p>	<p>Not injective</p>
	<p>You can see that for this graph any horizontal line intersects the graph only once.</p>	<p>Injective function</p>



Application activity 1.2

- Let A and B be two non-empty sets where $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Consider each of the following relations:

$$T = \{(1, a), (2, b), (2, c), (3, c), (4, b)\}$$

$$U = \{(1, a), (2, b), (4, b)\}$$

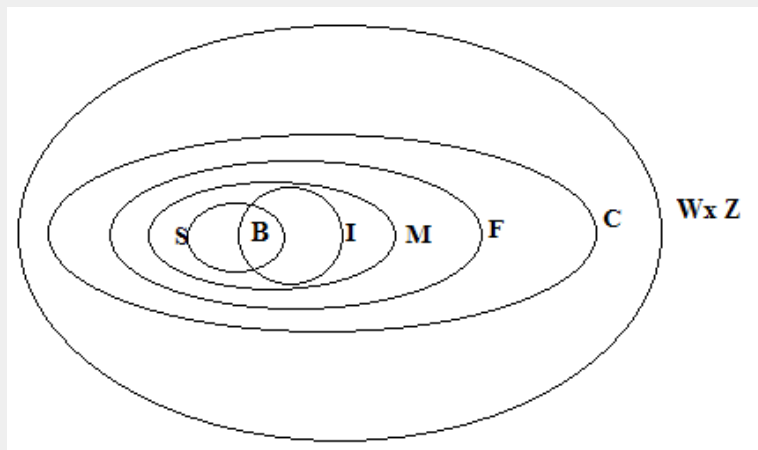
$$V = \{(1, a), (2, b), (3, c), (4, b)\}$$

Which of these relations (T , U and V) qualify as functions?

- Although the relation in Question 1 above is a function, it is not a one-to-one (or injective) function. Why?
 - Is the relation V above defined an onto (surjective) function? Why?
 - Does the function f , defined by the relation V , have an inverse?
- Find a function $f: \mathbb{N} \rightarrow \mathbb{N}$ that is injective, but not surjective.
 - Find a function $f: \mathbb{N} \rightarrow \mathbb{N}$ that is surjective, but not injective.

4. Discuss whether the following functions are bijective:
 $f(x) = 3^x$ and $g(x) = x^3$ if no, which condition fails?
5. Let W and Z be sets; $W \times Z$ the Cartesian product of W and Z ;
 C be the set of all correspondences (relations) from W to Z ;
 F be the set of all functions from W to Z ;
 M be the set of all mappings from W to Z ;
 I be the set of all one to one mapping from W to Z ;
 S be the set of all onto mappings from W to Z ;
 B be the set of all bijective mappings from W to Z ;

Then we have the following sequence of inclusion of sets



Using examples, explain in your own words the relationship amongst these sets and decide on the following inclusions: $S \subset M \subset F \subset C \subset (W \times Z)$

1.3. Domain and range of a numerical function

1.3.1. Existence condition for a given function

Activity 1.3.1



Consider the following functions and calculate their numerical value at the given value(s) of x . Discuss your findings

a. $f(x) = \frac{1}{x}$ at $x = 0$

b. $f(x) = \frac{2+x}{x-3}$ at $x = 1, x = 2, x = 3$

c. $f(x) = \frac{2(x-1)}{x-1}$ at $x = 0, x = 1$

d. $f(x) = \frac{x+1}{x^2-1}$ at $x = -1, x = 1$ and $x = 2$

e. $f(x) = \frac{2(x-1)}{x^2+1}$ at $x = -1, x = 1$ and $x = 2$

Content summary

The conditions or restrictions on the independent variable can be formulated as follow:

-Is there any denominator containing the independent variable? If yes, we set any denominator containing the independent variable to be different from zero, and we solve to find the restrictions on the independent variable.

-Is there any n^{th} root, with even index, greater or equal to two, where the radicand is containing the independent variable? If yes, we set the radicand to be greater or equal to zero, and we solve the inequality to find the restrictions on the independent variable.

The goal to succeed in getting the restrictions on the independent variable is to know to solve equations and inequalities.

Remember: $ax + b \neq 0 \Leftrightarrow x \neq -\frac{b}{a}; a \neq 0$

For $ax^2 + bx + c$:

If $b^2 - 4ac > 0$, then $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$; $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c \neq 0 \Leftrightarrow x \neq x_1$ and $x \neq x_2$

If $b^2 - 4ac = 0$, then $ax^2 + bx + c \neq 0 \Leftrightarrow x \neq -\frac{b}{2a}$

If $b^2 - 4ac < 0$, then $ax^2 + bx + c \neq 0$ for all real values of x .

To solve an inequality, first study the sign of the corresponding expression, then choose the convenient interval(s).

In particular, if $a \neq 0$, then:

x	$-\infty \dots \dots \dots -\frac{b}{a} \dots \dots \dots +\infty$
$ax + b$	Sign of $-a$ o sign of a

If $a \neq 0$ and $b^2 - 4ac > 0$, then:

x	$-\infty \dots \dots \dots x_1 \dots \dots \dots x_2 \dots \dots \dots +\infty$
$ax^2 + bx + c$	Sign of a o sign of $-a$ o sign of a

Where x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$

If $a \neq 0$ and $b^2 - 4ac = 0$, then:

x	$-\infty \dots \dots \dots -\frac{b}{2a} \dots \dots \dots +\infty$
$ax^2 + bx + c$	Sign of a o sign of a

Where x_1 and x_2 are the roots of the equation $ax^2 + bx + c = 0$

If $b^2 - 4ac < 0$, then:

x	$-\infty \dots\dots\dots +\infty$
$ax^2 + bx + c$	Sign of a

the equation $ax^2 + bx + c = 0$ has no real roots.

A radical function is a function that contains the independent variable under the square root sign. For example, $f(x) = \sqrt{7-x}$ is a radical function

An irrational function is a function that contains the independent variable in the radicand; the index may be any positive integer ≥ 2 . Thus all radical functions are irrational functions, but the converse is not true.

The following table gives examples of restrictions for irrational functions.

Function	Restriction
$f(x) = \sqrt{x}$	$x \geq 0$
$f(x) = \sqrt{x+10}$	$x+10 \geq 0$
$f(x) = \sqrt{-x}$	$-x \geq 0; x \leq 0$
$f(x) = \sqrt{x^2 - 1}$	$x^2 - 1 \geq 0$
$f(x) = \sqrt{x^2 + 1}$	$x^2 + 1 \geq 0$ (always true for all real values of x), therefore, there is no restriction for this function.

Examples

For each of the following functions determine the restrictions on the independent variable

1. $f(x) = \sqrt{x^2 - 1} + \frac{1}{x}$

2. $f(x) = \tan x$

3. Let $f(x) = x+2$ and $g(x) = \sqrt{x+1}$, hence find the restrictions for the

$$\text{function } h(x) = \frac{g(x)}{f(x)},$$

$$4. \quad k(x) = \frac{3x^2}{5x-1}$$

Solution:

1. $x^2 - 1 \geq 0$ and $x \neq 0$. This is equivalent to $x \leq -1$ or $x \geq 1$

2. $\cos x \neq 0$, that is $x \neq \pm \frac{\pi}{2} + 2k\pi; k \in \mathbb{Z}$

3. For $f(x)$: no restriction

For $g(x)$: $x+1 \geq 0$; that is $x \geq -1$

For $\frac{f(x)}{g(x)}$: $x+1 \geq 0$ and $x+2 \neq 0$, result to then $x \geq -1$

4. $5x-1 \neq 0; x \neq \frac{1}{5}$



Application activity 1.3.1

Find out the condition (s) or the restriction(s) for the existence of the image

under the following functions: i) $h(x) = \frac{x^2 - 3}{x - 4x^3}$ ii) $f(x) = \frac{9}{\sqrt{4-x^2}}$

iii) $f(x) = \frac{\sqrt[3]{x^3+x}}{5x}$ iv) $f(x) = \frac{3x}{x^2+4}$

1.3.2. Domain and range of polynomial functions

Activity 1.3.2



- Given the function $f(x) = x - 4$. Plot the graph of $f(x)$ and discuss whether it continues endlessly or not.
- Consider the function $f(x) = \frac{1}{2}(x-2)^2 - 4$. complete the following table

and use it to plot the graph of $f(x)$. Can you predict all values of x which can make $f(x)$ not defined.

x	-5	-4	-3	-2	-1	0	1	2	3	4
y										

3. Consider the function $f(x) = x^3 - 3x^2 + 3x - 2$. complete the following table and use it to plot the graph of $f(x)$. Can you predict all values of x which can make $f(x)$ defined?

x	-8	-4	-2	0	1	2	3	4	5
y									

4. For which value(s) of x is the function

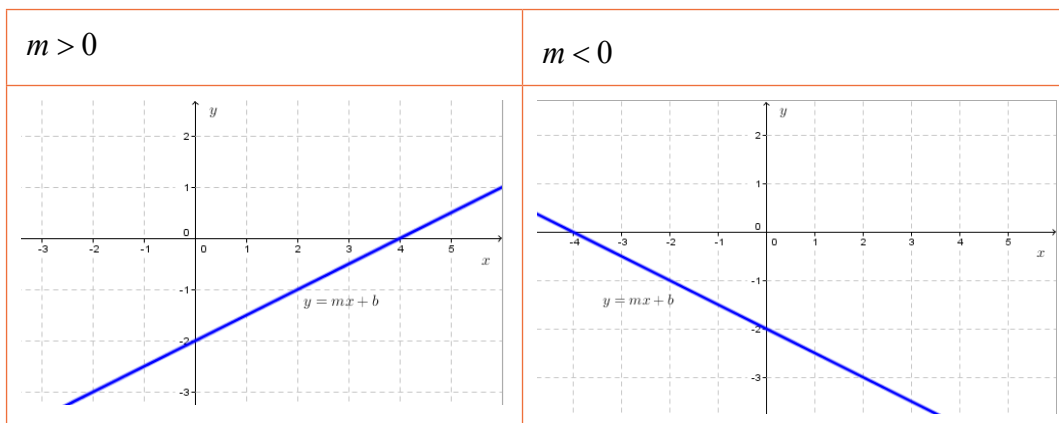
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ defined? $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real coefficients and $n = 0, 1, 2, 3, 4, \dots$

Content summary

The domain of any **linear functions** is the set of all real numbers,

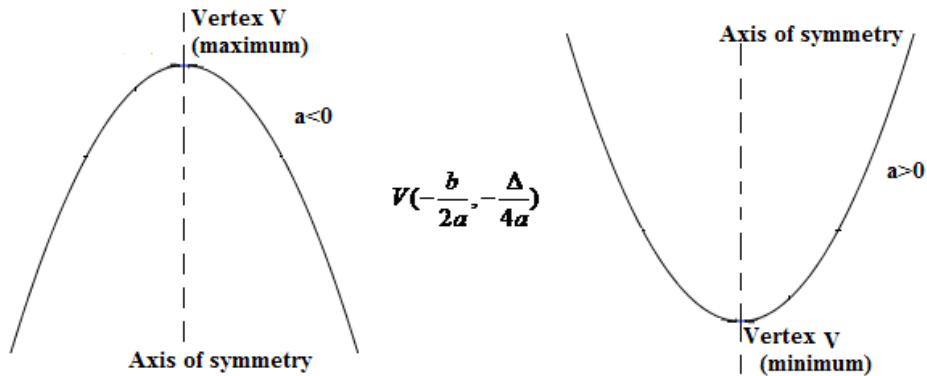
that is $Domf = \mathbb{R}$ or $Domf =]-\infty, +\infty[$. Similarly, the range of a linear function f , denoted Imf , is the set of all real numbers, that is $Imf =]-\infty, +\infty[$

Depending on the sign of m in the equation $y = mx + b$, the trend of the graph is as follows:



From the graphs, one can observe that each value of x has its corresponding y value.

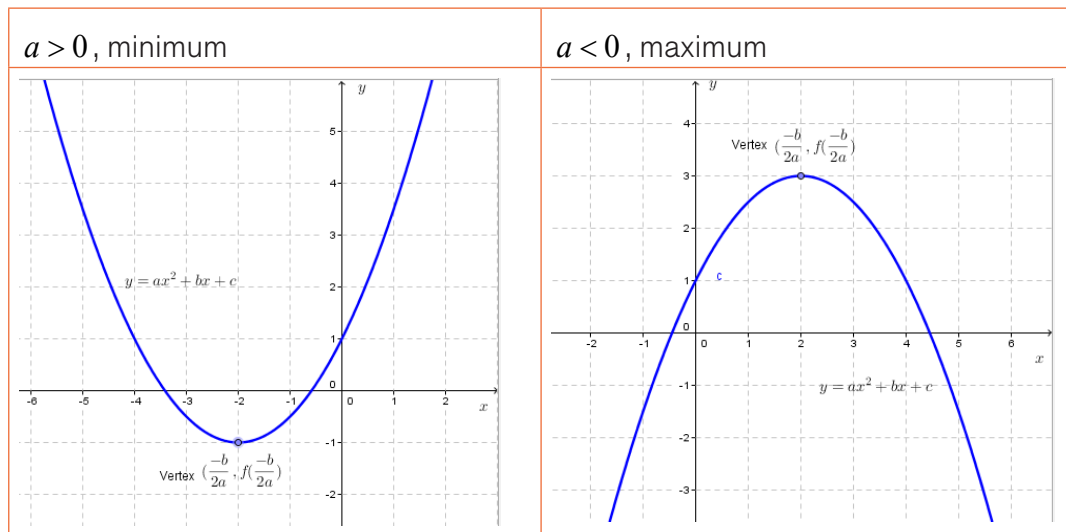
For quadratic functions $y = ax^2 + bx + c$, the main features are summarized on the graph below:



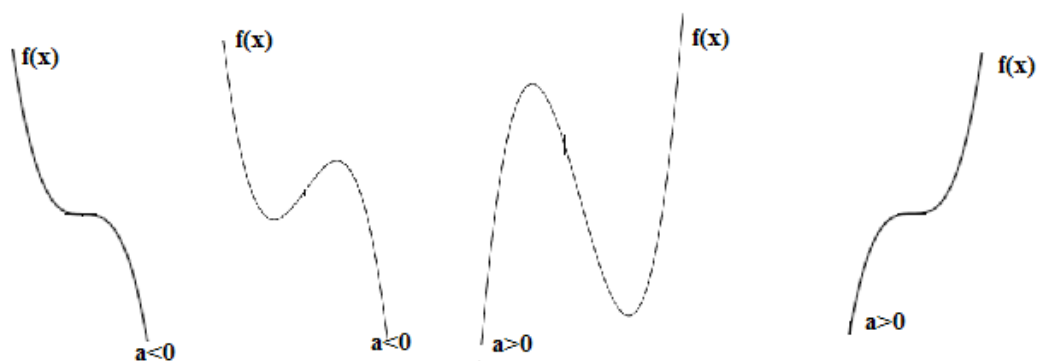
Not that $v = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right) \right) \Leftrightarrow v = \left(\frac{-b}{2a}, -\frac{\Delta}{4a} \right)$

If $a > 0$, then the range of function $y = ax^2 + bx + c$ is $[-\frac{\Delta}{4a}, +\infty[$

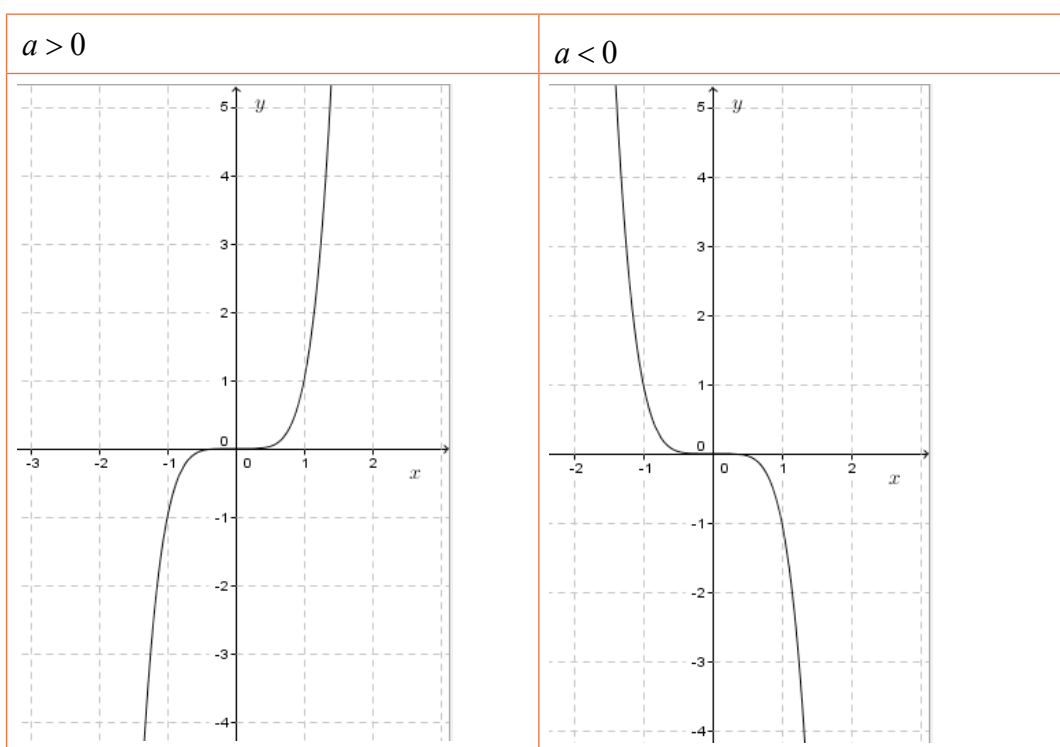
If $a < 0$, then the range of function $y = ax^2 + bx + c$ is $] -\infty, -\frac{\Delta}{4a}]$



For cubic functions $f(x) = ax^3 + bx^2 + cx + d; a \neq 0$, the trends of the graphs are as shown below:

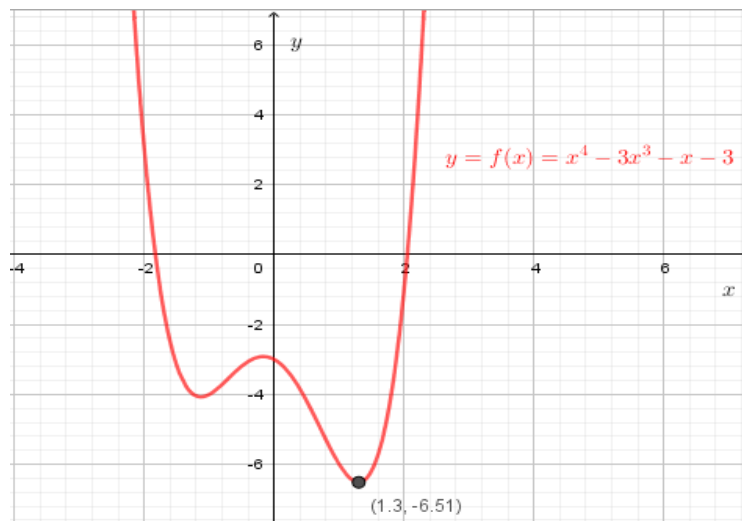


In each case, the domain is $]-\infty, +\infty[$ and the range is $]-\infty, +\infty[$



If for polynomials of odd degrees the range is the set of all real numbers, it is not the case for polynomials of even degree, greater or equal to 4. The determination of the range is not easy unless the function is given by its graph; in this case, find by inspection, on the y -axis, the set of all points such that the horizontal lines through those points cut the graph.

Determine the domain and range of $f(x) = x^4 - 3x^3 - x - 3$ shown on the graph below:



Solution:

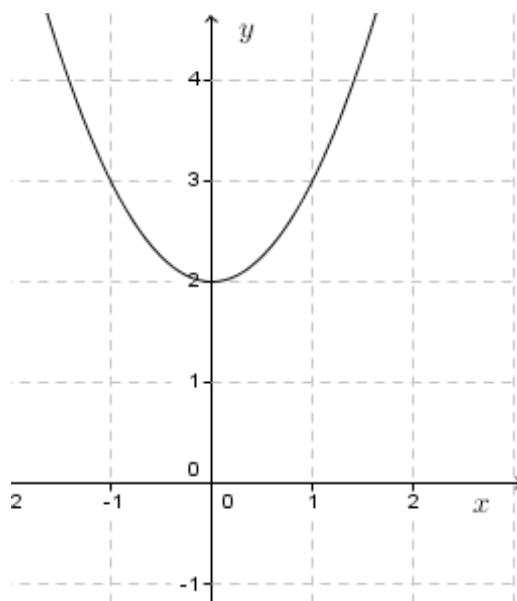
$\text{dom } f =]-\infty, +\infty[$ and $\text{Im } f = [-6.51, +\infty[$

Example 2:

Find the range for the function $f(x) = x^2 + 2$

Solution:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function given by $f(x) = x^2 + 2$. Then the range is $[2, +\infty[$ as shown on the graph below:





Application activity 1.3.2

Find the domain and range of the following functions.

a) $f(x) = -x^2 + 1$

b) $f(x) = 2x^3 - x + 1$

c) $f(x) = 3x + 1$

1.3.3. Domain and range of rational functions

Activity 1.3.3



For which value(s) the following functions are not defined

a. $f(x) = \frac{1}{x-1}$

b. $f(x) = \frac{x}{(x+1)(x-3)}$

c. $f(x) = \frac{x}{(x-1)(x+3)}$

Content summary

Given that $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomials, then the domain of definition is the set of real numbers excluding all values where the denominator is zero. That is $Domf = \{x \in \mathbb{R} : h(x) \neq 0\}$

From activity 1.3.3, one can determine the intervals for which the given functions are valid.

Example 1

Find the domain of each of the functions:

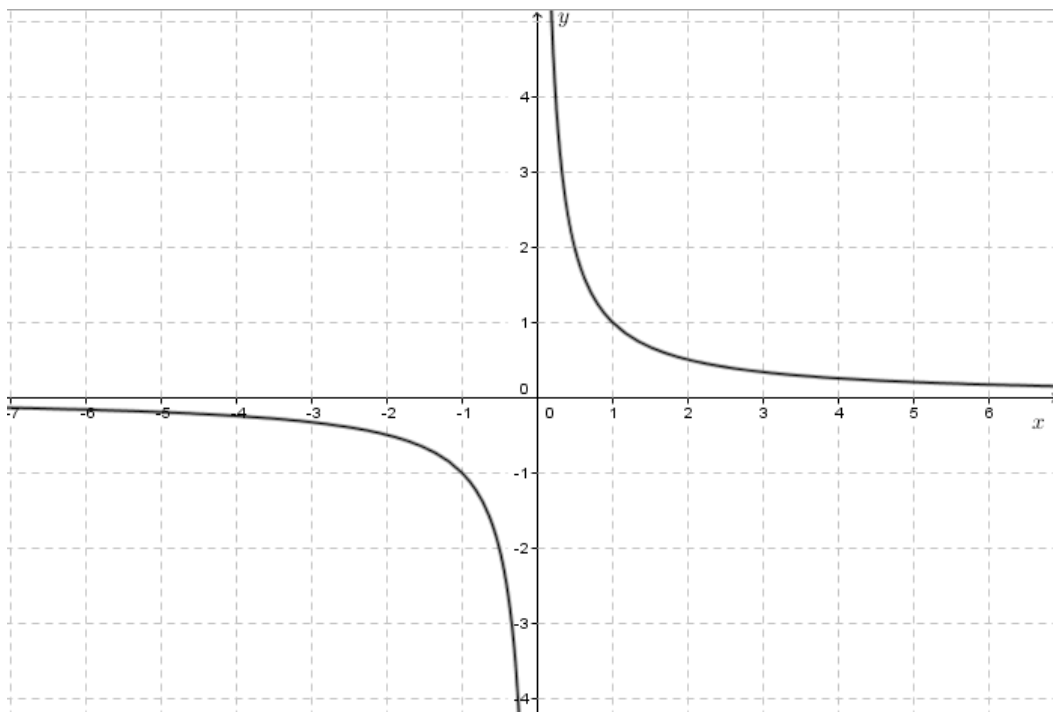
a. $f(x) = \frac{1}{x}$

b. $f(x) = \frac{x}{(x-1)(x+3)}$

c. $f(x) = \frac{x+1}{3x+6}$

Solution:

- a. The denominator should be different from zero ($x \neq 0$), domain of definition is $\{x \in \mathbb{R} : x \neq 0\}$ or $\mathbb{R}^* \text{ or } \mathbb{R}^+$. The domain can be written as an interval as follows: $]-\infty, 0[\cup]0, +\infty[$. Observing the graph of the function $f(x) = \frac{1}{x}$, one can early realize that the function has no value only if $x = 0$



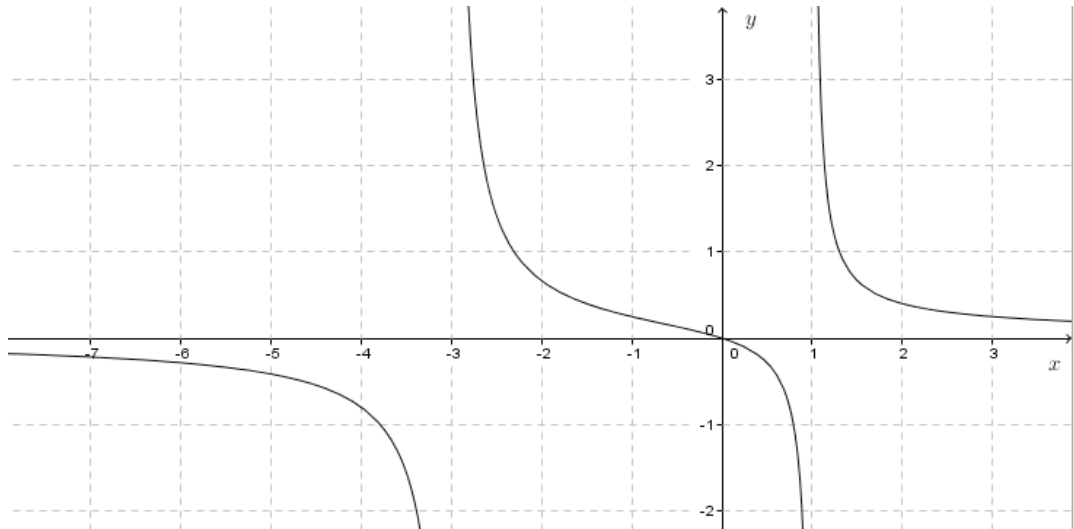
b. $f(x) = \frac{x}{(x-1)(x+3)}$

The denominator should be different from zero, $(x-1)(x+3) \neq 0$.

Domain of definition is $\{x \in \mathbb{R} : x \neq 1 \text{ and } x \neq -3\}$ or simply $]-\infty, -3[\cup]-3, 1[\cup]1, +\infty[$

.Observing the graph of the function $f(x) = \frac{x}{(x-1)(x+3)}$, one can early realize that

the function has no value only if $x = 1$ and $x = -3$



c. Condition: $3x + 6 \neq 0$

$$3x + 6 = 0 \Rightarrow x = -2$$

Then, $Domf = \mathbb{R} \setminus \{-2\}$ or $Domf =]-\infty, -2[\cup]-2, +\infty[$

Example 2:

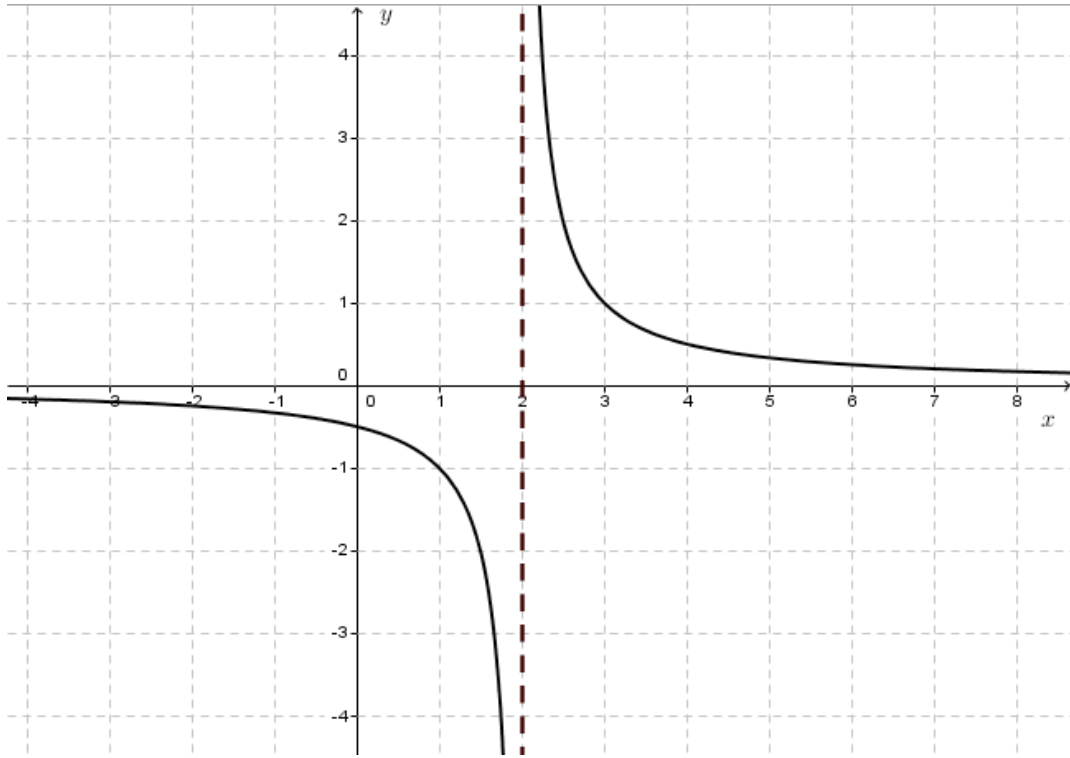
Find the range for the function $f(x) = \frac{1}{x-2}$

Solution

1. Put $y = f(x) = \frac{1}{x-2}$
2. Solve for x , $y = \frac{1}{x-2} \Leftrightarrow x = \frac{1}{y} + 2$. Note that x can be solved if and only if $y \neq 0$.

The range of $f(x)$ is $\{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} \setminus \{0\}$.

Alternatively, one can see on the graph that the range of $f(x)$ is $\mathbb{R} \setminus \{0\}$.



Example 3:

Find the range for the function $f(x) = \frac{2x+1}{x^2+2}$

Solution

1. Put $y = f(x) = \frac{2x+1}{x^2+1}$
2. Solve for x , $y = \frac{2x+1}{x^2+1} \Leftrightarrow yx^2 + y = 2x+1$.
 $yx^2 + y = 2x+1 \Leftrightarrow yx^2 - 2x + (y-1) = 0$

$$x = \frac{2 \pm \sqrt{4 - 4y(y-1)}}{2y} \text{ if } y \neq 0, \quad x = -\frac{1}{2} \text{ if } y = 0$$

$$= \frac{1 \pm \sqrt{1 - y^2 + y}}{y} \text{ if } y \neq 0$$

Comparing the two, we see that x exists in the set of real numbers if and only if $1 - y^2 + y \geq 0$, that is $y^2 - y - 1 \leq 0$

The range of $f(x)$ is $\{y \in \mathbb{R} : y^2 - y - 1 \leq 0\}$. Solving the inequality $y^2 - y - 1 \leq 0$, we get $y = \frac{1 \pm \sqrt{5}}{2}$. Then studying the sign of the quadratic expression,

$$y^2 - y - 1 = \left(y - \frac{1 - \sqrt{5}}{2}\right) \left(y - \frac{1 + \sqrt{5}}{2}\right)$$

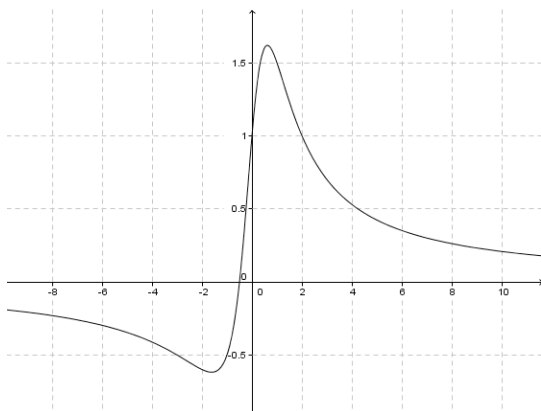
	$y < \frac{1 - \sqrt{5}}{2}$	$y = \frac{1 - \sqrt{5}}{2}$	$\frac{1 - \sqrt{5}}{2} < y < \frac{1 + \sqrt{5}}{2}$	$y = \frac{1 + \sqrt{5}}{2}$	$y > \frac{1 + \sqrt{5}}{2}$
$y - \frac{1 - \sqrt{5}}{2}$	-----	0	+++++	+++	+++
$y - \frac{1 + \sqrt{5}}{2}$	-----	-----	-----	0	+++
$y^2 - y - 1$	+++	0	-----	0	+++

From the table, we see that the range of the function $f(x) = \frac{2x+1}{x^2+2}$ is:

$$R(f) = \left\{ y \in \mathbb{R} : \frac{1 - \sqrt{5}}{2} \leq y \leq \frac{1 + \sqrt{5}}{2} \right\} = \left[\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right]$$

One can see on the graph that the range of $f(x)$ is

$$\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right] = [-0.618034; 1.61803]$$



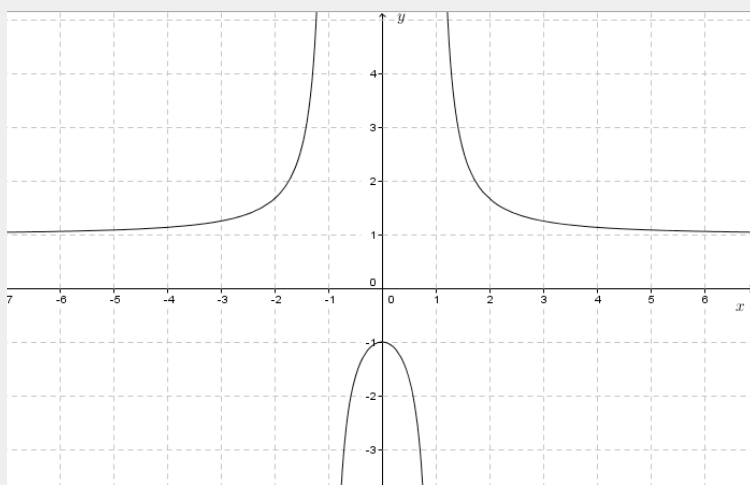
Application activity 1.3.3

1. Find the domain and range of the following functions

$$a) f(x) = \frac{2x+1}{6x^2-x-2} \quad b) f(x) = \frac{-6}{25x^2-4}$$

$$c) f(x) = \frac{2x-9}{x^3+2x^2-8x} \quad d) f(x) = \frac{5}{x-2}$$

2. Given the function $f(x) = \frac{x^2+1}{x^2-1}$, observe its graph and write down x values where the function is not defined.



Therefore $Domf =]-\infty, -1] \cup [1, +\infty[$

Example 2:

Find the domain of the function $h(x) = \frac{\sqrt{1-x^2}}{x}$

Solution

Conditions of existence $1-x^2 \geq 0$ and $x \neq 0$

- For the corresponding equation is $1-x^2 = 0$ and solving for the variable, we

$$\text{get: } (1-x)(1+x) = 0 \Rightarrow \begin{cases} 1-x = 0 \text{ or } x=1 \\ 1+x=0 \text{ or } x=-1 \end{cases}$$

- For $x \neq 0$ all real numbers are accepted except from zero. Combining the two conditions we get:

x	$-\infty$	-1	0	1	$+\infty$
$1+x$	- - -	0 + +	+ + +	+ + +	+ + +
$1-x$	+ + +	+ + +	+ + +	0 + +	- - -
$1-x^2$	- - -	0 + +	+ + +	0 - -	- - -
x	- - -	- - -	0 + +	+ + +	+ + +
$f(x) = \frac{\sqrt{1-x^2}}{x}$	Undefined	0 - -	+ +	0 Undefined	

$-1 \leq x \leq 1$ and $x \neq 0$. Therefore, $domf = [-1, 0[\cup]0, 1]$

Example 3

Find domain of definition of $f(x) = \sqrt[3]{x+1}$

Solution

Since the index in radical sign is odd number, then $Domf = \mathbb{R}$

Example 4

Find the domain of definition of $g(x)$ if $g(x) = \sqrt[4]{x^2 + 1}$

Solution

Condition: $x^2 + 1 \geq 0$

Clearly $x^2 + 1$ is always positive. Thus $Domg = \mathbb{R}$

Example 5

Find domain of $f(x) = \frac{x}{\sqrt{x^3 - 4x^2 + x + 6}}$

Solution

Conditions: $x^3 - 4x^2 + x + 6 \geq 0$ and $x^3 - 4x^2 + x + 6 \neq 0$. The two conditions are combined in one: $x^3 - 4x^2 + x + 6 > 0$

$$x^3 - 4x^2 + x + 6 = (x+1)(x-2)(x-3)$$

x	$-\infty$	-1	2	3	$+\infty$
$x+1$	- - -	0 + +	+ + +	+ + +	+ + +
$x-2$	- - -	- - -	0 + +	+ + +	+ + +
$x-3$	- - -	- - -	- - -	0 + +	+ + +
$x^3 - 4x^2 + x + 6$	- - -	0 + +	0 - -	0 + +	+ + +

Then, $Domf =]-1, 2[\cup]3, +\infty[$

Example 6

Find the range for the function $f(x) = \sqrt{1+5x}$

Solution

$$1 + 5x \geq 0 \text{ (Restrictions on } x\text{);}$$

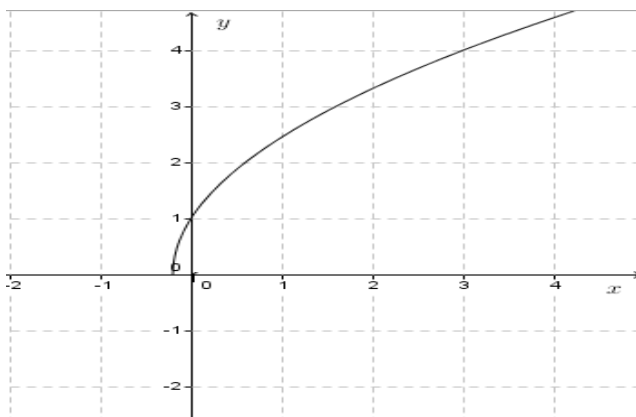
$$\sqrt{1 + 5x} \geq 0 \text{ (Taking the square roots);}$$

$$\text{But } f(x) = \sqrt{1 + 5x};$$

$$\text{Therefore, } f(x) \geq 0$$

The range of $f(x)$ is $\text{Im } f = [0, +\infty[$.

The graph below illustrates the range:



Application activity 1.3.3

1. Find the domain of definition for each of the following functions

$f(x) = \sqrt{4x-8}$	$g(x) = \sqrt{x^2+5x-6}$	$h(x) = \frac{x^3+2x^2-2}{\sqrt[3]{x+4}}$	$f(x) = \frac{x-2}{\sqrt[4]{x^2-25}}$
$f(x) = \sqrt{\frac{(x-1)^2}{x+4}}$	$h(x) = \sqrt{\frac{(x-1)(x+3)}{8-2x}}$	$f(x) = \frac{x-1}{\sqrt{2-x}}$	$f(x) = \sqrt{4-x^2}$

2. Find the range of each of the following functions

a) $f(x) = \sqrt{1-x}$

b) $f(x) = \sqrt{-x-3}$

1.4. Composition and inverse of numerical functions

1.4.1. Composition of functions

Activity 1.4.1



Given two functions $f(x) = 2x^2 + 2x - 5$ and $g(x) = 3x + 2$

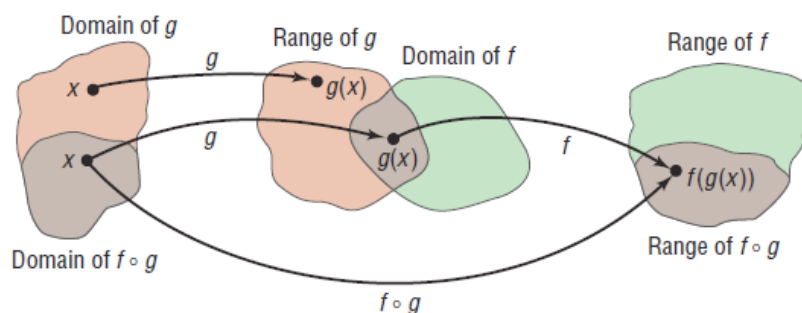
- Substitute the variable in $f(x)$ by the value of $g(x)$. Is the obtained function a function of x ?
- Substitute the variable in $g(x)$ by the value of $f(x)$. What do you notice?
- Name the obtained functions by $h(x)$ and $j(x)$ respectively. Compare the two new functions and explain if they are the same or different.

Content summary

Given two functions $f(x)$ and $g(x)$, one can “combine” them by substituting one function into the other as follow: $f[g(x)]$ or $g[f(x)]$.

Definition:

Let $f(x)$ and $g(x)$ be two functions of x such that the codomain of $f(x)$ is a subset of the domain of $g(x)$. The *composition* of $f(x)$ with $g(x)$, denoted by $(f \circ g)(x)$ is given by $(f \circ g)(x) = f(g(x))$. Indicates that $f(x)$ is a function from A to B and $g(x)$ is a function from C to D where $B \subseteq C$. The function given by $(f \circ g)(x) = f(g(x))$ is the composite of $f(x)$ and $g(x)$. The domain of $(f \circ g)(x)$ is the set of all x in the domain of $g(x)$ such that $g(x)$ is in the domain of $f(x)$ as shown by the figure below.



Only the x 's in the domain of g for which $g(x)$ is in the domain of f , can be in the domain of $f \circ g$. The reason is that if $g(x)$ is not in the domain of f then $f[g(x)]$ is not defined. Because of this, the domain of $f \circ g$ is a subset of the domain of g ; the range of $f \circ g$ is a subset of the range of f .

The domain of $(f \circ g)(x)$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f .

Here, we note that $(f \circ g)(x) \neq (g \circ f)(x)$

Examples

1. Let $f(x) = x^2$ and $g(x) = 2x + 1$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution

By the definition of composition, we have,

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = f(2x + 1) \\ &= (2x + 1)^2 = 4x^2 + 4x + 1 \end{aligned}$$

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = g(x^2) \\ &= 2x^2 + 1. \end{aligned}$$

Remark If the range of $f(x)$ is not contained in the domain of $g(x)$, then we have to restrict $f(x)$ to a smaller set so that for every x in that set, $f(x)$ belongs to the domain of $g(x)$. The domain of $(g \circ f)(x)$ is taken to be the following:

$$\mathbf{dom (g \circ f)(x) = \{x \in \mathbf{dom (f(x)) : f(x) \in \mathbf{dom (g(x))}\}.$$

2. Let $f(x) = x + 1$ and $g(x) = \sqrt{x}$. Find $(g \circ f)(x)$

Solution

Note that the domain of $f(x)$ is \mathbb{R} and the domain of $g(x)$ is $[0, +\infty[$. Thus, the domain of $(g \circ f)(x)$ is $\text{Dom}(g \circ f) = \{x \in \mathbb{R}; x + 1 \in [0, +\infty[\} = \{x \in \mathbb{R}; x \geq -1\} = [-1, +\infty[$



Application activity 1.4.1

Given the two functions $f(x) = x^2 + 1$ and $g(x) = x + 1$. Find the following

a) $f \circ g(x)$ b) $g \circ f(x)$ c) $f \circ g(1)$ d) $g \circ f(1)$ e) $f \circ g(x^2)$ f) $g \circ f(\sqrt{x})$

1.4.2 Inverse Functions

Activity 1.4.2



Given the function $f(x) = 4x + 6$

- Substitute $f(x)$ by y and solve for x
- After solving for x , replace x on (a) by $f^{-1}(x)$ and y by x .
- Find $f[f^{-1}(x)]$ and $f^{-1}[f(x)]$ and compare the results. What have you noticed?

Content summary

Definition: Let $f(x)$ and $g(x)$ be two functions such that $f[g(x)] = x$ for each x in the domain of g and $g[f(x)] = x$ for each x in the domain of f .

Under these conditions, the function g is the inverse of f . The function g is denoted by $f^{-1}(x)$, which is read as “ f -inverse”. So $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Note:

- If $f(x)$ is inverse of $g(x)$, then $g(x)$ is inverse of $f(x)$.
- The domain of f must be equal to the range of f^{-1} , and the range of f must be equal to the domain of f^{-1} .
- The notation $f^{-1}(x) \neq \frac{1}{f(x)}$

Example

Let the function $f(x) = \sqrt{2x-3}$ find the inverse function $f^{-1}(x)$.

Solution

$f(x) = \sqrt{2x-3}$ Rewrite the original function

$y = \sqrt{2x-3}$ Replace $f(x)$ with y

$x = \sqrt{2y-3}$ Interchange x and y

$x^2 = 2y-3$ Square each side to find y

$$x^2 + 3 = 2y \Rightarrow y = \frac{x^2 + 3}{2}$$

$$\text{Therefore, } f^{-1}(x) = \frac{x^2 + 3}{2}$$



Application activity 1.4.2

- The demand function for a commodity is $p = \frac{14.75}{1+0.01x}, x \geq 0$. Where, P is the price per unit and x is the number of units sold.
 - Find x as a function of p
 - Find the number of units sold if the unit price is 10\$.
- For each of the following pair of functions $f(x)$ and $g(x)$ show that they are inverse to each other.

a. $f(x) = 5x+1$ and $g(x) = \frac{x-1}{5}$

b. $f(x) = 9-x^2, x \geq 0$ and $g(x) = \sqrt{9-x}, x \leq 9$

c. $f(x) = 1-x^3$ and $g(x) = \sqrt[3]{1-x}$

1.5. Even functions and odd functions

1.5.1. Even functions and symmetry

Activity 1.5.1



Consider the function $f(x) = x^2 - 1$

i. Complete the table below

x	-3	-2	-1	0	1	2	3
$y = x^2 - 1$											

ii. How are the images of any two opposite values of x ? Express this mathematically,

iii. Plot each pair of obtained points (x, y) in the Cartesian plane and deduce the graph.

iv. From the graph of $f(x) = x^2 - 1$, what can you say about the line $x = 0$

Content summary

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **even function** if and only if:

- $(\forall x \in Domf), -x \in Domf$
- $f(-x) = f(x)$, that is, any two opposite values of the independent variable have the same image under the function.

The graph of an even function is symmetrical about the y -axis.

Example 1

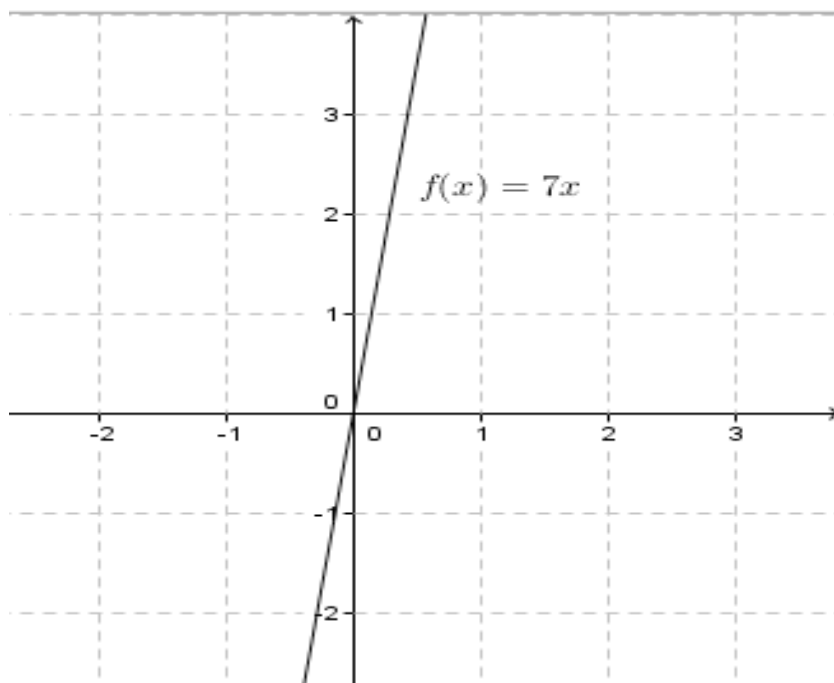
Determine whether the function $f(x) = 7x$ is even or not.

Solution

The domain of function f is the set of all real numbers. For any real number x , the opposite $-x$ is also a real number and $f(-x) = -7x \neq 7x = f(x)$

Since $f(-x) \neq f(x)$, function f is not even.

Graphically,



The graph is not symmetrical about the y -axis

Example 2

Determine whether the function $f(x) = 3x^2 - 4$ is even or not.

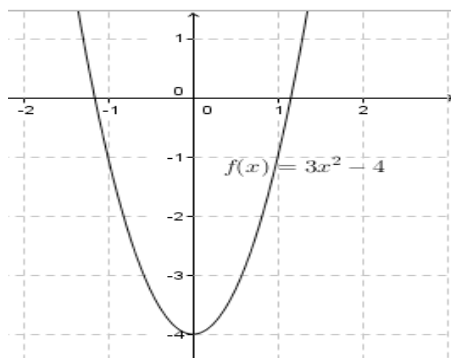
Solution

$$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$$

Since $f(-x) = f(x)$, function f is even.

Remember that $(-x)^n = \begin{cases} x^n, & \text{if } n \text{ is even} \\ -x^n, & \text{if } n \text{ is odd} \end{cases}$

The graph of the function is symmetrical about the y -axis as shown on the diagram below:



Example 3

Determine whether the function $g(x) = x^6 - x^4 + x^2 + 9$ is even or not.

Solution

$$\begin{aligned} g(-x) &= (-x)^6 - (-x)^4 + (-x)^2 + 9 \\ &= x^6 - x^4 + x^2 + 9 \\ &= g(x) \end{aligned}$$

Therefore, the function is even.

Example 4

Determine whether the function $f(x) = \frac{3x+1}{x^2-25}$ is even or not.

Solution

$$f(-x) = \frac{3(-x)+1}{(-x)^2-25} \Leftrightarrow f(-x) = \frac{-3x+1}{x^2-25}. \text{ Therefore, the function is not even since } f(-x) \neq f(x)$$

Example 5

Given functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x-4}$, find the $f(x) \cdot g(x)$ and determine if the result is an even function or not.

Solution

Provided $x \geq 0$ and $x - 4 \geq 0$,

$$f(x).g(x) = \sqrt{x} \cdot \sqrt{x-4} \Leftrightarrow f(x).g(x) = \sqrt{x^2 - 4x}$$

$f.g(-x) = \sqrt{(-x)^2 - 4(-x)} \Leftrightarrow f.g(-x) = \sqrt{x^2 + 4x}$. Therefore, the function is not even since

$$f.g(x) \neq f.g(-x)$$

Notice that the conclusion could have been drawn from the fact that $x \geq 4$ does not imply $-x \geq 4$, thus function f is not even



Application activity 1.5.1

Determine whether the following functions are even or not.

1. $f(x) = \frac{x^2 + 1}{x^4 + 3}$ 2. $f(x) = x^{\frac{2}{3}}(x - 4)$ 3. $f(x) = x\sqrt{9 - x}$

1.5.2. Odd functions and symmetry

Activity 1.5.2



Given the function $f(x) = x^3$

i. Complete the table below

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

How are the images of any two opposite values of x ? Express this mathematically ($f(x)$ and $f(-x)$),

ii. Plot each pair obtained in the Cartesian plane.

iii. How is the graph with respect to the origin O?

iv. Which name can you give to the origin (0,0) with respect to the graph of $f(x)$?

Content summary

Let $f(x)$ be a numerical function whose domain is $Domf$.

$f(x)$ is said to be an **odd function** if and only if:

- i. $(\forall x \in Domf), -x \in Domf$
- ii. $f(-x) = -f(x)$, that is, any two opposite values of the independent variable have opposite images under the function.

The graph of an odd function is symmetrical about the *origin*.

Note: Some functions are neither even nor odd

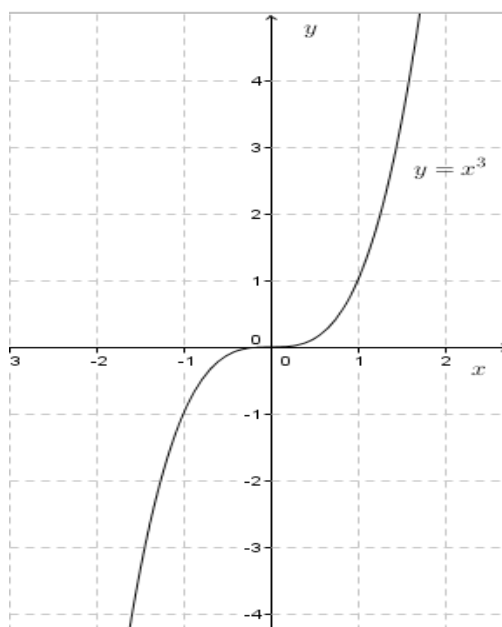
Example 1:

Determine whether the function $f(x) = x^3$ is odd or not

Solution

$f(-x) = (-x)^3 \Leftrightarrow f(-x) = -x^3$ and $-f(x) = -x^3$. Therefore, $f(-x) = -f(x)$ and the function $f(x) = x^3$ is odd.

Graphically, the point $(0,0)$ is the center of symmetry for the graph of the function $f(x) = x^3$.



Example 2:

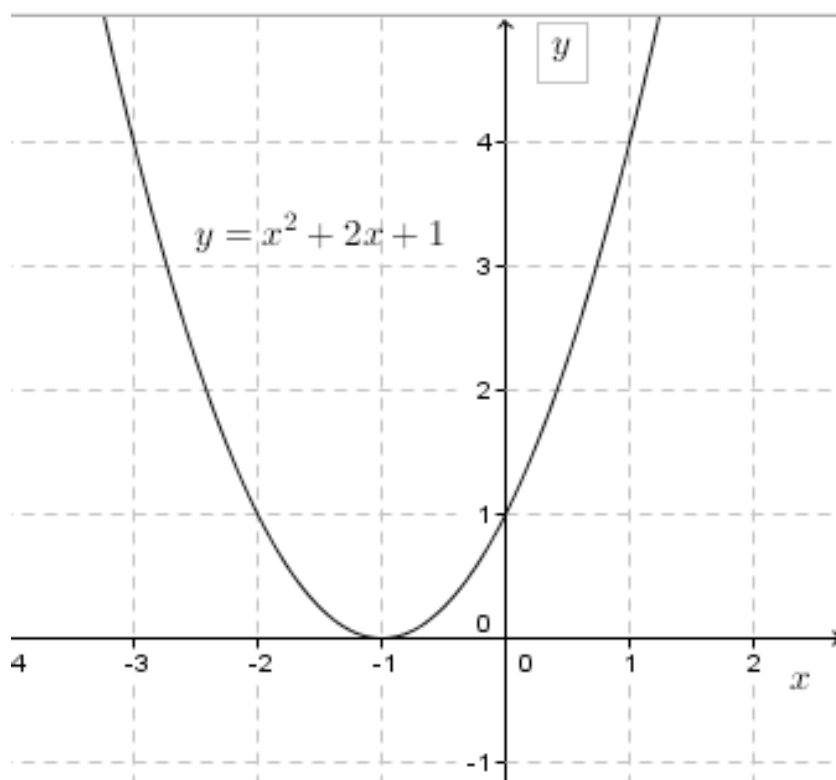
Determine whether the function $f(x) = x^2 + 2x + 1$ is odd, even or neither

Solution

$$f(-x) = (-x)^2 + 2(-x) + 1 \Leftrightarrow f(-x) = x^2 - 2x + 1 \text{ and } -f(x) = -x^2 - 2x - 1$$

$f(-x) \neq -f(x)$ and $f(-x) \neq f(x)$, therefore, the function $f(x) = x^2 + 2x + 1$ is not odd neither even.

Graphically, point $(0,0)$ is not the center of symmetry for the graph of the function $f(x)$, and the line $x=0$ is not the axis of symmetry for the graph of function $f(x) = x^2 + 2x + 1$.



Example 3:

Determine if the function $f(x) = \frac{x^3}{x^2 - 1}$ is even, odd, or neither and deduce the symmetry of its graph.

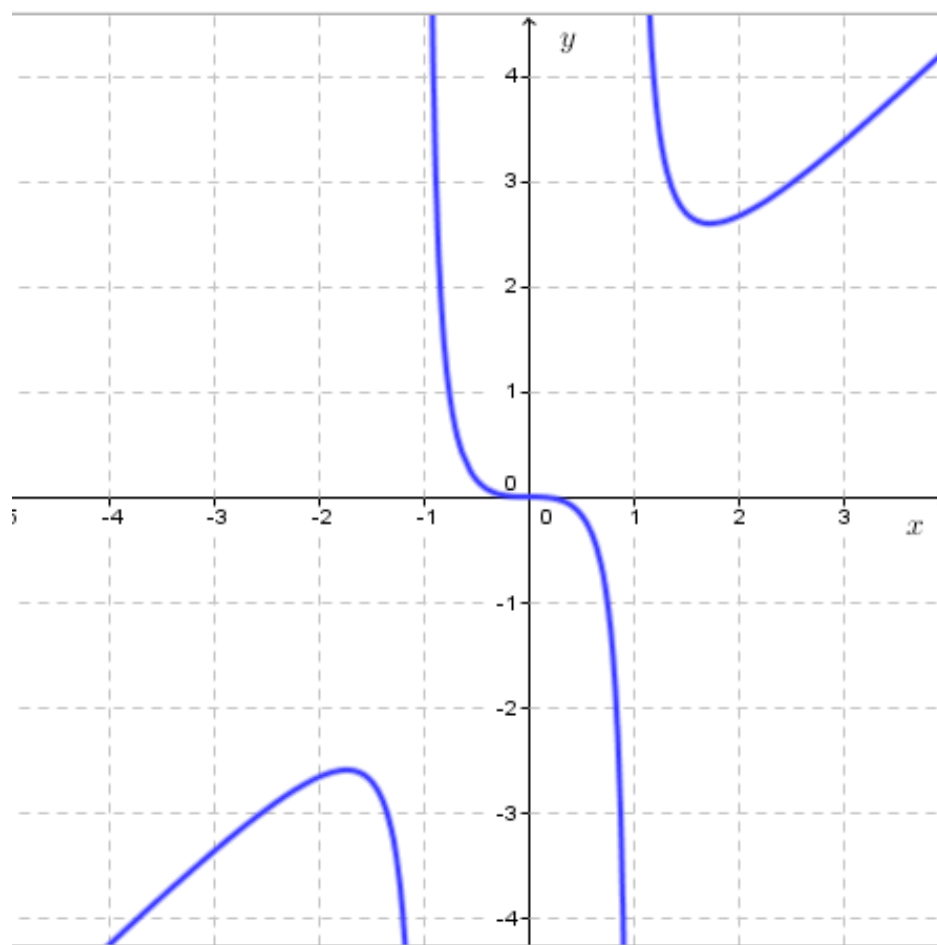
Solution

$$f(-x) = \frac{(-x)^3}{(-x)^2 - 1} = \frac{-x^3}{x^2 - 1} \neq f(x)$$

Therefore, the function is not even

But, $f(-x) = -f(x)$; it follows that f is an odd function.

The graph of $f(x) = \frac{x^3}{x^2 - 1}$ is shown below. It can be seen that point $(0, 0)$ is the center of symmetry for the graph





Application activity 1.5.2

1. Consider the function $f(x)$ defined in the set of real numbers $\mathbb{R} \rightarrow \mathbb{R}$

$$\text{by } f(x) = \frac{x^3}{9-x^2}$$

- i. Find the domain of definition of $f(x)$
 - ii. Determine if $f(x)$ is even, or odd function and deduce the symmetry of the graph of $f(x)$
2. For which value(s) of x the function $f(x) = \sqrt{x^2 + 5x + 6}$ is valid? Determine if $f(x)$ is an even or odd function

1.6 Introduction to limits of a function

1.6.1 Meaning of limit of a function

Activity 1.6.1



To find the value of a certain function $f(x)$ when x approaches 2, a student used a calculator and dressed a table as follows:

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

- a. Is it possible to put the values of x on a number line? Try to do it and locate the point $x = 2$
- b. Write 2 possible open intervals of the number line such that their centre is $x = 2$
- c. Try to approximate the value of $f(x)$ when x approaches 2.

Content summary

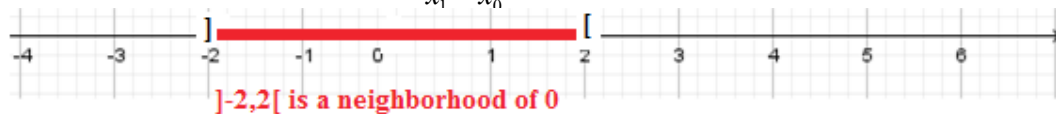
Neighborhood of a real number

Let x_0 be a real number. A δ -**neighborhood of** x_0 is any open interval centered at x_0 and with radius δ , that is the interval $]x_0 - \delta, x_0 + \delta[$

A **deleted δ -neighborhood of** x_0 is the set $]x_0 - \delta, x_0 + \delta[- \{x_0\}$, that is a δ -**neighborhood of** x_0 , excluding the center x_0 .

A neighborhood can be large or small; for example, the intervals $] -2, 2[$; $] -100, 100[$ and $] -0.001, 0.001[$ are all neighborhoods of 0, while the intervals

$] -2, 0[$ and $] -1.01; -0.99[$ are $\frac{y_1 - y_0}{x_1 - x_0}$ neighborhoods of -1.

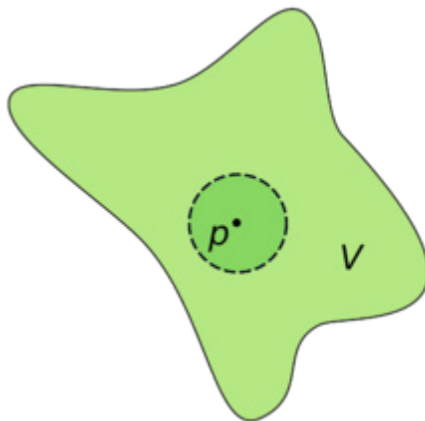


Note that the values of x are on both sides of $x = 2$ and that they become closer and closer to $x = 2$

Mathematically, a set N is called a neighborhood of point P if there exist an open interval I such that $x \in I$ and $I \subset N$. The collection of all neighborhoods of a point is called the **neighborhood system** at the point.

The center and the radius of the interval $]a, b[$ are $\frac{a+b}{2}$ and $\frac{b-a}{2}$, respectively.

A set V in the plane is a neighborhood of a point p if there exists a small disk around p is contained in V as illustrated below.

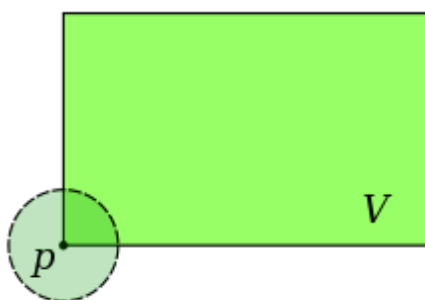


Example 1

- a. Determine whether interval I is a neighborhood of x_0 in each of the following cases”
- $I =]2, 3[$; x_0 is any element of I
 - $I = [2, 3]$; $x_0 = 3$
- b. Determine whether a rectangle is a neighborhood of any of its corners.

Solution

- a. i. I is a neighborhood of any of its elements x_0 , since it is always possible to center an open interval at x_0 , the interval being contained in I
- ii. I is not a neighborhood of x_0 , since it is not possible to center an open interval at x_0 , the interval being contained in I , that is any interval centered at x_0 , no matter how small it is, will have some elements outside I
- b. A rectangle is not a neighborhood of any of its corners.



In fact, it is not possible to center a disc at p so that the disc is entirely contained in the rectangle, no matter how small is the disc.

Example 2

Determine the ϵ -neighborhood of $x = 0$ and the deleted ϵ -neighborhood of $x = 0$

Solution:

The interval $] -1, 1[= \{y : -1 < y < 1\}$ is the ϵ -neighborhood of $x = 0$ on the real line
 $] -1, 0[\cup] 0, 1[=] -1, 1[\setminus \{0\}$ is the ϵ -deleted neighborhood of 0 .

Example 3

Find the center and the radius of the interval $]-1,3[$

Solution:

The center is $\frac{-1+3}{2}=1$ and the radius is $\frac{3-(-1)}{2}=2$, as shown on the diagram below:



In everyday language, people refer to “a speed limit”, “the limit of one’s endurance” or “stretching a spring to its limit”. All these phrases suggest that limit is a bound. This on some occasions may not be reached but on the other occasions may be reached or exceeded.

Consider a spring that will break if a weight of 10 newton or more is applied. To determine how far the spring will be stretched without breaking you could attach increasingly heavier weights, as long as they are less than 10 Newton, and measure the spring length s for each weight w . If the length s of the spring approaches a value L , then it is said that the limit of s as w approaches 10 is L .

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L .

This is written mathematically as $\lim_{x \rightarrow c} f(x) = L$

Example

You are given 24m of wire and asked to form a rectangle.

- Express the length l of the rectangle as function of the width w of the rectangle
- Express the area A of the rectangle as function of w .
- Complete the following table

Width (W)	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area (A)							

- Predict, from the table, the maximum area of the rectangle and the value of w for which it occurs.

Solution

- a. Let w be the width of the rectangle and l represent the length of the rectangle.

Because $2w + 2l = 24$ (perimeter is 24)

It follows that $l = 12 - w$ as shown below.



- b. Knowing that the area of the rectangle is $A = lw$, we have $A = (12 - w)w$
(Substitute $(12 - w)$ for l). $A = 12w - w^2$ After removing brackets.
- c. Using this model for area, you can obtain the following:

width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	37.75	35.99	36.00	35.99	35.75	35.00

- d. From the table, it appears that the maximum area is 36 m^2 and it occurs when $w = 6$.
- e. In limit terminology, we can say that **“the limit of A as w approaches 6 is 36.”** This is written and calculated as :

$$\begin{aligned}\lim_{w \rightarrow 6} A &= \lim_{w \rightarrow 6} (12w - w^2) \\ &= 12(6) - (6)^2 \\ &= 72 - 36 \\ &= 36\end{aligned}$$

To estimate numerically the $\lim_{x \rightarrow a} f(x)$ we have to construct a table that shows values of $f(x)$ for two sets of x -values: one set whose elements approach a from the left and another whose elements approach a from the right.



Application activity 1.6.1

- Apart of The Kingdom of Lesotho, give two examples of countries or Cities in the world that are surrounded by a single country or city.
- Give three examples of intervals that are neighborhoods of -5
- Is a circle a neighborhood of each of its points? Why
- Draw any plane and show three points on that plane for which the plane is their neighborhood.
- Use a table to estimate limits numerically.

a. $\lim_{x \rightarrow 2} (3x - 2)$

x	1.9	1.99	1.999	2.00	2.001	2.01	2.1
$f(x)$?			

b. $\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1}$

x	-0.1	-0.01	-0.001	-0.0001	\circ	0.0001	0.001	0.01	0.1
$f(x)$?				

c. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

x	1.9	1.99	1.999	2.00	2.001	2.01	2.1
$f(x)$?			

d. $\lim_{x \rightarrow 1} (2x^2 + x - 4)$

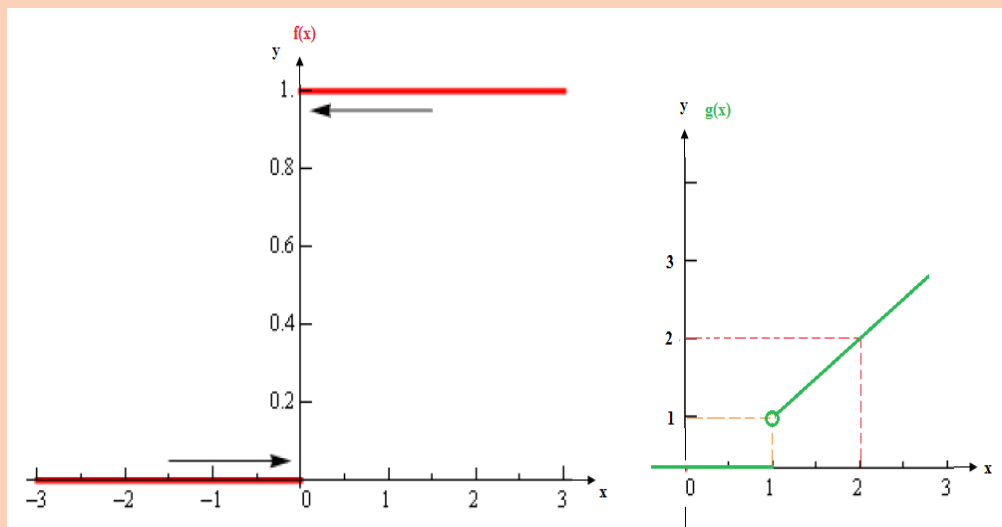
e. $\lim_{x \rightarrow -1} \frac{x+1}{x^2 - x - 2}$

1.6.2. Graphical interpretation and one-sided limits of a function

Activity 1.6.2



Consider the graphs of two functions $f(x)$ and $g(x)$ represented below:



a. Complete the following:

$$f(x) = \begin{cases} \dots & \text{for } x < 0 \\ \dots & \text{for } x \geq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \dots & \text{for } x \leq 1 \\ \dots & \text{for } x > 1 \end{cases}$$

- If we stay to the left side, as x approaches 0, $f(x)$ gets closer to
- If we stay to the right side, as x approaches 0, $f(x)$ gets closer to
- If we stay to the left side, as x approaches 1, $g(x)$ gets closer to
- If we stay to the right side, as x approaches 1, $g(x)$ gets closer to

Content summary

Right-handed limit

We say that $\lim_{x \rightarrow a^+} f(x) = L$ provided that we can make $f(x)$ as close to L , as we want, for all x sufficiently close to a and $x > a$, without actually letting x be a .

Left-handed limit

We say $\lim_{x \rightarrow a^-} f(x) = L$ provided that we can make $f(x)$ as close to L , as we want, for all x sufficiently close to a and $x < a$ without actually letting x be a .

For the right-handed limit, we now have $x \rightarrow a^+$ (note the “+”) which means that we will only look at $x > a$.

Likewise, for the left-handed limit, we have $x \rightarrow a^-$ (note the “-”) which means that we will only be looking at $x < a$.

So, when we are calculating limits, it’s important to pay attention to whether we are dealing with one-sided limit or not.

Condition of existence for a limit

If the value of $f(x)$ approaches L_1 as x approaches x_0 from the right side we write $\lim_{x \rightarrow x_0^+} f(x) = L$ and we read **“the limit of $f(x)$ as x approaches x_0 from the right equals L_1 ”**.

If the value of $f(x)$ approaches L_2 as x approaches x_0 from the left side we write $\lim_{x \rightarrow x_0^-} f(x) = L_2$ and we read **“the limit of $f(x)$ as x approaches x_0 from the left equals L_2 ”**.

If the limit from the left side is the same as the limit from the right side, that is, $\lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x) = L$, then we write $\lim_{x \rightarrow x_0} f(x)$ and we read “the limit of $f(x)$ as x approaches x_0 equals L ”.

Note that $\lim_{x \rightarrow x_0} f(x)$ exists if and only if $\lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x)$

Always recall that the value of a limit does not actually depend upon the value of the function at the point into consideration. The value of a limit depends only on the behavior of the values of the function around the point into consideration. Therefore, even if the function doesn’t exist at this point the limit can still have a value. To find graphically the limit $\lim_{x \rightarrow a} f(x)$, we must consider the behavior of the values of the independent variable x in the neighborhood of a , and analyze the

corresponding behavior of $f(x)$, and then draw conclusion about the limit of the function.

Example 1:

Calculate the limit of function $f(x) = \frac{|x-2|}{x^2+x-6}$ as x approaches 2

Solution

$$\lim_{x \rightarrow 2} \frac{|x-2|}{x^2+x-6}$$

$$|x-2| = \begin{cases} x-2, & \text{if } x \geq 2 \\ -(x-2), & \text{if } x < 2 \end{cases} \text{Applying the properties of absolute value}$$

$$\bullet \lim_{x \rightarrow 2^+} \frac{x-2}{x^2+x-6}$$

$$= \lim_{x \rightarrow 2^+} \frac{x-2}{(x-2)(x+3)} \text{ Factorizing the denominator}$$

$$= \lim_{x \rightarrow 2^+} \frac{1}{x+3} \text{ Simplifying like terms.}$$

$$= \frac{1}{(2)+3} = \frac{1}{5} \text{ Replacing } x \text{ by } 2 \text{ and evaluating.}$$

$$\bullet \lim_{x \rightarrow 2^-} \frac{-(x-2)}{x^2+x-6}$$

$$= \lim_{x \rightarrow 2^-} \frac{-(x-2)}{(x-2)(x+3)} \text{ Factorizing the denominator}$$

$$= \lim_{x \rightarrow 2^-} \frac{-1}{x+3} \text{ Simplifying like terms.}$$

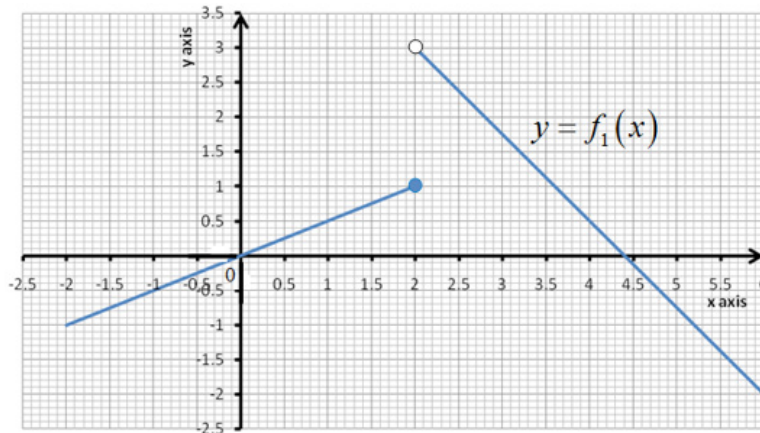
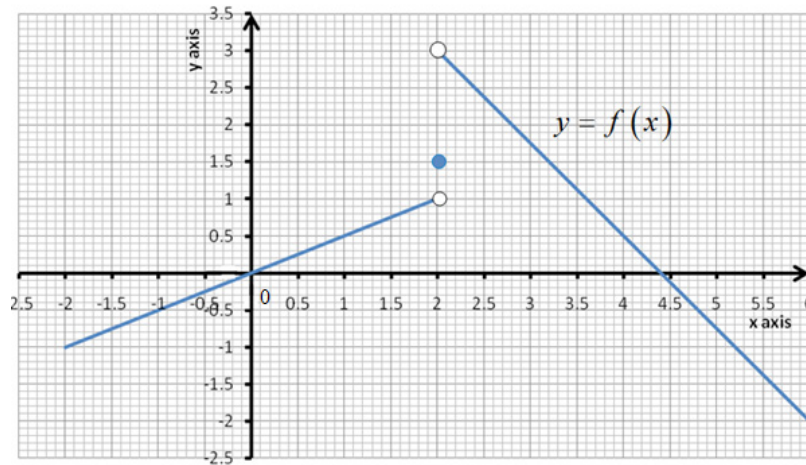
$$= \frac{-1}{(2)+3} = \frac{-1}{5} \text{ Replacing } x \text{ by } 2 \text{ and evaluating.}$$

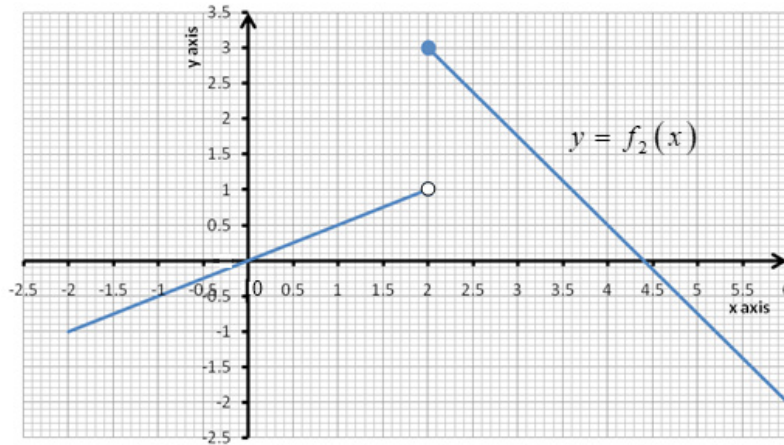
Note: $2^- \neq -2$

Conclusion $\lim_{x \rightarrow 2} f(x)$ does not exist since $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$

Example 2

Let $y = f(x)$, $y = f_1(x)$ and $y = f_2(x)$ be the functions whose graphs are shown below





a. Determine:

i. $f(2)$

ii. $f_1(2)$

iii. $f_2(2)$

b. Determine the limit of each function as:

i. x approaches 2 from left side

ii. x approaches 2 from right side

c. Determine whether the limit of function f , as x approaches 2 exist or not.

Solution:

a. We have:

i. $f(2) = 1.5$

ii. $f_1(2) = 1$

iii. $f_2(2) = 3$

b. i. As x approaches 2 from the left, $f(x)$ approaches 1, so $\lim_{x \rightarrow 2^-} f(x) = 1$.

As x approaches 2 from the left, $f_1(x)$ approaches 1, so $\lim_{x \rightarrow 2^-} f_1(x) = 1$

As x approaches 2 from the left, $f_2(x)$ approaches 1, so $\lim_{x \rightarrow 2^-} f_2(x) = 1$

ii. As x approaches 2 from the right, $f(x)$ approaches 3, so $\lim_{x \rightarrow 2^+} f(x) = 3$

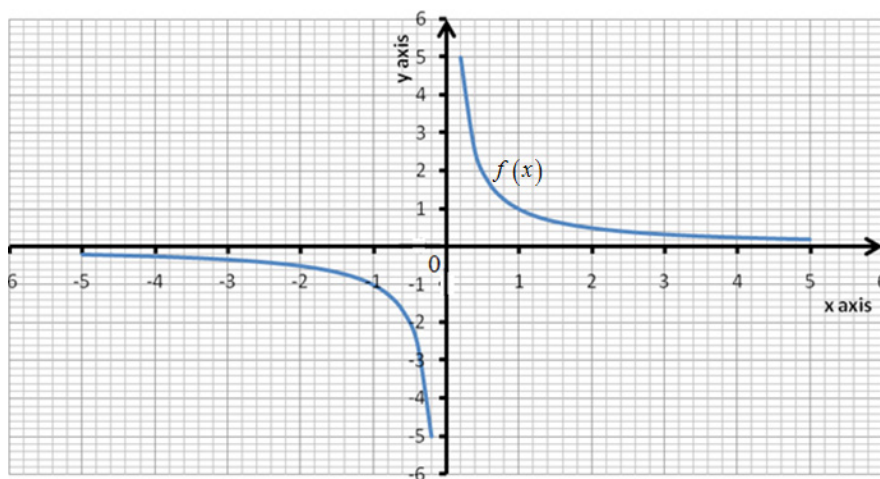
As x approaches 2 from the right, $f_1(x)$ approaches 3, so $\lim_{x \rightarrow 2^+} f_1(x) = 3$

As x approaches 2 from the right, $f_2(x)$ approaches 3, so $\lim_{x \rightarrow 2^+} f_2(x) = 3$

c. Since the left hand and right-hand limits at 2 are different, the limit of function f at $x=2$ does not exist, though the function f is defined at 2; $f(2) = 1.5$

Notice

Let f be the function whose graph is shown below:



As x approaches 0 from the right side, $f(x)$ gets larger and larger without bound and consequently approaches no fixed value. Thus, $\lim_{x \rightarrow 0^+} f(x)$ **does not exist.**

In this case, we would write $\lim_{x \rightarrow 0^+} f(x) = +\infty$ to indicate that the limit fails to exist because $f(x)$ is increasing without bound.

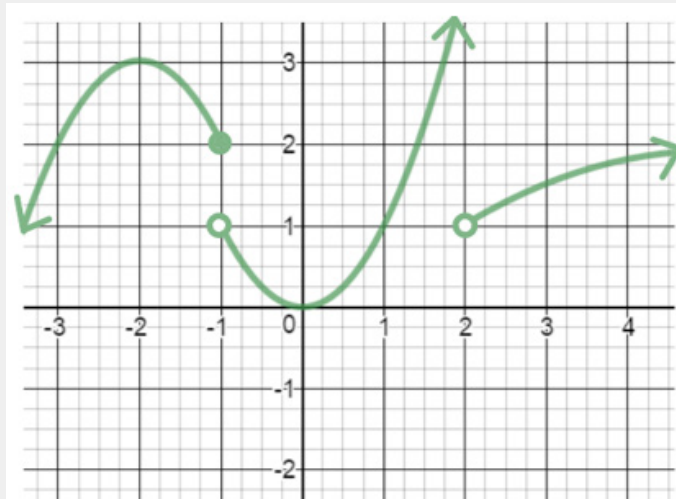
As x approaches 0 from the left side, $f(x)$ becomes more and more negative without bound and consequently approaches no fixed value. Thus, $\lim_{x \rightarrow 0^-} f(x)$ **does**

not exist. In this case, we write $\lim_{x \rightarrow 0^-} f(x) = -\infty$ to indicate that the limit fails to exist because $f(x)$ is decreasing without bound. As x gets larger and larger, $f(x)$ gets close to zero. Also, as x becomes more and more negative, $f(x)$ is close to zero. Thus, $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$



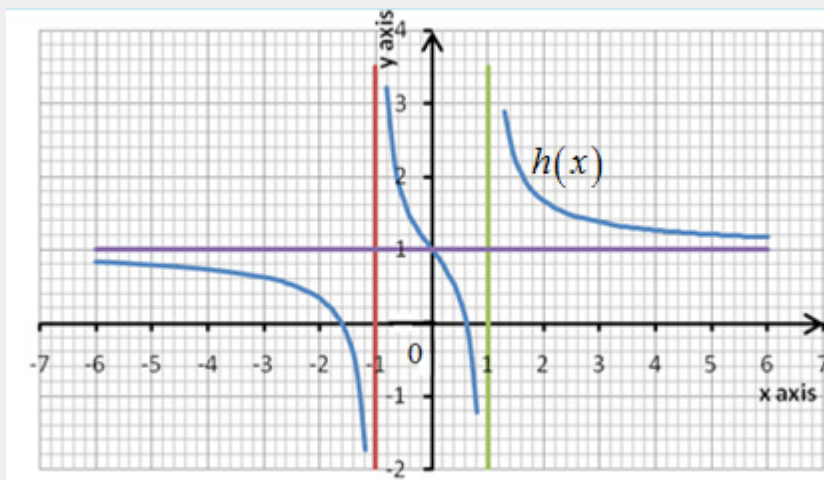
Application activity 1.6.2

1. Given the graph of the function illustrated below,



From the graph

- a. $\lim_{x \rightarrow -1^+} f(x)$ and $\lim_{x \rightarrow -1^-} f(x)$ and conclude about $\lim_{x \rightarrow -1} f(x)$.
- b. $\lim_{x \rightarrow 2^+} f(x)$ and $\lim_{x \rightarrow 2^-} f(x)$ and conclude about $\lim_{x \rightarrow 2} f(x)$.
2. Evaluate $\lim_{x \rightarrow -1} \frac{|x+1|}{x^2-1}$
3. Find $\lim_{x \rightarrow -1} h(x)$, $\lim_{x \rightarrow 1} h(x)$, $\lim_{x \rightarrow -\infty} h(x)$, $\lim_{x \rightarrow +\infty} h(x)$ using the following graph of $h(x)$



4. In the same Cartesian plane sketch the curves of $f(x) = x^2 + 5$, $g(x) = -x^2 + 5$, $h(x) = 5$. What can you say about the three curves referring to their curves and limits at point zero?

1.6.3. Properties of limits

Activity 1.6.3



Evaluate the following limits and make a comparison of the answers of each sub-question.

- a. $\lim_{x \rightarrow 0} [3(3x - 1)]$, $3 \left[\lim_{x \rightarrow 0} (3x - 1) \right]$
- b. $\lim_{x \rightarrow 0} (x^2)$, $\lim_{x \rightarrow 0} (3x - 1)$, $\lim_{x \rightarrow 0} (x^2 + 3x - 1)$
- c. $\lim_{x \rightarrow 1} (x^2 + 3x - 6)$, $\lim_{x \rightarrow 1} (x + 4)$, $\lim_{x \rightarrow 1} \frac{x^2 + 3x - 6}{x + 4}$
- d. $\lim_{x \rightarrow 2} (x - 1)$, $\lim_{x \rightarrow 2} (x + 4)$, $\lim_{x \rightarrow 2} (x^2 + 3x - 4)$
- e. $\lim_{x \rightarrow -4} [(x^2 + 1)^2]$, $\left[\lim_{x \rightarrow -4} (x^2 + 1) \right]^2$

Content summary

Assume that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist and that c is any constant. Then,

1. $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
2. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$, the limit of a sum or difference is equal to the sum or difference of the limits of the terms. This is also true for more than two functions.
3. $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$

The limit of a product is the same as the product of limits of the factors. This property applies for more than two factors.

$$4. \quad \lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ provided that } \lim_{x \rightarrow a} g(x) \neq 0$$

$$5. \quad \lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$$

In this property n can be any real number (positive or negative, integer, fraction, irrational, zero, etc.). In the case that n is an integer this rule can be thought of as an extended case of 3.

$$6. \quad \lim_{x \rightarrow a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

provided $f(x) \geq 0$ in a δ -neighborhood of a for some $\delta > 0$, if n is an even positive integer.

$$\begin{aligned} \lim_{x \rightarrow a} \left[\sqrt[n]{f(x)} \right] &= \lim_{x \rightarrow a} [f(x)]^{\frac{1}{n}} \\ &= \left[\lim_{x \rightarrow a} f(x) \right]^{\frac{1}{n}} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \end{aligned}$$

$$7. \quad \lim_{x \rightarrow a} c = c \text{ In other words, the limit of a constant is just the constant.}$$

$$8. \quad \lim_{x \rightarrow a} x = a$$

$$9. \quad \lim_{x \rightarrow a} x^n = a^n$$

It can be observed that the calculation of the limit of a function is equivalent to the calculation of the numerical value of the function, that is $\lim_{x \rightarrow a} f(x) = f(a)$ provided $f(a)$ is meaningful.

Example:

Evaluate the following limits

$$a. \quad \lim_{x \rightarrow 1} (4x^3 - 3x^2 + 2x - 1)$$

Solution

$$\lim_{x \rightarrow 1} (4x^3 - 3x^2 + 2x - 1)$$

$$= 4(1)^3 - 3(1)^2 + 2(1) - 1$$

Direct substitution of x by 1

$$= 4 - 3 + 2 - 1 = 2$$

$$\text{b. } \lim_{x \rightarrow 0} \frac{x^2 - 2x - 3}{x + 6}$$

Solution

$$\lim_{x \rightarrow 0} \frac{x^2 - 2x - 3}{x + 6}$$

$$= \frac{(0)^2 - 2(0) - 3}{(0) + 6} \quad \text{Direct substitution of } x \text{ by } 0$$

$$= \frac{-3}{6} = \frac{-1}{2}$$

$$\text{c. } \lim_{x \rightarrow 1} \frac{\sqrt{x+3}}{\sqrt{3x-2}}$$

Solution

$$= \frac{\sqrt{(1)+3}}{\sqrt{3(1)-2}} = \frac{2}{1} = 2 \quad \text{Direct substitution and evaluating}$$



Application activity 1.6.3

If $\lim_{x \rightarrow 3} f(x) = 3$ and $\lim_{x \rightarrow 3} g(x) = -3$, find

$$\text{a. } \lim_{x \rightarrow 3} [f(x) + g(x)]$$

$$\text{b. } \lim_{x \rightarrow 3} [f(x)g(x)]^3$$

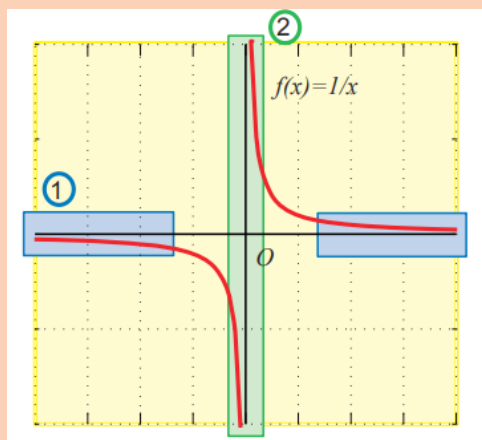
$$\text{c. } \lim_{x \rightarrow 3} 5f(x)$$

1.6.4. Finite and infinite limits

Activity 1.6.3



- Consider the function $f(x) = \frac{x+1}{x-1}$. Find
 - $f(0.97)$
 - $f(0.98)$
 - $f(0.99)$
 - $f(1.01)$
 - $f(1.02)$
 - $f(1.03)$
- Observe the following graph of $f(x) = \frac{1}{x}$ and hence complete the following table under it.



x	-100	-10	-1	-0.1	0	0.1	1	10	100
$f(x)$									

- What can you say about y -values as x increases without bound or decreases without bound?
 - What can you say about y -values as x approaches to zero from the right and left?
- Evaluate the following:
 - $-2^{+\infty}$
 - $3^{+\infty}$
 - $\frac{+\infty}{-\infty}$
 - $\frac{3}{\infty}$
 - $+\infty - \infty$
 - $-\infty \times (+\infty)$
 - $\frac{\infty}{-2}$

Content summary

Finite and Infinity limits

A function whose values grow arbitrarily large are said to have an infinite limit. Infinity is not a real number, infinite limits provide a way of describing the behaviour of functions that grow arbitrarily large, in absolute value, positive or negative.

Example 1

Describe the behaviour of the function $f(x) = \frac{1}{x^2}$ near $x = 0$.

Solution

As x approaches 0 from either side, the values of $f(x) = \frac{1}{x^2}$ are positive and grow larger and larger, so the limit of $f(x)$ as x approaches 0 is $+\infty$. It is, nevertheless, convenient to describe the behaviour of f near 0 by saying that $f(x)$ increases without bound as x approaches zero. We write

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty \quad \text{And} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$$

Example 2

Describe the behaviour of the function $f(x) = \frac{1}{x}$ near $x = 0$.

Solution

Let x successively assumes x - values $= 1, \frac{1}{10}, \frac{1}{100}, \dots$, then $\frac{1}{x} = 1, 10, 100, \dots$ successively. As x approaches 0 from the right the value of $\frac{1}{x}$ gets larger and larger without bound, then $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$

Let x successively assumes values $x = -1, -\frac{1}{10}, -\frac{1}{100}, \dots$, then $\frac{1}{x} = -1, -10, -100, \dots$ successively. As x approaches 0 from the left the value of $\frac{1}{x}$ decreases and

becomes more and more negative without bound, then $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

Another way to see this is to construct the sign table:

x	$-\infty$	0										$+\infty$						
1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+			
x	-	-	-	-	-	-	-	-	-	-	-	-	-	+	+	+	+	+
$\frac{1}{x}$	-	-	-	-	-	-	-		-	-	-	-	-	+	+	+	+	+
								∞										

Considering the last row, we see that for $x = 0$ the value of $\frac{1}{x}$ does not exist (∞). At the left there is a negative sign, thus $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$. At the right there is a positive sign, thus $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$.

It follows that $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist because the one-sided limits as x approaches zero do not exist.

Example 3: Evaluate

a. $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$ b. $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$

If we use direct substitution, we will get $\frac{1}{(1)-1} = \frac{1}{0} = \infty$

To determine whether this infinity is positive or negative, this will depend on the signs of the denominator.

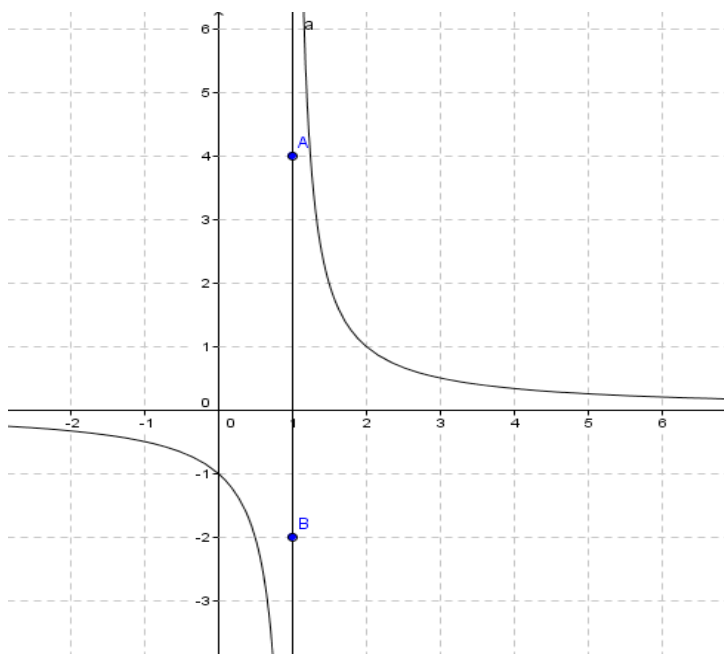
Sign table of the denominator.

x	$-\infty$	1										$+\infty$						
$x-1$	-	-	-	-	-	-	-	0	+	+	+	+	+	+	+	+	+	+

We can see that the signs of the denominator at the left of Zero is negative this means at the left of zero we would be dividing 1 by a negative number while at the right we would be dividing 1 by a positive.

Therefore $\lim_{x \rightarrow 1^-} \frac{1}{x-1} = \frac{1}{0^-} = -\infty$ and $\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \frac{1}{0^+} = +\infty$

See on the graph



Example 4

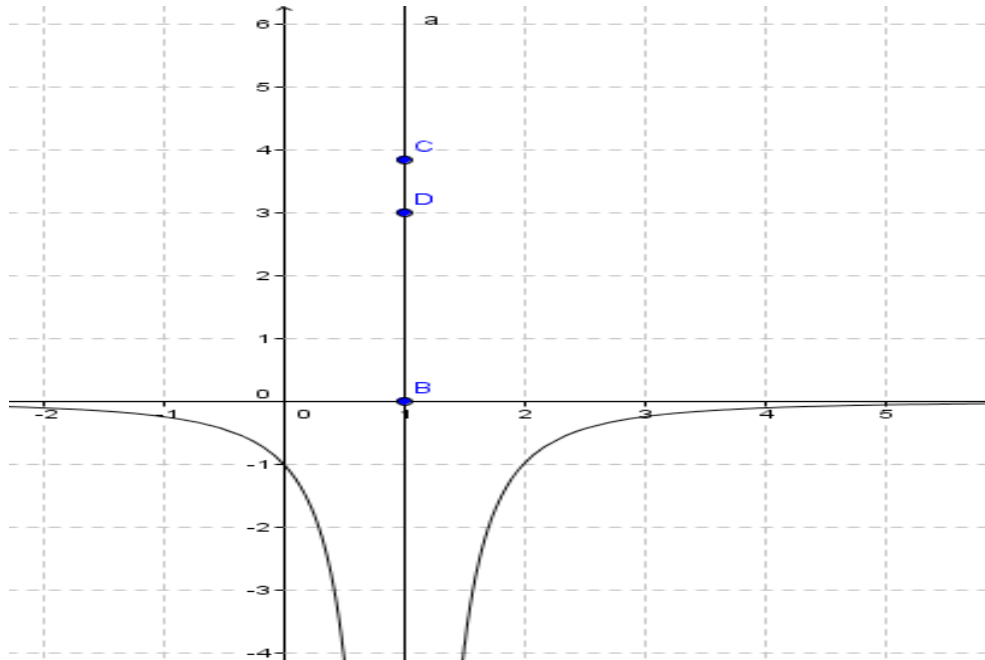
Evaluate a) $\lim_{x \rightarrow 1^-} \frac{-1}{(x-1)^2}$ b) $\lim_{x \rightarrow 1^+} \frac{-1}{(x-1)^2}$

The foremost key is to find the dividing signs. That is simply studying the signs of the denominator.

x	$-\infty$	1												$+\infty$				
$x-1$	-	-	-	-	-	-	-	0	+	+	+	+	+	+	+	+	+	+

Because $(x-1)^2$ is always positive either at the left or right, then we will be dividing -1 by a positive number.

$$\text{That is } \lim_{x \rightarrow 1^-} \frac{-1}{(x-1)^2} = \frac{-1}{0^+} = -\infty \text{ and } \lim_{x \rightarrow 1^+} \frac{-1}{(x-1)^2} = \frac{-1}{0^+} = -\infty$$

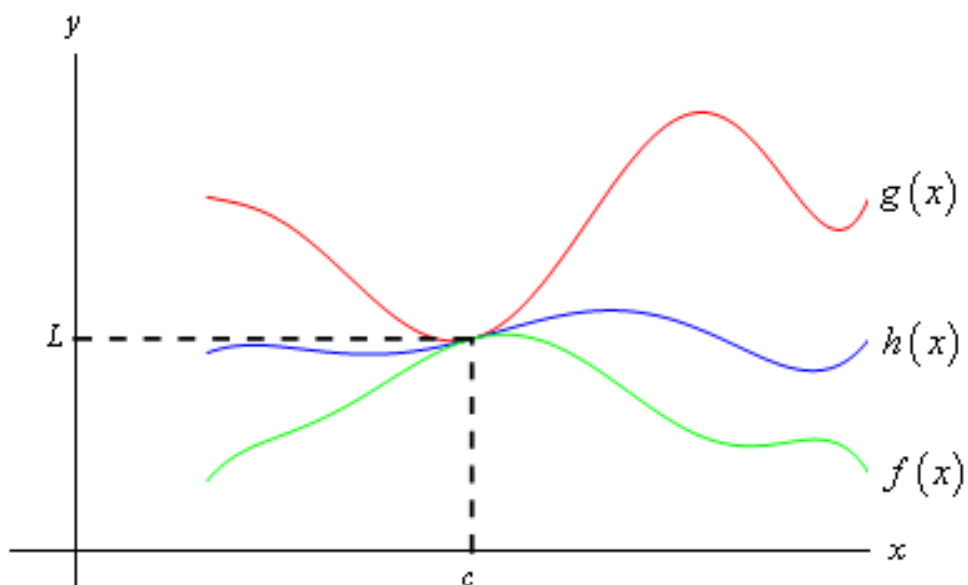


From the graph you can see that as x approaches to 1 from left or right is always $-\infty$

The squeeze theorem (or Sandwich theorem or Pinching theorem)

Suppose that $f(x) \leq h(x) \leq g(x)$. If $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = L$, then $\lim_{x \rightarrow c} h(x) = L$

The following figure illustrates what is happening in this theorem



From the figure we can see that if the limits of $f(x)$ and $g(x)$ are equal at $x = c$ then the function values must also be equal at $x = c$. However, because $h(x)$ is “**squeezed**” between $f(x)$ and $g(x)$ at this point then $h(x)$ must have the same value. Therefore, the limit of $h(x)$ at this point must also be the same. Similar statements hold for left and right limits.

Examples:

1. Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$. Find $\lim_{x \rightarrow 0} u(x)$.

Solution

Since $\lim_{x \rightarrow 0} 1 - \frac{x^2}{4} = 1$ and $\lim_{x \rightarrow 0} 1 + \frac{x^2}{2} = 1$, then according to Sandwich theorem

$$\lim_{x \rightarrow 0} u(x) = 1$$

2. Show that if $\lim_{x \rightarrow a} |f(x)| = 0$ then $\lim_{x \rightarrow a} f(x) = 0$

Solution

Since $-|f(x)| \leq f(x) \leq |f(x)|$, and both $-|f(x)|$ and $|f(x)|$ have limit 0 as x approaches a , so does $f(x)$ by the Sandwich theorem.



Application activity 1.6.4

1. Evaluate the following limits by using direct replacement.

a. i. $\lim_{x \rightarrow 2^+} \frac{3}{x-2}$ ii. $\lim_{x \rightarrow 2^+} \frac{-3}{x-2}$ iii. $\lim_{x \rightarrow 2^-} \frac{-3}{x-2}$

b. i. $\lim_{x \rightarrow -2^-} \frac{5}{x+2}$ ii. $\lim_{x \rightarrow -2^-} \frac{-5}{x+2}$ iii. $\lim_{x \rightarrow -2^+} \frac{-5}{x+2}$

c. $\lim_{x \rightarrow 3} (-3x^2 + 7x)$ d. $\lim_{x \rightarrow -1^+} \frac{4x-5}{3-x}$ e. $\lim_{x \rightarrow 5} \frac{\sqrt{x+4}-2}{x}$ f. $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x}$

g. $\lim_{x \rightarrow 4} \sqrt[3]{x+4}$ h. $\lim_{x \rightarrow +\infty} \frac{x}{x^2}$

2. The cost function for a new supermarket to recycle x tons of organic material is given by $C = 60x + 1650$, where C is the cost in dollars
 - a. Write a model for the average cost per ton of organic material recycled
 - b. Find the average cost of recycling 100 tons and 1000 tons of organic material.
 - c. Determine the limits of the average cost function as x approaches to infinity. Explain the meaning of this limit in the context of the problem.

1.6.5. Infinite limits and rational functions.

Activity 1.6.5



Evaluate the following limits

a. $\lim_{x \rightarrow \infty} \frac{5}{x+2}$ b. $\lim_{x \rightarrow 10} \frac{x}{2}$ c. $\lim_{x \rightarrow 0} \frac{x}{x^2}$ d. $\lim_{x \rightarrow \infty} \frac{x}{x^2}$

Content summary

We can evaluate limits by direct replacement method. But in some cases we need, to factorise and simplify the common factor in the case we have a rational function.

Examples

1. Evaluate the following limits $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$

Solution

$$\begin{aligned} & \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} \\ &= \lim_{x \rightarrow 2} \frac{(x-2)}{(x-2)(x+2)} \end{aligned}$$

Factorising the denominator.

$$= \lim_{x \rightarrow 2} \frac{1}{x+2}$$

Simplifying like terms.

$$= \frac{1}{4}$$

Direct replacement.

2. Evaluate :

$$\lim_{x \rightarrow \infty} \frac{x-2}{x^2-4}$$

$$\lim_{x \rightarrow \infty} \frac{x \left(1 - \frac{2}{x}\right)}{x^2 \left(1 - \frac{4}{x^2}\right)}$$

$$\lim_{x \rightarrow \infty} \frac{\left(1 - \frac{2}{x}\right)}{x \left(1 - \frac{4}{x^2}\right)}$$

$$\frac{1}{\infty} = 0$$

Dividing through by the term with greatest power on both numerator and denominator and simplifying

Note: $\frac{c}{\infty} = 0$ if $c \in \mathbb{R}$



Application activity 1.6.5

1. Evaluate the following limits and say whether they are finite or infinite.

1. $\lim_{x \rightarrow -\infty} \frac{4x^3 + 5x - 3}{x^2 + 3x + 1}$

2. $\lim_{x \rightarrow \infty} \frac{5x + 2}{3x^2 + 1}$

3. $\lim_{x \rightarrow -\infty} (-6)$

II. Factorize the numerator and evaluate the limits.

1. $\lim_{x \rightarrow 7} \frac{49 - x^2}{x - 7}$

2. $\lim_{x \rightarrow -2} \frac{x^3 + 5x^2 + 6x}{x + 2}$

3. $\lim_{x \rightarrow 3} \frac{x^3 - 4x^2 + 5x - 6}{x - 3}$

III. Simplify the following expressions and evaluate the limits.

1. $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right)$

2. $\lim_{x \rightarrow 1} \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$

1.6.6. Limits of functions with indeterminate cases

Activity 1.6.6



1. Is it possible to evaluate the following limits by using direct substitution? If yes evaluate them. If no, what can be done before?

a. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

b. $\lim_{x \rightarrow 2} \frac{x^3 + x^2 - 5x - 2}{x^2 - 4}$

2. What is the conjugate of the irrational expression in each of the following functions?

a. $f(x) = \sqrt{x^2 - 2} + 3$

b. $f(x) = \frac{\sqrt{x-2} - 1}{x-3}$

Content summary

A limit is said to be in an indeterminate form if it cannot be evaluated using properties of elementary operations (addition, subtraction, multiplication and division). An indeterminate case, or form, hides the true value, if any, towards which the function is approaching. There are several types of indeterminate forms such

as: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty$ and ∞^0 . In this section we shall study the indeterminate

forms: $0 \times \infty, \frac{0}{\infty}, \frac{\infty}{0}, \infty - \infty$

To determine the true value towards which the function is approaching, it is necessary to remove the indetermination.

The indeterminate forms may be produced in the following ways:

- Suppose that $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$.

The limit of the product $f(x)g(x)$ has the indeterminate form, $0 \times \infty$ at $x = a$. To evaluate this limit, we change the limit into one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in this way:

$$f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}} = \frac{g(x)}{\frac{1}{f(x)}}$$

- If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = +\infty$, then $\lim_{x \rightarrow a} [f(x) - g(x)] = \infty - \infty$ (*IF*). To evaluate this limit, we perform the algebraic manipulations by converting the limit

into a form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $f(x)$ or $g(x)$ is expressed as a fraction, let first reduce them on the common denominator and evaluate the limit after.

Examples of indeterminate cases of rational functions

1. Evaluate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Solution:

$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$, this is the indeterminate form

By factoring the numerator and cancelling, we obtain

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x + 2) \\ &= 4 \end{aligned}$$

$$2. \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0} \text{ I.F}$$

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x+1)(x-1)}{x-1} \\ &= \lim_{x \rightarrow 1} (x+1) \\ &= 2 \end{aligned}$$

$$3. \quad \lim_{x \rightarrow +\infty} \frac{x^2 + 4x + 5}{4x^2 + 7x + 9} = \frac{\infty}{\infty} \text{ I.F}$$

Factor out x^2 , we have

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{x^2 \left(1 + \frac{4}{x} + \frac{5}{x^2} \right)}{x^2 \left(4 + \frac{7}{x} + \frac{9}{x^2} \right)} &= \lim_{x \rightarrow +\infty} \frac{1 + \frac{4}{x} + \frac{5}{x^2}}{4 + \frac{7}{x} + \frac{9}{x^2}} \\ &= \frac{1 + 0 + 0}{4 + 0 + 0} \\ &= \frac{1}{4} \end{aligned}$$

Or

Since we have a **rational function** and degree of numerator is equal to the degree of denominator, to find the limit as x tends to infinity we need to divide the coefficients of the highest degree for numerator and denominator. That is the limit is given by $\frac{1}{4}$

When we are computing **the limits of irrational functions**, in case of indeterminate form, we need to know the conjugate of the irrational expression in that function. We may need to find the domain of the given function.

Examples of indeterminate cases of irrational functions

$$1. \quad \text{Evaluate } \lim_{x \rightarrow 4} \frac{\sqrt{x-3} - 1}{x-4}.$$

Solution

Let us first substitute the independent variable for 4:

$$\lim_{x \rightarrow 4} \frac{\sqrt{x-3}-1}{x-4} = \frac{\sqrt{4-3}-1}{4-4} = \frac{0}{0}, \text{ which is an indeterminate form.}$$

To evaluate this limit, we multiply the numerator and denominator by the conjugate of $\sqrt{x-3}-1$ which is $\sqrt{x-3}+1$, then we get

$$\lim_{x \rightarrow 4} \frac{\sqrt{x-3}-1}{x-4} = \lim_{x \rightarrow 4} \frac{(\sqrt{x-3}-1)(\sqrt{x-3}+1)}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x \rightarrow 4} \frac{x-3-1}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x \rightarrow 4} \frac{x-4}{(x-4)(\sqrt{x-3}+1)}$$

$$= \lim_{x \rightarrow 4} \frac{1}{(\sqrt{x-3}+1)}$$

$$= \frac{1}{2}$$

2. Evaluate $\lim_{x \rightarrow \infty} (\sqrt{4x^2+2} - 2x)$

Solution

$$\lim_{x \rightarrow \infty} (\sqrt{4x^2+2} - 2x) = +\infty - \infty \text{ (IF)}$$

To evaluate this limit we multiply and divide by the conjugate of $\lim_{x \rightarrow \infty} (\sqrt{4x^2+2} - 2x)$.

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{4x^2+2} - 2x)(\sqrt{4x^2+2} + 2x)}{(\sqrt{4x^2+2} + 2x)}$$

Multiplying the conjugate on both numerator and denominator

$$= \lim_{x \rightarrow \infty} \frac{4x^2 + 2 - 4x^2}{(\sqrt{4x^2 + 2} + 2x)} = \lim_{x \rightarrow \infty} \frac{2}{(\sqrt{4x^2 + 2} + 2x)} \text{ After multiplying and simplifying.}$$

$$= \frac{2}{\infty} = 0$$

3. Evaluate $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 - 11x - 3}}{x}$

Solution

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} = \frac{\sqrt{\infty - \infty}}{\infty} \text{ (IF)}$$

To evaluate this limit, we try the algebraic manipulations such that the denominator will be cancelled.

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 \left(1 - \frac{11}{4x} - \frac{3}{4x^2}\right)}}{x}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2}}{x} \sqrt{\left(1 - \frac{11}{4x} - \frac{3}{4x^2}\right)}$$

Be careful!

$$\sqrt{x} = |x| \begin{cases} -x, & \text{if } x \geq 0 \\ x, & \text{if } x < 0 \end{cases}$$

Now, let us consider separately the limit at $-\infty$ and at $+\infty$

$$\text{We have: } \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2}}{x} \sqrt{1 - \frac{11}{4x} - \frac{3}{4x^2}} = \lim_{x \rightarrow -\infty} \frac{-2x}{x} \sqrt{1 - \frac{11}{4x} - \frac{3}{4x^2}} = -2,$$

$$\text{And } \lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2}}{x} \sqrt{1 - \frac{11}{4x} - \frac{3}{4x^2}} = \lim_{x \rightarrow +\infty} \frac{2x}{x} \sqrt{1 - \frac{11}{4x} - \frac{3}{4x^2}} = 2,$$

$$\lim_{x \rightarrow -\infty} f(x) = -2 \text{ and } \lim_{x \rightarrow +\infty} f(x) = 2$$



Application activity 1.6.6

1. Evaluate the following limits (if they exist).

a. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - x + 2}$ b. $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{x - 1}$

c. $\lim_{u \rightarrow 0} \frac{\sqrt{4+u} - 2}{u}$ d. $\lim_{v \rightarrow 1} \frac{\sqrt{2v+1} - \sqrt{3}}{v-1}$

2. Simplify the following expressions and evaluate the limits.

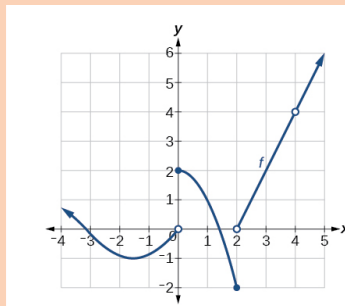
$$\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right)$$

1.6.7 Graphs and limits of function

Activity 1.6.7



Observe the following graph and answers about it.

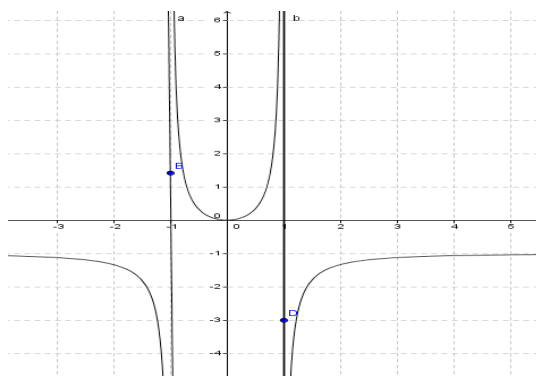


- How many parts does the graph of the function have?
- What is the limit of the function as x approaches 0 from the left?
- What is the limit of the function as x approaches 0 from the right?
- What is the limit of the function as x approaches 2 from left?
- What is the limit of the function as x approaches 2 from right?
- What is the limit of the function as x approaches 4?

Content summary

We can determine the limits by using the graph of a function. A graph which has a jump at a given point is said to be **discontinuous** at that point. On the other hand, if the graph has no jump is said to be **continuous** at that point.

Example: Observe the function whose graph is shown below:



Evaluate the following limits by using the graph

a. $\lim_{x \rightarrow -1^-} f(x)$ b. $\lim_{x \rightarrow -1^+} f(x)$ c. $\lim_{x \rightarrow -1^-} f(x)$ d. $\lim_{x \rightarrow 1^+} f(x)$ e. $\lim_{x \rightarrow +\infty} f(x)$ f. $\lim_{x \rightarrow -\infty} f(x)$

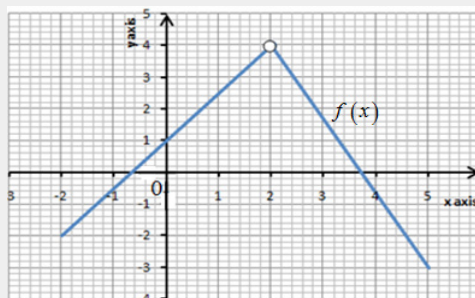
Solution

As x approaches to -1 at the left, the function decreases to $-\infty$ this implies that $\lim_{x \rightarrow -1^-} f(x) = -\infty$. $\lim_{x \rightarrow -1^+} f(x) = +\infty$, $\lim_{x \rightarrow -1^-} f(x) = +\infty$, $\lim_{x \rightarrow -\infty} f(x) = 0$ and $\lim_{x \rightarrow +\infty} f(x) = 0$.



Application activity 1.6.7

- Find $\lim_{x \rightarrow 2} f(x)$ using the following graph of $f(x)$



2. Given the functions:

a) $f(x) = x^3 - x$, use graph to find i) $\lim_{x \rightarrow -\infty} f(x)$ ii) $\lim_{x \rightarrow +\infty} f(x)$

b) $f(x) = \frac{1}{x-1}$ find i) $\lim_{x \rightarrow 1^+} f(x)$ ii) $\lim_{x \rightarrow 1^-} f(x)$ iii) $\lim_{x \rightarrow +\infty} f(x)$ iv) $\lim_{x \rightarrow -\infty} f(x)$

1.6.8. Limit and continuity of a function

Activity 1.6.8



Given the function $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$, find

a. $f(2)$

b. $\lim_{x \rightarrow 2} f(x)$

c. What can you say about $f(2)$ and $\lim_{x \rightarrow 2} f(x)$?

Content summary

1. Continuity of a function at a point or on interval I

A function $f(x)$ is said to be **continuous at point c** if the following conditions are satisfied:

a. $f(c)$ is defined

b. $\lim_{x \rightarrow c} f(x)$ exists

c. $\lim_{x \rightarrow c} f(x) = f(c)$

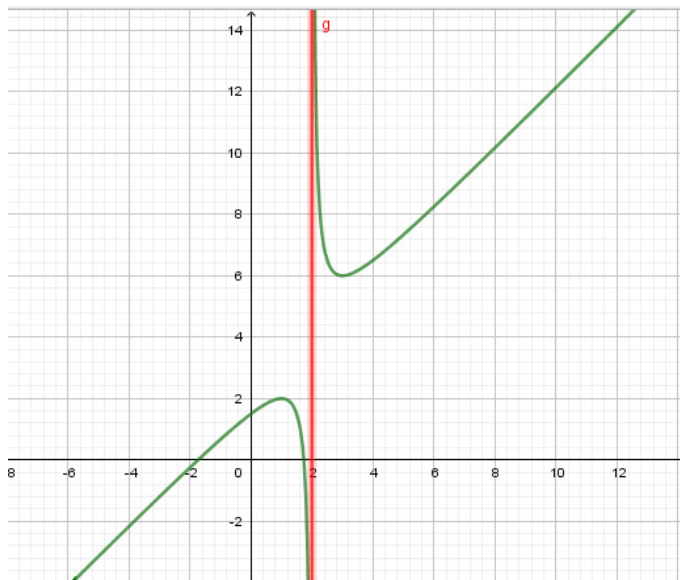
If one or more conditions in this definition fails to hold, then $f(x)$ is said to be **discontinuous at point c**, and c is called **a point of discontinuity** of $f(x)$.

If f is continuous at all point of an open interval $]a, b[$, then $f(x)$ is said to be continuous on $]a, b[$.

A function that is continuous on $]-\infty, +\infty[$ is said to be **continuous everywhere** or simply **continuous**.

Examples:

- The function $f(x) = \frac{x^2 - 3}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined, see the graph below.



- The function $g(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$ is continuous at 3 because $g(3)$

and $\lim_{x \rightarrow 3^+} g(x) = \lim_{x \rightarrow 3^-} g(x) = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \rightarrow 3} (x+3) = 6$ so that

$\lim_{x \rightarrow 3} g(x) = g(3) = 6$.

2. Continuity at the left and continuity at the right of a point

A function $f(x)$ is continuous at the left of point c if the following conditions are satisfied:

a. $f(c)$ is defined

b. $\lim_{x \rightarrow c^-} f(x)$ exists

c. $\lim_{x \rightarrow c^-} f(x) = f(c)$

A function f is continuous at the right at of point c if the following conditions are satisfied:

a. $f(c)$ is defined

b. $\lim_{x \rightarrow c^+} f(x)$ exists

c. $\lim_{x \rightarrow c^+} f(x) = f(c)$

Example

Find the value of the constant a such that the function $f(x) = \begin{cases} -x+1, & x \leq 3 \\ 2x+a, & x > 3 \end{cases}$ is continuous on the entire line.

Solution

The function will be continuous if and only if $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$

$$\lim_{x \rightarrow 3^+} (2x + a) = 6 + a$$

$$\lim_{x \rightarrow 3^-} (-x + 1) = -3 + 1 = -2$$

Equating the two limits we get, $6 + a = -2 \Rightarrow a = -8$

Therefore, the function is continuous if $a = -8$

Continuity on an interval

We say that $f(x)$ is **continuous on the interval** I if it is continuous at each point of I . In particular, we will say that $f(x)$ is a continuous function if $f(x)$ is continuous at every point of its domain.

Examples

1. The function $f(x) = \sqrt{x}$ is a continuous function on its domain. Its domain is $[0, +\infty[$. It is continuous at the left endpoint 0 because it is right continuous

there. Also, f is continuous at every number $c > 0$. Since $\lim_{x \rightarrow c} \sqrt{x} = \sqrt{c}$.

2. Discuss the continuity of the function $f(x) = \frac{|x-1|}{x^2-1}$ at $x=1$

Solution

$$|x-1| = \begin{cases} x-1, & x \geq 1 \\ -(x-1), & x < 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} \frac{-(x-1)}{(x+1)(x-1)} = \frac{-1}{2} \quad \text{And} \quad \lim_{x \rightarrow 1^+} \frac{(x-1)}{(x+1)(x-1)} = \frac{1}{2}$$

It can be seen that the limits at $x=1$ does not exist, since the left limit is different from right limit. Hence, the function is discontinuous at $x=1$.

On the other hand, the function is continuous at the right of 1.

Theorem 1

- a. Polynomials are continuous functions
- b. If the functions f and g are continuous at c , then
 - i. $f(x) + g(x)$ is continuous at c
 - ii. $f(x) - g(x)$ is continuous at c
 - iii. $f(x)g(x)$ is continuous at c
 - iv. $\frac{f(x)}{g(x)}$ is continuous at c if $g(c) \neq 0$, and is discontinuous at c if $g(c) = 0$.
- c. A rational function is continuous everywhere except at the point where the denominator is zero.
- d. Piecewise functions (functions that are defined on a sequence of intervals by different formulas) are continuous if every function is in its interval of definition, and if the functions match their side limits at the boundary's points of their component intervals.



Application activity 1.6.8

1. Discuss the continuity of the function defined by $f(x) = \begin{cases} 1, x \leq 1 \\ x, 1 < x \leq 3 \\ -x + 6, 3 < x \leq 6 \\ 0, x > 6 \end{cases}$

2. Determine the points at which the function below is not continuous.

$$f(x) = \frac{4x + 10}{x^2 - 2x - 15}$$

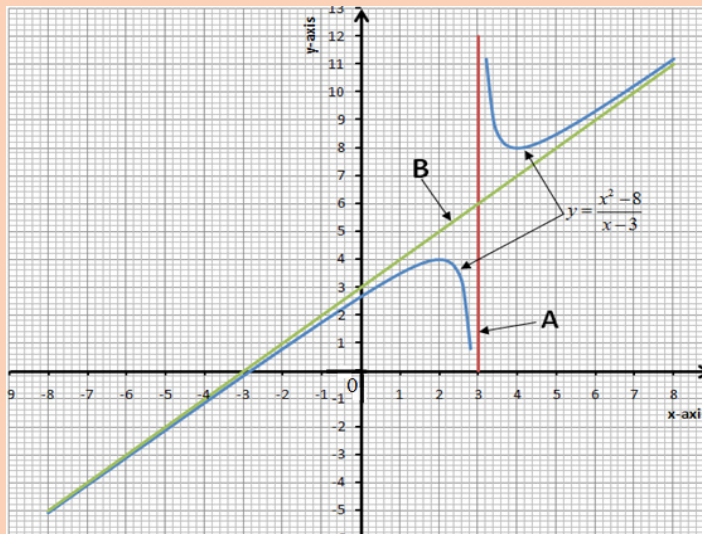
3. For which value of k is the function: $f(x) = \begin{cases} x^2 - 9, x \neq 3 \\ k, x = 3 \end{cases}$ continuous at $x = 3$.

1.6.9. Asymptotes to curve of a function

Activity 1.6.9



1. Consider the following curve of function y



- What is the position of the line A ?
- What is the position of line B ?
- What is the other type of line that you know?
- Do the lines A and B cross the curve of the function?

2. Let the function $f(x)$ be given by $f(x) = \frac{x}{x^2 - 1}$
- Find the domain of $f(x)$
 - Find the limits at the endpoints of the domain.
 - Discuss on the values obtained.

Content summary

Recall that if $P(x)$ and $Q(x)$ are polynomials, then their ratio $f(x) = \frac{P(x)}{Q(x)}$ is called a rational function of x . The discontinuity occurs at points where $Q(x) = 0$.

An asymptote on the curve is a straight line that is closely approached by that curve so that the perpendicular distance between them decreases to zero.

To find any asymptote of the function first we need to determine its domain of definition and evaluate the limits at the boundaries of the domain.

- **Types of asymptotes: Vertical asymptote, horizontal and oblique asymptotes.**

A function $f(x)$ admits a **vertical asymptote** at a given point $x = a, a \in \mathbb{R}$, if $f(x)$ is undefined at $x = a$ that is $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ and $\lim_{x \rightarrow a^-} f(x) = \pm\infty$. The vertical asymptote is denoted by $V \equiv x = a$

A function $f(x)$ admits an horizontal asymptotes at a given point $y = b, b \in \mathbb{R}$, if $\lim_{x \rightarrow \pm\infty} f(x) = b$ it is denoted by $H.A \equiv y = b$.

A function $f(x)$ admits an oblique asymptotes $O.A \equiv y = ax + b$ where a is the gradient and b is the y -axis intercept if $a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x}, a \in \mathbb{R}^*$ and $b = \lim_{x \rightarrow \pm\infty} f(x) - ax, a \in \mathbb{R}^*$ and $b \in \mathbb{R}$

Examples

- Given the function $f(x) = \frac{1}{4-x^2}$
 - Find the domain of definition
 - Find the limits at the endpoints of the domain and deduce all possible asymptotes.

Solution

$$\text{dom}f = \{x \in \mathbb{R} / 4 - x^2 \neq 0\}$$

$$(2-x)(2+x) \neq 0 \Rightarrow \begin{cases} 2-x \neq 0 \text{ or } x \neq 2 \\ 2+x \neq 0 \text{ or } x \neq -2 \end{cases}$$

$$\text{dom}f = \{x \in \mathbb{R} \setminus \{-2, 2\}\}$$

$$\begin{cases} \lim_{x \rightarrow 2^+} f(x) = -\infty \\ \lim_{x \rightarrow 2^-} f(x) = -\infty \end{cases} \Rightarrow V.A \equiv X = 2 \text{ and } \begin{cases} \lim_{x \rightarrow 2^+} f(x) = -\infty \\ \lim_{x \rightarrow 2^-} f(x) = -\infty \end{cases} \Rightarrow V.A \equiv X = -2$$

$$\begin{cases} \lim_{x \rightarrow -\infty} f(x) = 0 \\ \lim_{x \rightarrow +\infty} f(x) = 0 \end{cases} \Rightarrow H.A \equiv y = 0 \text{ There is no } O.A$$

- Find the oblique (Slant) asymptote for the following function $f(x) = \frac{x^3}{x^2-1}$

Solution.

Slant asymptote has for equation $y = ax + b$

$$a = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x^3}{x(x^2-1)} = 1, \quad b = \lim_{x \rightarrow \infty} f(x) - ax = \lim_{x \rightarrow \infty} \frac{x^3}{x^2-1} - (1)x$$

$$b = \lim_{x \rightarrow \infty} \frac{x^3 - x^3 + x}{x^2-1} = \lim_{x \rightarrow \infty} \frac{x}{x^2-1} = 0, \text{ hence the oblique asymptote has for equation } y = x.$$



Application activity 1.6.9

For each of the following functions find

- The domain
- Limits at the endpoints of the domain and deduce all possible asymptotes.

i. $f(x) = \frac{x^2 + 1}{x^2 - x}$

ii. $f(x) = \frac{x^3}{1 - x^2}$

iii. $f(x) = \frac{1}{x}$

1.6.10. Solving problems involving limits in real life

Activity 1.6.10



The total cost evolution of a small business is modeled by the function $C = ax + b$ where C is the cost and x is the units sold. If $a = 0.5$ and $b = 5000$

- Find the model function for the evolution of the business
- Find the average cost function \bar{C} , if x units are sold.
- Using the function on (b) find the average if 100 units are sold, if 1000 units are sold and if 100000 units are sold.
- Using the values obtained at (c), find the average cost as x approaches to infinity.
- Discuss the meaning of the limit obtained at (d).

Content summary

Limits can be applied in different fields in real life. In economics the average cost per unit sold is calculated using limits.

1. Instantaneous rate of change of a function in physics

The **instantaneous rate of change** of $f(x)$ at a , also called the **rate of change** of $f(x)$ at a , is defined to be the limit of the average rate of change of

$f(x)$ over shorter and shorter intervals around a .

Since the average rate of change is a difference quotient of the form $\frac{\Delta y}{\Delta x}$, the instantaneous rate of change is a limit of difference quotient. In practice, we often approximate a rate of change by one of these difference quotients.

Example:

1. The quantity (in mg) of a drug in the blood at time t (in minutes) is given by $Q = 25(0.8)^t$. Estimate the rate of change of the quantity at $t = 3$ and interpret your answer.

Solution:

We estimate the rate of change at $t = 3$ by computing the average rate of change over intervals near $t = 3$. We can make our estimate as accurate as we like by choosing our intervals small enough.

Let's look at the average rate of change over the interval $3 \leq t \leq 3.01$:

$$\text{Average rate of change} = \frac{\Delta Q}{\Delta t} = \frac{25(0.8)^{3.01} - 25(0.8)^3}{3.01 - 3.00} = -2.85$$

A reasonable estimate for the rate of change of the quantity at $t = 3$ is -2.85

Since Q is in mg and t in minutes, the units of $\frac{\Delta Q}{\Delta t}$ are mg/minute. Since the rate of change is negative, the quantity of the drug is decreasing. After 3 minutes, the quantity of the drug in the body is decreasing at 2.85 mg / minute

2. Instantaneous velocity

Instantaneous velocity of a moving body is the limit of average velocity over an infinitesimal interval of time.

$$v = \lim_{t \rightarrow 0} \frac{\Delta s}{\Delta t}$$

3. Instantaneous acceleration

Instantaneous acceleration for a moving body is the limit of average acceleration

over an infinitesimal interval of time $a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t}$

(Here and elsewhere, if motion is in a straight line, vector quantities can be substituted by scalars in the equations.)

Examples

1. The cost of removing $p\%$ in a given place is given by the model

$$C(x) = \frac{80000p}{100 - p}, 0 \leq p < 100$$

- a. What is the vertical asymptote for this function?
- b. What does this vertical asymptote mean in the context of the problem?

Solution

Vertical asymptote is found at $p = 100$

To assess the behavior of the model, let us choose some values

Example: the cost of removing 85% of pollutants is

$$C(85\%) = \frac{80000(85)}{100 - 85} \approx 453.333$$

But the cost of removing 90% is

$$C(90\%) = \frac{80000(90)}{100 - 90} = 720.$$

We can see that the higher and higher percentage of pollutants is removed, the cost increases dramatically.



Application activity 1.6.10

1. The cost C (in millions dollars) for the federal government to seize $p\%$ of a type of illegal drug as it enters the country is modeled by the

$$C = \frac{528p}{100-p}, 0 \leq p < 100$$

- a. Find the cost of seizing (stopping) 25%, 50% and 75%.
 - b. Find the limit as $p \rightarrow 100^-$, interpret this limit in the context of the problem.
2. A business has a cost in dollars of $C = 0.5x + 500$ for producing x units.
- a. Find the average cost function \bar{C} .
 - b. Find \bar{C} when $x=250$ and when $x=1250$
 - c. What is the limit of \bar{C} as x approaches to infinity? Interpret the results in the context of the problem.



1.7. End of unit assessment

1. Use limits to find the slope of the tangent line to the graph of $s = t^2$ at the point (1,1).

2. Given the function $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$

Find the :

- domain of $f(x)$
- Limits at the endpoints of the domain and deduce all possible asymptotes.

3. The cost and revenue functions for a product are
 $C = 25.5x + 1000$ and $R = 75.5x$

$$\bar{P} = \frac{R - C}{x}$$

- Find the average profit function \bar{P} .
 - Find the average profits when x is 100, 500 and 1000.
 - What is the limit of the average profit as x approaches to infinity? Explain your reasoning.
4. You are given 24m of wire and are asked to form a rectangle whose area is as large as possible. What dimensions should the rectangle have? (Hint: use numerical approach).

5. given a function $f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$ find

i) $\lim_{x \rightarrow 2^+} f(x)$ ii) $\lim_{x \rightarrow 2^-} f(x)$ iii) $\lim_{x \rightarrow 2} f(x)$

iv) Discuss the continuity of $f(x)$ at the point where $x = 2$.

UNIT 2

DIFFERENTIATION

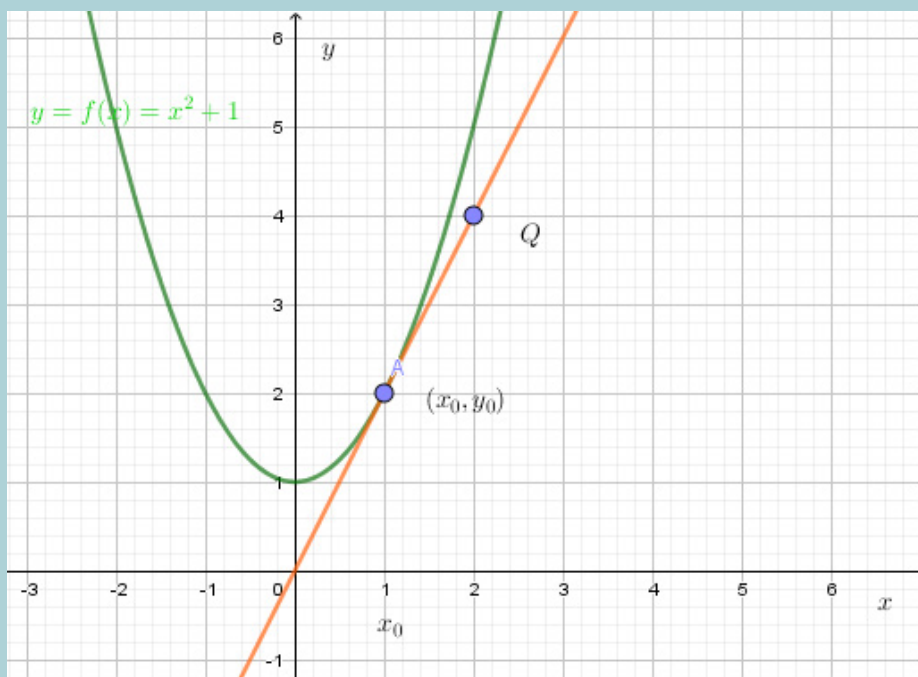
Key Unit Competence:

Differentiate a real function and apply derivatives to sketch the graphs of the given functions



2.0. Introductory activity

1. Consider the function $f(x) = x^2 + 1$ illustrated on the following graph;



It is defined that the slope m_P of the tangent of the curve of $f(x)$ at a point

$$P(x_0, y_0) \text{ is obtained by } \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

- Determine the slope of $f(x)$ at the point for which $x_0 = 1$.
- Deduce the value of the function $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ for

$f(x) = x^2 + 1$ and compare the slope m_p and $f'(x_0)$ for $x_0 = 1$.

- Go in library or computer lab, do research and make a short presentation on the following:
 - Derivative of a function
 - Find 2 examples of applications of derivatives.

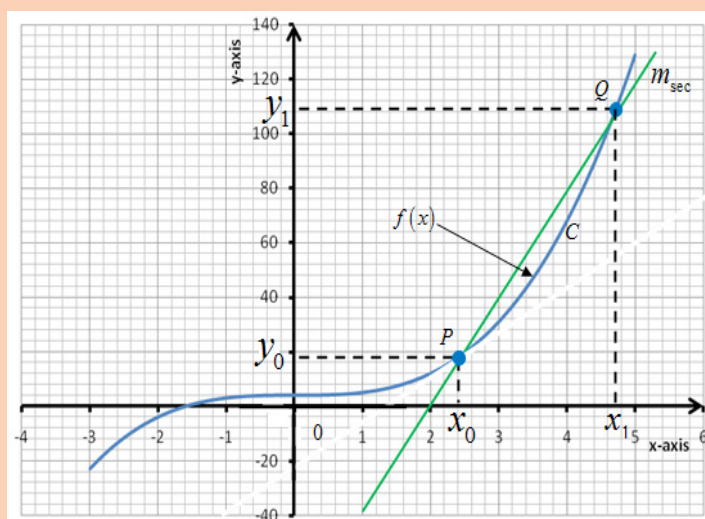
2.1 Concepts of derivative of a function

2.1.1 Definition and graphical interpretation

Activity 2.1.1



- Consider the figure below, analyse it and answer the questions that follows



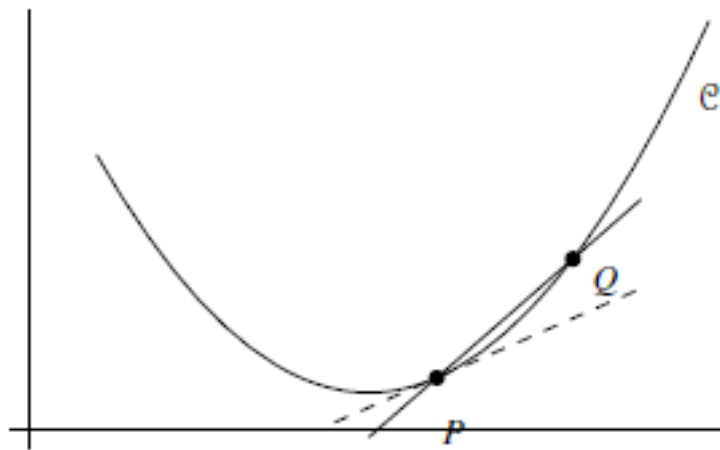
- If $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two points on the graph of a function f , refer to what you learned in S3 and find the slope of secant line (m_{sec}) passing through P and Q . Since $y_0 = f(x_0)$ and $y_1 = f(x_1)$, express the slope in terms of $f(x_0)$ and $f(x_1)$.

- b. If we let x_1 approach x_0 , how can you conclude about position of Q to P ?
- c. Let $m_{\tan} = \lim_{x_1 \rightarrow x_0} m_{\sec}$, write down expression of m_{\tan} in terms of $f(x_0)$ and $f(x_1)$.
- d. After letting $h = x_1 - x_0$, rewrite m_{\tan} in terms of $f(x_0)$ and $f(x_0 + h)$.

Content summary

Slope of a function at a point

To define the slope of the curve C at P , take a point Q on the curve different from P . The line PQ is called a secant line at P . Its slope, denoted by m_{PQ} , can be found using the coordinates of P and Q . If we let Q move along the curve, the slope m_{PQ} changes.



Suppose that as Q approaches P , the number m_{PQ} approaches a fixed value and the increment $h = x_Q - x_P$ of x approaches 0. This value, denoted simply by m_P , is called the slope of C at P ; and the line with slope m_P and passing through P is called the tangent line to the curve C at P .

In view of the concept "limit of a function at a point", the slope m_{PQ} of the secant line PQ is
$$m_{PQ} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Note that as Q approaches P , the number h approaches 0. From these, we see that the slope of C at P (denoted by m_p) is $m_p = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$

If this limit exists, the number m_p is called **slope of tangent line** to the graph of f at P or at $x = x_0$

Definition of derivative of a function

The slope m_p has a special notation, we denote it by $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ and $f'(x_0)$ is read f prime of x_0 .

Dropping the subscript on x_0 in notation of m_p , we get one of the most important concept in mathematics, the **derivative of function f at $x = x_0$**

The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ or

$\frac{d}{dx} f(x)$ and defined as $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ provided that the limit exists.

Examples:

1. Let $f(x) = x^2 + 1$

The derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 1 - x^2 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 + 1 - x^2 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2x + h) \\ &= 2x \end{aligned}$$

Thus, $f'(x) = 2x$

2. By using definition, calculate the first derivative for $y = x^2 - 3$

Solution:

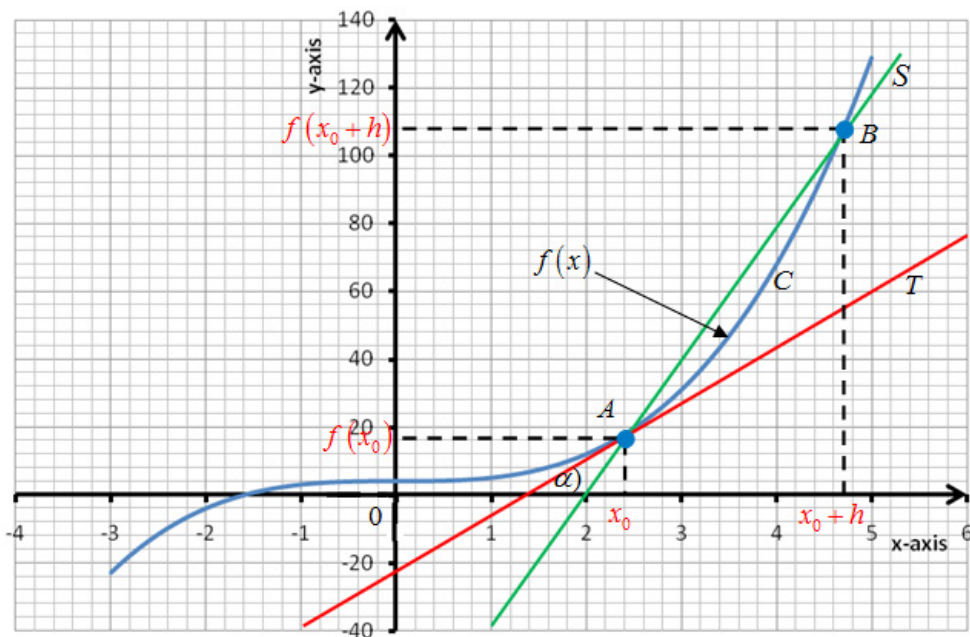
$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}, \quad y' = \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3 - (x^2 - 3)}{h}$$

$$y' = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} 2x + h = 2x$$

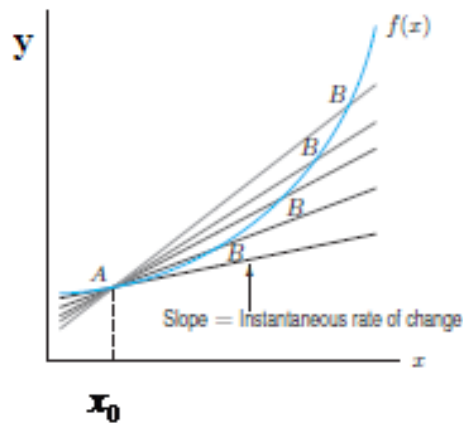
Remarks

- If $t \rightarrow f(t)$ represents the law of a moving object, then the derivative number of f represents the instantaneous speed of that moving object at instant t .
- The process of finding derivative of a function is called **differentiation** of that function.

Graphical interpretation of derivative and the slope of a function using differentiation



The graph shows the **average rate of change of a function** represented by the slope of the secant line joining points A and B.



The derivative is found by taking the average rate of change over smaller and smaller intervals. In the figure above, as point B moves toward point A , the secant line becomes the tangent line at point A . Thus, **the derivative is represented by the slope of the tangent line to the graph at the point.**

The derivative of a function at the point A is equal to:

- The slope of the graph of the function at A .
- The slope of the line tangent to the curve at A .

Then AB has slope of $\frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$. When h approaches

to zero, the point B approaches the point A .

Then by definition $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$ for $y = f(x)$ at point $(x_0, f(x_0))$

Alternative notations

The derivative: $f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$ where $\frac{dy}{dx}$ and $\frac{d}{dx}$ are called

differentiation operators but $\frac{dy}{dx}$ should not be regarded as a ratio.

If $x = t$ is time and $y = s(t)$ is the displacement function of a moving object then

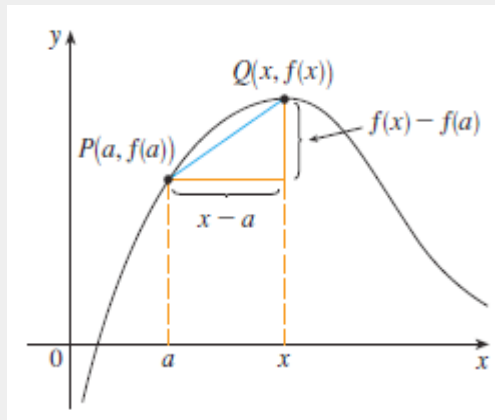
$s'(t_0)$ or $\left. \frac{ds}{dt} \right|_{t=t_0}$ is the rate of change of displacement with respect to time when t

$= t_0$, that is, the (instantaneous) velocity at $t = t_0$.



Application activity 2.1.1

- Find the slope of the curve given by $f(x) = x^2$ at the point $P(3, 9)$.
- By the use of definition, find the derivative of:
 - $f(x) = x^2 + 1$
 - $f(x) = x^2 + 2x - 1$
- Observe the graph and answer the questions that follow:



- From your observation, interpret the graph by indicating the rate of change given that the variable x changes from a to x
 - Deduce the formula of derivative;
 - Use your interpretation and the formula deduced to find out the derivative of $f(x) = x + 1$.
- Find the derivative of the function $f(x) = x^2 - 8x + 9$ at (the number) a .

2.2 Rules of differentiation

Activity 2.2



- Find the derivative of
 - $f(x) = x^2$
 - $h(x) = 3x - 1$
 - $g(x) = x^2 + 3x - 1$

Deduce the derivative of $S(x) = g(x) + h(x)$

- Calculate the derivative of $f(t) = \frac{1}{t}$, $t \neq 0$

Content summary

Suppose that f is a function differentiable at every point belonging to an open interval $]a, b[$ or (a, b) . Then we say that f is differentiable on $]a, b[$ or (a, b) .

1) Derivative of a constant function

If f is a constant function, $f(x) = c$, for all x then $\frac{df}{dx} = \frac{d}{dx}(c) = 0$

Example: Calculate the derivative of $f(x) = 8$

Solution: $\frac{df}{dx} = \frac{d}{dx}(8) = 0$

2) Derivative of identity function

The derivative of the identity function is the constant function 1, that is

if $f(x) = x$, $\frac{df}{dx} = \frac{dx}{dx} = 1$

3) Multiplication by a scalar

If f is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cf(x)) = c \frac{d}{dx}f(x)$$

Example: find the derivative of $f(x) = 3x$

Solution: $f'(x) = 3(x)' = 3x^{1-1} = 3x^0 = 3$, knowing that $x^0 = 1$, provided $x \neq 0$

4) Derivative of a power

If n is any real number,

then $\frac{d}{dx}x^n = nx^{n-1}$ for all x where the powers x^n and x^{n-1} are defined.

This holds for any function with power. Let $D(I, \mathbb{R})$ be the set of functions differentiable on interval I . Thus, if $f \in D(I, \mathbb{R})$ for positive and negative, and

fractional value of n , then $[f^n(x)]' = nf^{n-1}(x)f'(x)$.

Example:

Differentiate $f(x) = (2x + 1)^4$ respecting the value of x .

Solution:

$$f'(x) = ((2x + 1)^4)' = 4(2x + 1)'(2x + 1)^{4-1}$$

$$f'(x) = 4(2)(2x + 1)^3$$

$$f'(x) = 8(8x^3 + 12x^2 + 6x + 1)$$

$$f'(x) = 64x^3 + 96x^2 + 48x + 8$$

5) Sum rule

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Example: Calculate the derivative of $y = x^4 + 12x$ with respect to x

Solution: $y' = 4x^{4-1} + 12x^{1-1}$

$$y' = 4x^3 + 12$$

6) The Difference Rule

$$\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$$

Example

$$y = x^4 - 2x^2 + 2 \quad \frac{dy}{dx} = 4x^3 - 4x$$

7) Product rule

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Notice that this is not just the product of two derivatives

Example

$$\frac{d}{dx}[(x^2 + 3)(2x^3 + 5x)] = (x^2 + 3)(6x^2 + 5) + (2x^3 + 5x)(2x)$$

$$= 6x^4 + 5x^2 + 18x^2 + 15 + 4x^4 + 10x^2$$

$$= 10x^4 + 33x^2 + 15$$

8) Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \text{ or can be written as } d\left(\frac{u}{v}\right) = \frac{v du - u dv}{v^2}$$

Examples:

1. Find out the derivative for $f(x) = \frac{2x^3 + 5x}{x^2 + 3}$

Solution:

$$f'(x) = \frac{(x^2 + 5)(2x^3 + 5x)' - (2x^3 + 5x)(x^2 + 3)'}{(x^2 + 3)^2}$$

$$f'(x) = \frac{(x^2 + 5)(6x^2 + 5) - (2x^3 + 5x)(2x)}{(x^2 + 3)^2}$$

2. Find the derivative of $f(x) = \frac{2x^2 + 3x}{4x^3 + x + 1}$

$$f'(x) = \left(\frac{2x^2 + 3x}{4x^3 + x + 1} \right)'$$

$$= \frac{(2x^2 + 3x)'(4x^3 + x + 1) - (2x^2 + 3x)(4x^3 + x + 1)'}{(4x^3 + x + 1)^2}$$

$$= \frac{(4x + 3)(4x^3 + x + 1) - (2x^2 + 3x)(12x^2 + 1)}{(4x^3 + x + 1)^2}$$

$$= \frac{16x^4 + 4x^2 + 4x + 12x^3 + 3x + 3 - 24x^4 - 2x^2 - 36x^3 - 3x}{(4x^3 + x + 1)^2}$$

$$= \frac{-8x^4 - 24x^3 + 2x^2 + 4x + 3}{(4x^3 + x + 1)^2}$$

Derivative of polynomial

Let $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x^1 + a_0$ be a polynomial.

Then, we have $\frac{dy}{dx} = na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \dots + 2a_2 x^1 + a_1$

9) Derivative of the reciprocal function

Let $D(I, \mathbb{R})$ be the set of functions differentiable on I . If $f \in D(I, \mathbb{R})$, then

$\frac{1}{f} \in D(I, \mathbb{R})$ $f(x) \neq 0$. Moreover $\frac{d}{dx} \left(\frac{1}{f} \right) = -\frac{\frac{df}{dx}}{f^2}$ or $\frac{d}{dx} \left(\frac{1}{f} \right) = -\frac{f'}{f^2}$

Example

Calculate the derivative of $f(x) = \frac{1}{x}$ respecting the value of x

$$\text{Solution: } \left(\frac{1}{x} \right)' = -\frac{(x)'}{x^2} = -\frac{1}{x^2}$$

10) Derivative of a composite function: Chain rule

If f and g are both differentiable and F is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x)) \cdot g'(x)$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions,

$$\text{then } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Example:

Find the derivative of $f \circ g$ if $f(x) = x^2 + 3x + 3$ and $g(x) = \frac{2x+1}{x}$

Solution:

$$\begin{aligned}
 (f \circ g)'(x) &= f'[g(x)]g'(x) = \left[2\left(\frac{2x+1}{x}\right) + 3\right]\left(\frac{2x+1}{x}\right)' \\
 &= \left(\frac{4x+2+3x}{x}\right)\left(\frac{2x-2x-1}{x^2}\right) = \left(\frac{7x+2}{x}\right)\left(\frac{-1}{x^2}\right) \\
 &= \frac{-7x-2}{x^3}
 \end{aligned}$$

11) Differentiation of radical functions

i) If we take any function in the square root function, then

$$y = f(x) \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{f(x)}} \frac{d}{dx} f(x) = \frac{1}{2\sqrt{f(x)}} f'(x)$$

Example1: $y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

Example2: $y = \sqrt{4x} \Rightarrow \frac{dy}{dx} = \frac{4}{2\sqrt{x}} = \frac{2}{\sqrt{x}}$

Example3: Find the derivative of $y = \sqrt{2x^2 + 5}$

We have the given function as $y = \sqrt{2x^2 + 5}$ and Differentiating with respect to variable x , we get $\frac{dy}{dx} = \frac{d}{dx} \sqrt{2x^2 + 5}$.

Now using the formula derivative of a square root, we have $\frac{dy}{dx} = \frac{1}{2\sqrt{2x^2 + 5}} \frac{d}{dx} (2x^2 + 5)$

$$\frac{dy}{dx} = \frac{4x}{2\sqrt{2x^2 + 5}} = \frac{2x}{\sqrt{2x^2 + 5}}$$

ii) $\frac{d}{dx} (\sqrt[n]{f(x)}) = \frac{1}{n} \frac{\frac{d}{dx} (f(x))}{\sqrt[n]{f^{n-1}(x)}}$

Example: $y = \sqrt[7]{x^5 - 2x} \Rightarrow y' = \frac{5x^4 - 2}{7\sqrt[7]{(x^5 - 2x)^6}}$

12) Implicit differentiation

$y = f(x) = \frac{x}{x^2+1}$ is **explicit function** when it is expressed directly in terms of

x , and when this example $y = \frac{x}{x^2+1}$ is in the form $x^2y + y - x = 0$ it becomes an implicit function and y is said to be defined implicitly as a function of x . The derivative of y respect to x may be found by considering y as a function of x and differentiating term by term:

$$\frac{d}{dx}(x^2y + y - x) = 0$$

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(y) - \frac{d}{dx}(x) = 0$$

$$\frac{d}{dx}(x^2)y + x^2 \frac{d}{dx}(y) + \frac{dy}{dx} - 1 = 0$$

$$2xy + x^2 \frac{dy}{dx} + \frac{dy}{dx} - 1 = 0$$

$$(x^2 + 1) \frac{dy}{dx} = 1 - 2xy$$

$$\frac{dy}{dx} = \frac{1 - 2xy}{x^2 + 1}$$

13. Differentiation of function of function

Let $y = f(v)$ with $v = \Psi(x)$ and derivative of y respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

Example: Let $y = 3v^2 - v$ with $v = x^2$, find $\frac{dy}{dx}$

Answer: $\frac{dy}{dv} = 6v - 1 = 6x^2 - 1$

$$\frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx} = (6x^2 - 1) \cdot (2x) = 12x^3 - 2x$$

$$\frac{dy}{dx} = 12x^3 - 2x$$



Application activity 2.2

- Given the function $f(x) = x^2 + 3x - 4$ and $g(x) = x + 1$. Find
 - $(f[g(x)])'$
 - $f'[g(x)]$
 - $f'[g(x)] \cdot g'(x)$
- Differentiate $y = (x^3 - 1)^{100}$
- A body is moving along the x -axis such that its displacement is given by

$$x(t) = t^3 - 3t$$
 . What will the acceleration $a(t)$ of the body be given that

$$a(t) = \frac{dv}{dt} \quad \text{and} \quad v(t) = \frac{dx}{dt}$$
- Find $\frac{dy}{dx}$ given that $x^2 - 2y^3 + 4x = 2$

2.3 Differentiation of trigonometric functions

Activity 2.3



Consider $y = f(x) = \sin x$

Based on definition of derivative and trigonometric formula of transformation of sum to product : $\sin a - \sin b = 2 \sin\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)$,

Establish the formula of $\frac{dy}{dx}$

e. Knowing that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$,

$\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$, deduce the derivative of

$\cos x$, $\tan x$, $\sec x$ and $\operatorname{csc} x$.

Content summary

1. Derivative of of function $y = f(x) = \sin x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2 \sin\left(\frac{x+h-x}{2}\right) \cos\left(\frac{x+h+x}{2}\right)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2 \sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \lim_{h \rightarrow 0} \cos\left(x + \frac{h}{2}\right)$$

$$f'(x) = 1 \cdot \lim_{h \rightarrow 0} \cos\left(x + \frac{h}{2}\right)$$

$$f'(x) = \cos x$$

In general:

$$[\sin u(x)]' = u'(x) \cos u(x) \quad \text{i.e. if } y = \sin u \text{ with } u = \psi(x) \text{ then } \frac{dy}{dx} = \frac{du}{dx} \cdot \cos u$$

Example: given that $y = \sin(2x^4 + x)$, find $\frac{dy}{dx}$

$$\text{Answer: } \frac{dy}{dx} = (2x^4 + x)' \cos(2x^4 + x) = (8x^3 + 1) \cos(2x^4 + x)$$

Derivative of $y = \cos x$

$$\text{We know that } \cos(x) = \sin\left(\frac{\pi}{2} - x\right) \Rightarrow \frac{dy}{dx} = -\sin x$$

In general, $(\cos u(x))' = -u'(x) \sin u(x)$

$$[\cos u(x)]' = -u'(x)\sin(u(x)) \quad \text{i.e. if } y = \cos(u(x)) \text{ with } u = \psi(x) \text{ then } \frac{dy}{dx} = -\frac{du}{dx} \cdot \sin(u(x))$$

Example:

$$y = \cos 2x \Rightarrow y' = -(2x)' \sin 2x$$

$$y' = -2 \sin 2x$$

Derivative of function $y = \tan x$

$$y = \tan x \text{ can be written as } y = \frac{\sin x}{\cos x}$$

then

$$\frac{dy}{dx} = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2} = \frac{\cos x \cos x + \sin x \sin x}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{(\cos x)^2}$$

$$\frac{dy}{dx} = \frac{1}{(\cos x)^2} \quad \text{or} \quad \frac{dy}{dx} = \sec^2 x$$

$$\text{In general, } [\tan u(x)]' = \frac{u'(x)}{\cos^2 u(x)} \quad \text{or} \quad [\tan u(x)]' = u'(x) \sec^2 u(x)$$

1. Derivative of function $y = \cot x$

$$\text{In the same way with derivative of tangent, } \frac{d}{dx} \cot x = \frac{-1}{(\sin x)^2} \quad \text{or}$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$= -\csc^2 x$$

$$\text{In general, } [\cot u(x)]' = -\frac{u'(x)}{\sin^2 u(x)} \quad \text{or} \quad [\cot u(x)]' = -u'(x) \operatorname{cosec}^2 u(x)$$

2. Derivative of $y = \sec x$

$$y = \sec x = \frac{1}{\cos x} \Rightarrow y' = \frac{\sin x}{\cos^2 x} \quad \text{or} \quad y' = \sin x \sec^2 x$$

In general,

$$(\sec u(x))' = u'(x) \frac{\sin u(x)}{\cos^2 u(x)} \quad \text{or} \quad (\sec u(x))' = u'(x) \sin u(x) \sec^2 u(x)$$

3. Derivative of $y = \csc x$

$$y = \csc x = \frac{1}{\sin x} \Rightarrow y' = \frac{-\cos x}{\sin^2 x} \text{ or } y' = -\cos x \csc^2 x$$

In general,

$$(\csc u(x))' = -u'(x) \frac{\cos u(x)}{\sin^2 u(x)} \text{ or } (\csc u(x))' = -u'(x) \cos u(x) \csc^2 u(x)$$



Application activity 2.3

1. Find the derivative of the following functions

a. $f(x) = \sin^3(x^2 + 4)$

b. $f(x) = \cos 3x^2$

2. Find the derivative of the following functions

a. $f(x) = \sin^3(x^2 + 4)$

b. $f(x) = \cos 3x^2$

3. Find the derivative of the following functions

a. $f(x) = x \tan x$

b. $f(x) = \tan(3x + 2)$

c. $f(x) = \cot(x^2 - 5)$

d. $f(x) = \sin x \cot 4x$

4. Find the derivative of the following functions

a. $f(x) = \sec(3x + 2)$

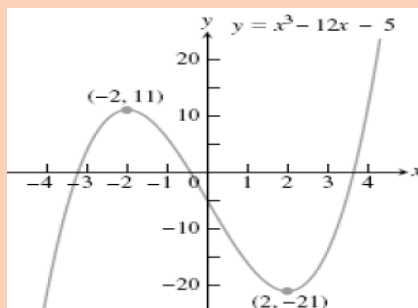
b. $f(\theta) = \theta^3 \csc 2\theta$

2.4 Derivative and the variation of a function

Activity 2.4



1. Observe the graph below and answer the questions that follow:



- a. Is the given function $y = x^3 - 12x - 5$ differentiable? Justify your answer.
 - b. Find y' by respecting x
 - c. Show the interval on which the function is increasing, and decreasing if $y = x^3 - 12x - 5$
2. Find where the function $f(x) = x^3 - 3x^2$ is concave up or down.

Content summary

Increasing and decreasing function

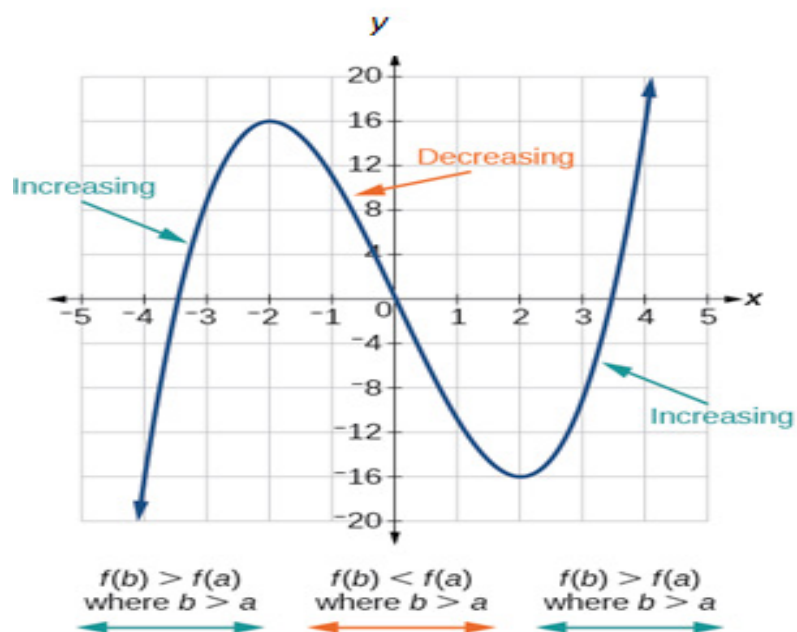
Theorem:

Let f be a function differentiable on an interval $]a, b[$

- a) If $f'(x) > 0$ on each point x of $]a, b[$, then f is increasing on $]a, b[$
- b) If $f'(x) < 0$ on each point x of $]a, b[$, then f is decreasing on $]a, b[$
- c) If $f'(x) = 0$ for all $x \in]a, b[$, then f is constant on this interval, that is $f(x_1) = f(x_2)$ for all $x_1, x_2 \in]a, b[$, or equivalently, there exists a real c such that $f(x) = c$ for all $x \in]a, b[$.

Given x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) < f(x_2)$ then $f(x)$ is increasing on I

Given any x_1 and x_2 from an interval I with $x_1 < x_2$ if $f(x_1) > f(x_2)$ then $f(x)$ is decreasing on I .



If the derivative of a function is positive at a point then the function is increasing at that point and if the derivative is negative at a point then the function is decreasing at that point. Also, the fact that the derivative of a function is zero at a point then the function is not changing at that point. These ideas used previously to identify the intervals in which a function is increasing and decreasing. This can be summarized as follows:

- If $f'(x) > 0$ for every x on some interval I , then $f(x)$ is increasing on that interval
- If $f'(x) < 0$ for every x on some interval I , then $f(x)$ is decreasing on the interval.
- If $f'(x) = 0$ for every x on some interval I , then $f(x)$ is constant on the interval.

Example:

1. Let us determine all intervals where the following function is increasing or

decreasing. $f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$

To determine if the function is increasing or decreasing we will need the derivative.

$$\begin{aligned} f'(x) &= -5x^4 + 10x^3 + 40x^2 \\ &= -5x^2(x^2 - 2x - 8) \\ &= -5x^2(x - 4)(x + 2) \end{aligned}$$

From the factored form of the derivative we see that there are three critical points: $x = -2$, $x = 0$, and $x = 4$. We now need to determine where the derivative is positive and where it is negative by drawing a sign table of $f'(x)$, graphing the critical points and picking test points from each region to see if the derivative is positive or negative in each region.

x	$-\infty$	-2	0	4	$+\infty$
$f'(x)$	-	0	+	0	-
$f(x)$	$+\infty$	$f(-2)$	$f(0)$	$f(4)$	$-\infty$

Increase is symbolized by the arrow ↗ on $-2 < x < 0$ and $0 < x < 4$

Decrease is symbolized by the arrow ↘ on $-\infty < x < -2$ and $4 < x < +\infty$

First derivative test for local extrema

Knowing where a function increases and decreases also tells us how to test for the nature of **local extrema values**.

Theorem

Suppose that c is a critical point of a continuous function f , and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. If f' changes from negative to positive at c , then f has a **local minimum** at c
2. If f' changes from positive to negative at c , then f has a **local maximum** at c
3. If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has **no** local extremum.

It means that, the relative extrema of a continuous function occur at those critical

points where the first derivative changes sign.

Definition: Let f be a function and let x_0 be a real number such that f is defined on an open interval containing x_0 .

- If $f'(x_0) = 0$, then we say that x_0 is a stationary number of f . For **example:** If x is the time and $y = f(t)$ is the displacement (function) of a moving object, then $\frac{dy}{dt} = f'(t)$ is the velocity (function). Thus $f'(t_0) = 0$ means that the velocity at time t_0 is 0, that is, the object is stationary at that moment.

Definition:

Let f be a function and let x_0 be a real number such that f is defined on an open interval containing x_0 . We say that :

- f has a relative maximum at $x = x_0$ if $f(x_0) \geq f(x)$ for all x sufficiently close to x_0 .
- f has a relative minimum at $x = x_0$ if $f(x_0) \leq f(x)$ for all x sufficiently close to x_0 .

Example

Locate the relative extreme points of $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$

Solution:

$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{-\frac{1}{3}}(x-2)$$

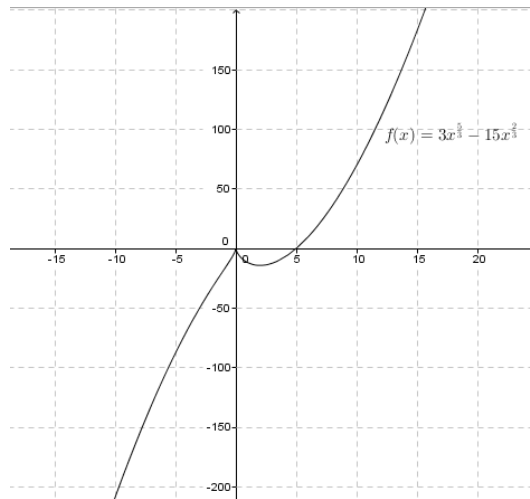
- If $x = 0$ then, $f'(x)$ does not exist at $x = 0$ but $f(0)$ exists and $f(0) = 0$
- if $x = 2$, the critical points are $(0, 0)$ and $(2, -14.28)$

Sign table of $f'(x)$

x		0		2	
$x-2$	- - -	- - -	- - -	0 + + +	+ + +
$\frac{-1}{x^3}$	- -	+ + +	+ + +	+ + +	+ + +
$f'(x)$	+ + +	- - -	- -	0 + + +	+ + +
$f(x)$	\nearrow	0 \searrow	\searrow	-14.28 \nearrow	\nearrow

There is a relative maximum at 0 and a relative minimum at 2.

Graph of the function $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$ is as follows:



Application activity 2.4

Find the intervals where the following functions are increasing or decreasing, and locate the relative extreme points for each function;

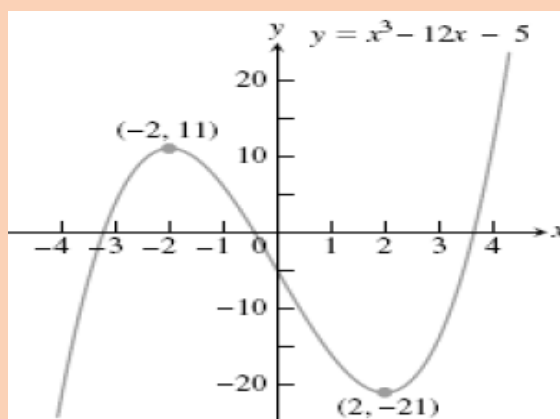
- $f(x) = 27x - x^3$
- $f(x) = x^4 - 4x^3 + 5$.

2.5 Derivative and Concavity of a function

Activity 2.5



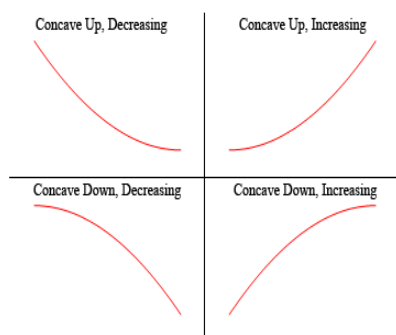
Observe the graph below and answer the questions that follow



- Give the intervals where the curve of the function opens up or down?
- Find second derivative of $y = x^3 - 12x - 5$ with respect to x and study the sign of $\frac{d^2y}{dx^2}$
- What is your conclusion from the intervals obtained in a) and positive or negative second derivative in b)

Content summary

The second derivative of a function can give us information about the graph of a function such as concave up or concave down. The following figure gives us the idea of concavity



Function is **turned up (concavity up)** if it “opens” up and the function is **turned down (concavity down)** if it “opens” down. Notice as well that concavity has nothing to do with increasing or decreasing. A function can be concave up and

either increasing or decreasing. Similarly, a function can be concave down and either increasing or decreasing.

Given the function $f(x)$ then

- $f(x)$ is **concave up** on an interval I if all of the tangents to the curve on I are below the graph of $f(x)$.
- $f(x)$ is **concave down** on an interval I if all of the tangents to the curve on I are above the graph of $f(x)$

Theorem:

Let f be a function that is defined and is twice differentiable on an open interval $]a, b[$.

1. If $f''(x) > 0$ for all $x \in]a, b[$, then f is convex or concave up on $]a, b[$

2. If $f''(x) < 0$ for all $x \in]a, b[$, then f is concave down on $]a, b[$

Notice that this fact tells us that a list of possible inflection points will be those points where the second derivative is zero or doesn't exist. Be careful however to not make the assumption that just because the second derivative is zero or doesn't exist that the point will be an inflection point.

We will only know that it is an inflection point once we determine the concavity on both sides of it. It will only be an inflection point if the concavity is different on both sides of the point.

Example:

1. Let us find where the function $f(x) = x^3 - 3x^2$ is concave up or down.

We need the second derivative so that we will find where concave is up or down

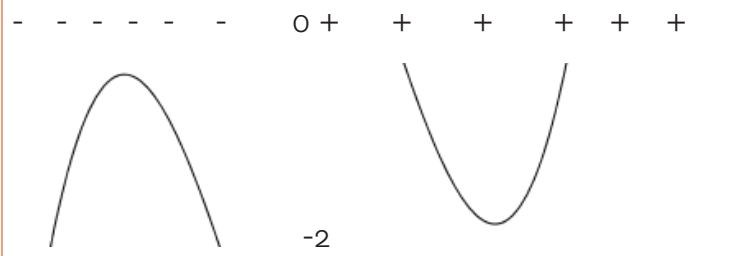
$$f'(x) = 3x^2 - 6x,$$

$$f''(x) = 6x - 6$$

$$f''(x) = 0, x = 1$$

Make a table of sign for second derivative

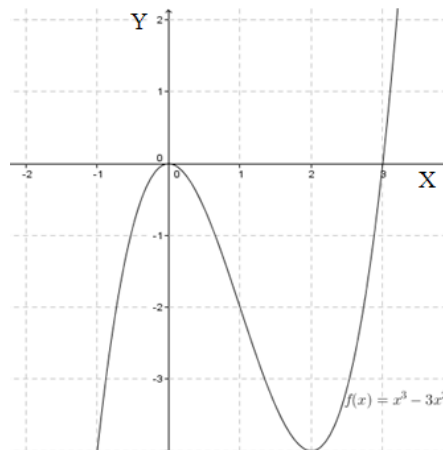
Sign of $f''(x)$

x	1
$f''(x)$	- - - - - 0 + + + + +
$f(x)$	

Thus, $f(x)$ is concave up if $x > 1$ or in interval $]1, +\infty[$

$f(x)$ is concave down if $x < 1$ or in interval $] -\infty, 1[$

The graph of the function $f(x) = x^3 - 3x^2$ is as follows:

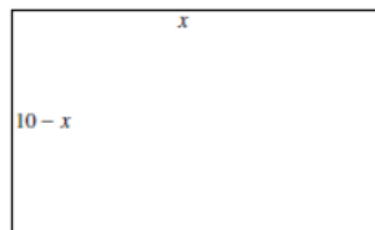


- Find the dimensions of the rectangle that has maximum area if its perimeter is 20 cm.

Solution:

Let x be the length, the width w is such that $2(x + w) = 20$

i.e, $w = 10 - x$





we will consider where the area A is increasing or decreasing where $A = x(10 - x)$, $0 < x < 10$.

We want to find the value of x at which A attains its maximum. Differentiating

$$A(x), \text{ we get } A'(x) = \frac{d}{dx}(10x - x^2) = 10 - 2x \quad (0 < x < 10)$$

Solving $A'(x) = 0$, we obtain the critical number of A : $x_1 = 5$.

Since A is increasing on $(0, 5)$ and decreasing on $(5, 10)$, it follows that A attains its absolute maximum at $x_1 = 5$. The dimensions of the largest rectangle is $5\text{cm} \times 5\text{cm}$.

	$(0, 5)$	$(5, 10)$
A'	+ + + + + +	- - - - - -
A		



Application activity 2.5

1. Let f be the function $f(x) = x^3 - 12x + 1$
 - a. Calculate extreme points
 - b. Show the interval of increasing and decreasing
 - c. Show where the function is concave up or down

2.6 Derivative and the table of variation for a function

Activity 2.6



Let f be the function $f(x) = x^3 - 12x - 5$

- a. Calculate the first and second derivative of $f(x)$
- b. In one table, present the sign table of $f'(x)$ and $f''(x)$ showing also the intervals where $f(x)$ is increasing or decreasing and the sense of concavity.
- c. Try to analyse the function according to the above table

Content summary

Variation table or synthetic table summarize all information about the function: domain of definition, limits at the boundaries of domain, asymptotes to the curve, extrema points, variation of function (increasing or decreasing), inflection points and the sense of concavity (opens up or down).

Example: Establish the variation table and sketch of the function $f(x) = \frac{x^2 + 1}{x}$

Condition of existence: $x \neq 0$

1. $Domf = \mathbb{R} \setminus \{0\} =]-\infty, 0[\cup]0, +\infty[$
2. Limits at the boundaries of $Domf$

a)

$$\lim_{x \rightarrow -\infty} \frac{x^2 + 1}{x} = \frac{\infty}{\infty} \quad (IF)$$

T.V: $\lim_{x \rightarrow -\infty} \frac{x^2 + 1}{x} = \lim_{x \rightarrow -\infty} \frac{x^2(1 + \frac{1}{x^2})}{x} = \lim_{x \rightarrow -\infty} x = -\infty \Rightarrow$ There is no Horizontal asymptote

$$\lim_{x \rightarrow 0} \frac{x^2 + 1}{x} = \frac{1}{0}$$

$$\lim_{x \rightarrow 0^-} \frac{x^2 + 1}{x} = \frac{1}{0^-} = -\infty \Rightarrow \text{There is a vertical asymptote : } x = 0$$

$$\lim_{x \rightarrow 0^+} \frac{x^2 + 1}{x} = \frac{1}{0^+} = +\infty$$

$$\lim_{x \rightarrow +\infty} \frac{x^2 + 1}{x} = \frac{\infty}{\infty} \quad (IF)$$

b)

T.V: $\lim_{x \rightarrow +\infty} \frac{x^2 + 1}{x} = \lim_{x \rightarrow +\infty} \frac{x^2(1 + \frac{1}{x^2})}{x} = \lim_{x \rightarrow +\infty} x = +\infty$

\Rightarrow There is no Horizontal asymptote

Let find the oblique asymptote (OA)

$$OA \equiv y = ax + b \text{ with } a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} \text{ and } b = \lim_{x \rightarrow \pm\infty} [f(x) - ax]$$

$$a = \lim_{x \rightarrow \pm\infty} \frac{x^2 + 1}{x^2} = \lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2} = 1$$

$$b = \lim_{x \rightarrow \pm\infty} \left(\frac{x^2 + 1}{x} - x \right) = \lim_{x \rightarrow \pm\infty} \left(\frac{x^2 + 1 - x^2}{x} \right) = \lim_{x \rightarrow \pm\infty} \left(\frac{1}{x} \right) = 0$$

$a = 1, b = 0 \Rightarrow$ there is no OA

3. Parity of $f(x)$

- $f(x)$ is even function if $f(x) = f(-x)$

$$f(-x) = \frac{(-x)^2 + 1}{-x} = -\frac{x^2 + 1}{x} \neq f(x), \text{ so } f(x) = \frac{x^2 + 1}{x} \text{ is not even}$$

- $f(x)$ is odd function if $-f(x) = f(-x)$

$$f(-x) = \frac{(-x)^2 + 1}{-x} = -\frac{x^2 + 1}{x}$$

- $-f(x) = -\frac{x^2 + 1}{x}$, so $-f(x) = f(-x)$ and $f(x)$ is odd function

4. Parity of $f(x)$

The given function is not periodic

5. First derivative of $f(x)$

$$f'(x) = \left(\frac{x^2 + 1}{x} \right)' = \frac{2x \cdot x - 1 \cdot (x^2 + 1)}{x^2} = \frac{2x^2 - x^2 - 1}{x^2}$$

$$f'(x) = \frac{x^2 - 1}{x^2}$$

Critical points:

- Condition of existence of $f'(x)$

$$x^2 \neq 0 \Leftrightarrow x \neq 0$$

$$f'(x) = 0 \Leftrightarrow \frac{x^2 - 1}{x^2} = 0 \Leftrightarrow x^2 - 1 = 0$$

$$\Rightarrow x = -1 \text{ or } x = 1$$

Extrema points: if $x = -1 \Rightarrow y = -2$, function has a maximum point $(-1, -2)$
 $x = 1 \Rightarrow y = 2$, function has a minimum point $(1, 2)$

Sign table of $f'(x)$

x	$-\infty$	-1	0	1	∞
$f'(x)$	+++	0---	- - -	0 +++	+++
$f(x)$	$-\infty$	-2	$-\infty$	2	$+\infty$

6. Second derivative of $f(x)$



$$f''(x) = \left(\frac{x^2 - 1}{x^2} \right)' = \frac{2x \cdot x^2 - 2x \cdot (x^2 - 1)}{x^4} = \frac{2x^3 - 2x^3 + 2x}{x^4} = \frac{2x}{x^4}$$

$$f''(x) = \frac{2}{x^3}$$

Critical points:

- Existence condition of $f''(x)$: $x^3 \neq 0 \Rightarrow x \neq 0$
- $f''(x) = 0 \Leftrightarrow \frac{1}{x^3} = 0$ It doesn't exist in \mathbb{R}

Sign table of $f''(x)$

x	$-\infty$	0	$+\infty$
$f''(x)$	---	+ + +	+
$f(x)$			

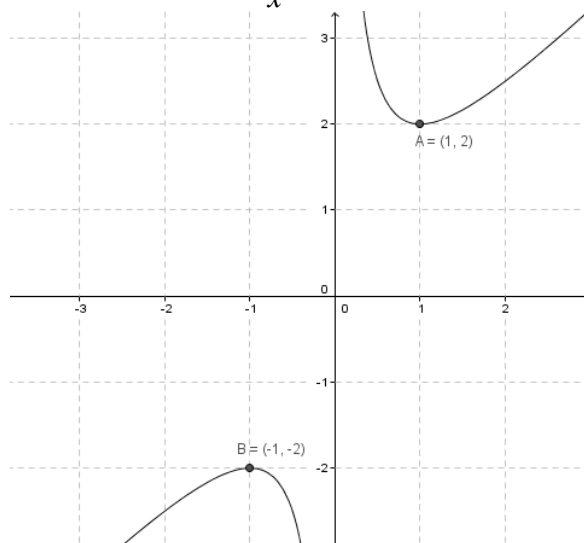
7. Variation table or synthetic table

x	$-\infty$	-1	0	1	$+\infty$						
$f'(x)$	$+$	$+$	$+0$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f''(x)$	$-$	$-$	$-$	$-$	$-$	$ $	$+$				
$f(x)$	$-\infty$	-2	$ $	$ $	$+\infty$	2	$+\infty$				

8. Supplementary points

x	2	-2	-3	3	-4	4
$f(x)$	2.5	-2.5	$-\frac{10}{3}$	$\frac{10}{3}$	$-\frac{17}{4}$	$\frac{17}{4}$

9. Graph of function $f(x) = \frac{x^2 + 1}{x}$



Application activity 2.6

1. Study completely $y = x + \frac{1}{4x}$
2. Study completely $y = \frac{1}{x-2}$

2.7 Derivative and limit with indeterminate cases: Hospital's rule

Activity 2.7



Evaluate the following limits:

a. $\lim_{t \rightarrow 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^2}$

b. $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

c. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$

What happens if you calculate the limits of derivative of numerator over derivative of denominator for above limits?

Content summary

Hospital's rule states that, if $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$ with $g(x) \neq 0$ and x_0 a finite number

or infinity is indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then, it can be calculated by

$\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ with $g'(x) \neq 0$. If this result is indeterminate form, the procedure can be repeated.

Examples

By using Hospital's rule, evaluate the following limits

a. $\lim_{x \rightarrow 2} \frac{\sqrt{7+x} - 3}{x - 2} = \frac{0}{0}$ (I.F)

$$\lim_{x \rightarrow 2} \frac{\sqrt{7+x} - 3}{x - 2} = \lim_{x \rightarrow 2} \frac{(\sqrt{7+x} - 3)'}{(x - 2)'} = \lim_{x \rightarrow 2} \frac{\frac{1}{2\sqrt{7+x}}}{1} = \frac{1}{2} \lim_{x \rightarrow 2} \frac{1}{\sqrt{7+x}} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

b. $\lim_{x \rightarrow 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right) = \lim_{x \rightarrow 2} \left(\frac{4 - (x + 2)}{x^2 - 4} \right) = \lim_{x \rightarrow 2} \left(\frac{2 - x}{x^2 - 4} \right) = \frac{0}{0}$ I.F

$$\lim_{x \rightarrow 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right) = \lim_{x \rightarrow 2} \frac{(2-x)'}{(x^2 - 4)'} = \lim_{x \rightarrow 2} -\frac{1}{2x} = -\frac{1}{4}$$



Application activity 2.7

By using Hospital rule , evaluate the following limits

1. $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x}$

2. $\lim_{x \rightarrow \infty} \frac{2x + 7}{3x^2 - 5}$

3. $\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 + 3x - 4}$

4. $\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$

2.8 Derivative, tangent line equation and Normal line equation

Activity 2.8



Consider the function $f(x) = -x^3 + 3x$ and the line $y = 3x$ passing through point $(0,0)$.

1. Show that $(0,0)$ is the intersection of $f(x) = -x^3 + 3x$ and $y = 3x$
2. Find $f'(0)$
3. Compare the result in 2. and the gradient of the given line.

Content summary

Tangent line

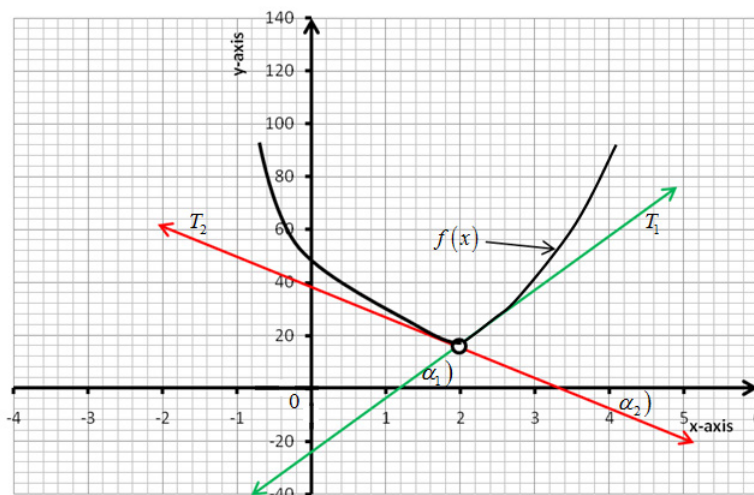
Recall that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by $y - y_0 = m(x - x_0)$. Note that the slope of the

tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $f'(x_0) = \left. \frac{dy}{dx} \right|_{(x_0, y_0)}$.

Replacing m by $f'(x_0)$ in the equation $y - y_0 = m(x - x_0)$ we get the equation of the tangent line defined by $T \equiv y - y_0 = f'(x_0)(x - x_0)$

Remember that the function $f(x)$ can have distinct right-hand and left-hand derivatives at x_0 ; that is, $f'(x_0^-) \neq f'(x_0^+)$.

In this case we say that the point x_0 is a **sharp**. The curve has no tangent line at x_0 . Centrally, it has a half tangent at the left and another at the right with different slopes (see the following figure).



$$\tan \alpha_1 = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad \tan \alpha_2 = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

Normal line

We call **normal line** to the curve at point (x_0, y_0) , the perpendicular line to the tangent line of the curve at point (x_0, y_0) . The slope of the normal line to the

curve $y = f(x)$ at (x_0, y_0) is $\frac{-1}{f'(x_0)}$, if $f'(x_0) \neq 0$. Therefore, the equation of

the normal line is given by $N \equiv y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Example

Given the parabola $f(x) = x^2$

- Find the point where the tangent line is parallel to the bisector of the first quadrant.
- Find the tangent line to the curve of this function at point $(2, 4)$

Solution

- The bisector of the first quadrant has the equation $y = x$ so its slope is $m = 1$. Since the two lines are parallel they have the same slope. So $f'(x_0) = 1$. Since the slope of the tangent line to the curve is equal to the derivative at $x = x_0$

$$\begin{aligned} f'(x_0) &= \lim_{h \rightarrow 0} \frac{(x_0 + h)^2 - x_0^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x_0^2 + 2x_0h + h^2 - x_0^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x_0h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (2x_0 + h) \\ &= 2x_0 \end{aligned}$$

$$\text{But } f'(x_0) = 1 \Rightarrow 2x_0 = 1 \Rightarrow x_0 = \frac{1}{2} \text{ and } y_0 = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Thus, the needed point is $\left(\frac{1}{2}, \frac{1}{4}\right)$.

- The given point is $(2, 4)$, then $x_0 = 2$, $y_0 = 4$. $f'(x_0) = 2x_0 \Rightarrow f'(2) = 4$

The tangent line is

$$T \equiv y - 4 = 4(x - 2)$$

$$T \equiv y - 4 = 4x - 8$$

$$T \equiv y = 4x - 4$$

$$T \equiv y = 4(x - 1)$$

Example

Let us determine the equation of the normal line to the curve with equation $y = x^2 - 5x + 6$ at point with abscissa $x = 0$.

$$f'(x) = 2x - 5, \quad f'(0) = -5, \quad f(0) = 6$$

The equation of normal line is $N \equiv y - 6 = -\frac{1}{-5}(x - 0)$ or $N \equiv y = \frac{1}{5}x + 6$



Application activity 2.8

- Determine the equation of the tangent and normal line to the curve of function
 - $f(x) = \sqrt{x^2 + 3}$ at the point $(-1, 2)$
 - $f(x) = x^3 - x + 5$ at the point $(1, 5)$
 - $f(x) = x^3$ at the point where $x = 2$
- Let $f(x) = x^2 - x$. Find the equation of tangent line with slope $m = -3$

2.9 Applications of differentiation in Economics and finance

Activity 2.9



1. Go in library or computer lab, do research on application of differentiation in economics and finance.
2. The marginal cost (MC) is the rate of change of the total cost (TC) function. In fact, in situations where one is dealing with the concept of a marginal increase, the marginal function is equal to the rate of change

of the original function. This means that $MC = \frac{dTC}{dq}$. Determine the function MC when $TC = 6 + 4q^2$.

Content summary

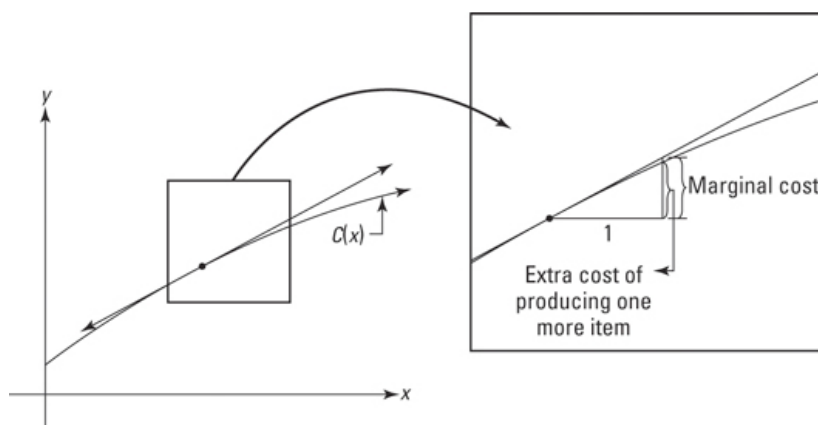
Marginal cost

Suppose a manufacturer produces and sells a product. Denote $C(q)$ to be the total cost for producing and marketing q units of the product. Thus C is a function of q and it is called the *(total) cost function*. The rate of change of C with respect to q is called the *marginal cost*, that is,

$$\text{Marginal Cost} = \frac{dC}{dq}$$

Marginal cost, marginal revenue, and marginal profit all involve how much a function goes up (or down) go through unit to the right — this is very similar to the way linear approximation works.

Say that you have a cost function that gives you the total cost, $C(x)$, of producing x items (shown in the figure below).



Example

A widget manufacturer determines that the demand function for her widgets is $p = \frac{1000}{\sqrt{x}}$ where x is the demand for widgets at a given price, p .

The cost of producing x widgets is given by the following cost function:
 $C(x) = 10x + 100\sqrt{x} + 10,000$

Determine the marginal cost at $x = 100$ widgets

Solution

Marginal cost is the derivative of the cost function, so take the derivative and evaluate it at $x = 100$.

$$C(x) = 10x + 100\sqrt{x} + 10,000$$

$$C'(x) = 10 + \frac{50}{\sqrt{x}} \text{ (power rule)}$$

$$C'(100) = 10 + \frac{50}{\sqrt{100}} = 10 + \frac{50}{10}$$

$$C'(100) = 15$$

Thus, the marginal cost at $x = 100$ is \$15 — this is the approximate cost of producing the 101st widget.

Marginal revenue

Denote $R(q)$ to be the total amount received for selling q units of the product. Thus R is a function of q and it is called the *revenue function*. The rate of change of R with respect to q is called the *marginal revenue*, that is,

$$\text{Marginal Revenue} = \frac{dR}{dq}$$

Denote $P(q)$ to be the **profit** of producing and selling q units of the product, that is,

$$P(q) = R(q) - C(q)$$

Thus P is a function of q and it is called the profit function.

In introductory economics texts, marginal revenue (MR) is sometimes defined as the increase in total revenue (TR) received from sales caused by an increase in output by 1 unit.

This only gives an approximate value for marginal revenue and it will vary if the units that output is measured in are changed. A more precise definition of *marginal revenue* is that it is the rate of change of total revenue relative to increases in output.

Denote q_{\max} to be the largest number of units of the product that the manufacturer can produce. Assuming that q can take any value between 0 and q_{\max} . Then for each of the functions C , R and P , the domain is $[0; q_{\max}]$. Suppose that the cost function and the revenue function are differentiable on $]0, q_{\max}[$ and suppose that producing 0 or q_{\max} units of the product will not give maximum profit. Then in order to have maximum profit, we need

$$\frac{dP}{dq} = 0$$

Or eventually,

$$\frac{dC}{dq} = \frac{dR}{dq},$$

that is, marginal cost = marginal revenue.

To have a maximum profit, marginal cost = marginal revenue

Example 1:

1. Given that $TR(q) = 80q - 2q^2$,
 - a. Find the function of the marginal revenue MR , and $P(q)$, given that $TR(q) = P \cdot q$
 - b. Find the exact value of the output at which TR is a maximum.

Solution:

a. $MR = \frac{dTR}{dq} = 80 - 4q$

We have $TR(q) = Pq \Leftrightarrow 80q - 2q^2 = Pq$

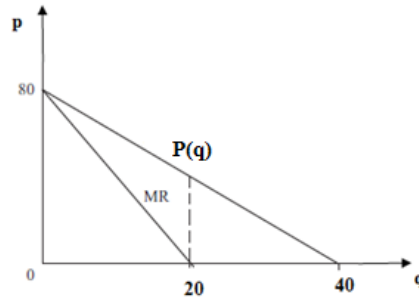
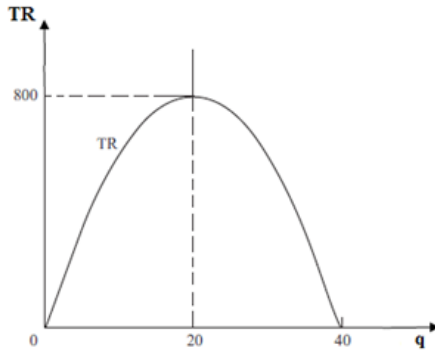
Therefore, the price function is $P(q) = 80 - 2q$

b. When TR is at its maximum, $\frac{dTR(q)}{dq} = MR = 0$
Thus,

$$MR = 80 - 4q = 0$$

$$80 = 4q$$

$$20 = q$$



Example 2: For the total revenue function $TR = 500q - 2q^2$, find the value of MR when $q = 80$.

Solution

$$MR = \frac{dTR}{dq} = 500 - 4q. \text{ Thus, when } q = 80, MR = 500 - 4(80) = 180$$

Example 3:

A widget manufacturer determines that the demand function for her widgets is

$$p = \frac{1000}{\sqrt{x}} \text{ where } x \text{ is the demand for widgets at a given price, } p.$$

The cost of producing x widgets is given by the following cost function:

$$C(x) = 10x + 100\sqrt{x} + 10,000$$

Determine the marginal revenue at $x = 100$ widgets

Solution

Revenue, $R(x)$, equals the number of items sold, x , times the price, p :

$$R(x) = x \cdot p = x \cdot \frac{1000}{\sqrt{x}} \text{ (using the above demand function)}$$

$$R(x) = \frac{1000x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{1000x\sqrt{x}}{x} = 1000\sqrt{x}$$

Marginal revenue is the derivative of the revenue function, so take the derivative of $R(x)$ and evaluate it at $x = 100$:

$$R(x) = 1000\sqrt{x}$$

$$R'(x) = \frac{500}{\sqrt{x}} \text{ (power rule)}$$

$$R'(100) = \frac{500}{\sqrt{100}} = 50$$

Thus, the approximate revenue from selling the 101st widget is \$50.

Example 4: Profit maximization

A monopoly faces the demand schedule $p = 460 - 2q$ and the cost schedule $TC = 20 + 0.5q^2$

How much should it sell to maximize profit and what will this maximum profit be? (All costs and prices are in \$.)

Solution

To maximize profits, the rule $MC = MR$ must be realized.

To find the output where $MC = MR$ we first need to differentiate the MC and MR functions.

$$\text{Given } TC = 20 + 0.5q^2, \text{ then } MC = \frac{dTC}{dq} = q$$

$$\text{As } TR = pq = (460 - 2q) \cdot q = 460q - 2q^2$$

$$\text{Then } MR = \frac{dTR}{dq} = 460 - 4q$$

To maximize profit $MR = MC$. Therefore, equating the two quantities, we find:

$$460 - 4q = q \text{ which implies that } q = 92$$

The actual maximum profit when the output is 92 will be:

$$TR - TC = (460q - 4q^2) - (20 + 0.5q^2)$$

$$= 460q - 2.5q^2 - 20 = 460 \cdot (92) - 2.5(8,464) - 20 = 21,140$$

The actual maximum profit when the output is 92 will be \$21,140

Example 5: A firm faces the demand schedule $p = 184 - 4q$ and the TC function $TC = q^3 - 21q^2 + 160q + 40$.

What output will maximize profit?

Solution

Given that $TR = pq = (184 - 4q)q = 184q - 4q^2$, we have $MR = \frac{dTR}{dq} = 184 - 8q$

And $MC = \frac{dTC}{dq} = 3q^2 - 42q + 160$.

To maximize the profit, $MC = MR$. Therefore,

$$3q^2 - 42q + 160 = 184 - 8q$$

$$(q - 12)(3q + 2) = 0$$

$$q - 12 = 0 \text{ or } 3q + 2 = 0$$

$$q = 12 \text{ or } q = -\frac{2}{3}$$

As one cannot produce a negative quantity, the firm must produce 12 units of output in order to maximize profits.

Example 6: Suppose that a company has estimated that the cost (in Rwandan francs) of producing x items is $C(x) = 10000 + 5x + 0.01x^2$. What is the marginal cost at the production level of 500 items?

Solution: Then the marginal cost function is $c'(x) = 5 + 0.02x$

The marginal cost at the production level of 500 items is $c'(500) = 5 + 0.02(500) = 15$ Rwandan Francs per item.

Example 7. The demand equation for a certain product is $q - 90 + 2p = 0$, $0 \leq q \leq 90$ where q is the number of units and p is the price per unit, and the average cost

function is $C_{av} = q^2 - 8q + 57 + \frac{2}{q}$, $0 \leq q \leq 90$. At what value of q will there be maximum profit? What is the maximum profit?

Solution:

Although the average cost function is undefined at $q = 0$, we may include 0 in the domain of the cost function. The cost function and the revenue function are differentiable on $]0; 90[$. However, we do not know whether maximum profit would be attained in $]0; 90[$ or at an endpoint. So we use the method for finding absolute

extrema for functions on closed and bounded intervals.

The cost function C is given by

$C(q) = q.C_{av} = q^3 - 8q^2 + 57q + 2, \quad 0 \leq q \leq 90$ and the total revenue function R is given by

$$R(q) = p.q = \frac{90-q}{2}.q; \quad 0 \leq q \leq 90$$

Therefore the profit function P is given by

$$\begin{aligned} p(q) &= R(q) - C(q) = \left(45q - \frac{q^2}{2}\right) - (q^3 - 8q^2 + 57q + 2) \\ &= -q^3 + \frac{15}{2}q^2 - 12q - 2, \quad (0 \leq q \leq 90) \end{aligned}$$

Differentiating $P(q)$, we get

$$p'(q) = \frac{d\left(-q^3 + \frac{15}{2}q^2 - 12q - 2\right)}{dq} = -3q^2 + 15q - 12, \quad (0 \leq q \leq 90)$$

Solving $P'(q) = 0$, that is,

$$-3q^2 + 15q - 12 = 0$$

$$-3(q-1)(q-4) = 0; \quad (0 < q < 90)$$

We get the critical number of P where $q_1 = 1$ and $q_2 = 4$.

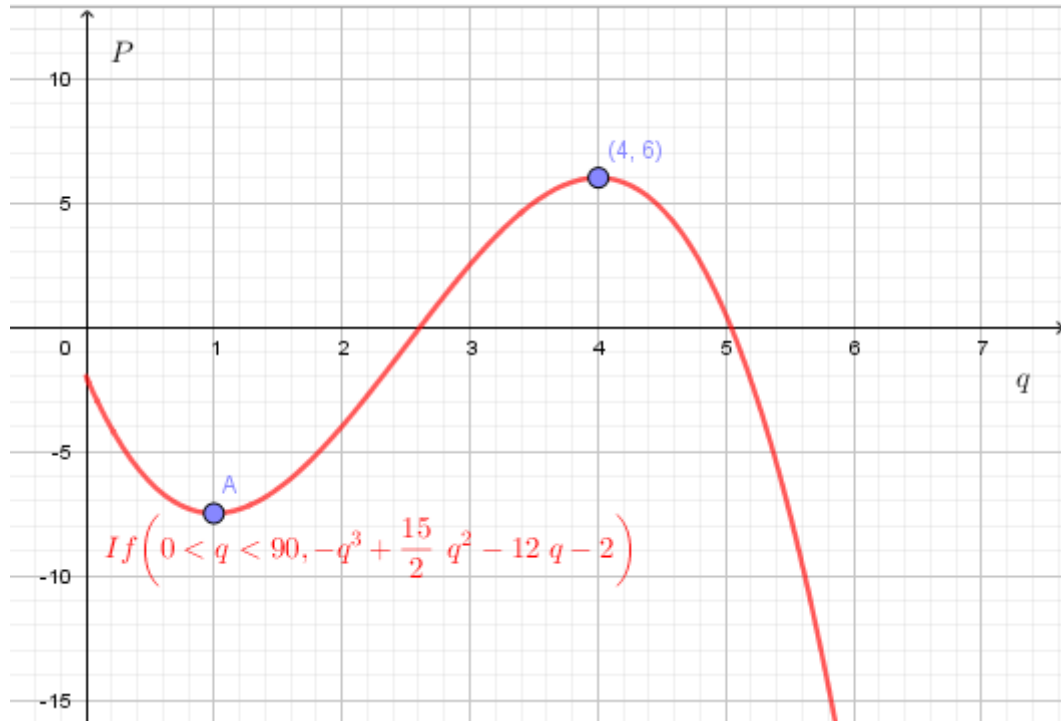
Comparing the values of P at the critical numbers as well as that at the endpoints:

q	0	1	4	90
$P(q)$	-2	$-\frac{15}{2}$	6	-669332

We see that maximum profit is attained at $q_2 = 4$ and the maximum profit is 6 (units of money).

Remarks:

If we know that maximum profit is not attained at the end points, we can simply compare the values of P at $q_1 = 1$ and $q_2 = 4$.



Example 8: A store has been selling 200 DVD burners a week at 350 dollars each. A market survey indicates that for each 10 dollars rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution

If x is the number of DVD burners sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units sold, the price is decreased by 10 dollars. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$

The demand function is $p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{1}{2}x$

The revenue function is $R(x) = xp(x) = 450x - \frac{1}{2}x^2$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$

This value of x gives an absolute maximum by the First Derivative Test

$$p(450) = 450 - \frac{1}{2}(450) = 225$$

The rebate is $350 - 225 = 125$, Therefore, to maximize revenue, the store should offer a rebate of 125 Rwandan francs.

Tax yield

Elementary supply and demand analysis tells us that the effect of a per-unit tax t on a good sold in a competitive market will effectively shift up the supply schedule vertically by the amount of the tax. This will cause the price paid by consumers to rise and the quantity bought to fall. The change in total revenue spent by consumers will depend on the price elasticity of demand.

Example 9:

A market has the demand schedule $p = 92 - 2q$ and the supply schedule $p = 12 + 3q$. What per-unit tax will raise the maximum tax revenue for the government? (All prices are in \$.).

Solution:

Let the per-unit tax be t . This changes the supply schedule to $p = 12 + t + 3q$

i.e the intercept on the price axis shifts vertically upwards by the amount t .

we now need to derive a function for q in terms of the tax t . In equilibrium, supply price equals demand price. Therefore,

$$12 + 3q + t = 92 - 2q$$

$$q = 16 - 0.2t$$

The tax yield is (amount sold) \times (per unit tax). Therefore,

$$TY = q \cdot t = (16 - 0.2t) \cdot t = 16t - 0.2t^2$$

And so the rate of change of TY with respect to t is

$$\frac{dTY}{dt} = 16 - 0.4t$$

If $\frac{dTY}{dt} > 0$, an increase in t will increase TY. However, from the formula for $\frac{dTY}{dt}$ derived above, one can see that as the amount of tax t is increased the value of $\frac{dTY}{dt}$ falls. Therefore in order to maximize TY, t should be increased until $\frac{dTY}{dt} = 0$

.Any further increases in t would cause $\frac{dTY}{dt}$ to become negative and cause TY to start to fall.

$$\frac{dTY}{dt} = 16 - 0.4t = 0$$

Thus,
 $t = 40$

Therefore a per-unit tax of \$40 will maximize the tax yield.

Example 10:

A widget manufacturer determines that the demand function for her widgets is

$$p = \frac{1000}{\sqrt{x}}$$
 where x is the demand for widgets at a given price, p .

The cost of producing x widgets is given by the following cost function:

$$C(x) = 10x + 100\sqrt{x} + 10,000$$

Determine the marginal profit at $x = 100$ widgets

Solution

Profit, $P(x)$, equals revenue minus costs. So,

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 1000\sqrt{x} - (10x + 100\sqrt{x} + 10,000) \\ &= -10x + 900\sqrt{x} - 10,000 \end{aligned}$$

Marginal profit is the derivative of the profit function, so take the derivative of $P(x)$ and evaluate it at $x = 100$.

$$P(x) = -10x + 900\sqrt{x} - 10,000$$

$$P'(x) = -10 + \frac{450}{\sqrt{x}} \text{ (power rule)}$$

$$\begin{aligned} P'(100) &= -10 + \frac{450}{\sqrt{100}} \\ &= -10 + 45 \\ &= 35 \end{aligned}$$

So, selling the 101st widget brings in an approximate profit of \$35.

Once you know the marginal cost and the marginal revenue, you can get marginal profit with the following simple formula: Marginal Profit = Marginal Revenue – Marginal Cost.



Application activity 2.9

A market faces the demand schedule $P = 58 - \frac{q}{2}$ and the cost schedule

$$TC = 97q - 17\frac{q^2}{2} + \frac{q^3}{3}$$

How much should it sell to maximize profit and what will be this maximum profit?

(All costs and prices are in Rwandan Francs)

2.10 Applications of differentiation: rate of change problems, optimization problems

Activity 2.10



Given the function $y = f(x) = x^2$, find

the ratio of the variation of y over the variation of x in each of the following intervals:

- [2;4]
- [-3;-1]

Content summary

Rate of Change

The purpose here is to remind us of one of the most important applications of derivatives. Recall that by the derivative $\frac{dS}{dt}$, we mean the rate of change of distance S with respect to time t . In similar fashion, whenever one quantity y varies with another quantity x , satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ or $f'(x)$

represents the rate of change of y with respect to x and $\left. \frac{dy}{dx} \right|_{x=x_0}$ or $f'(x_0)$

represents the change of rate of y with respect to x at $x = x_0$

If (x_0, y_0) is a point on the graph of $y = f(x)$, then we define $m = \frac{y_1 - y_0}{x_1 - x_0}$ to be

the average rate at which y changes with respect to x over the interval $[x_0, x_1]$.

If $y = f(x)$ and $f(x)$ is differentiable at x_0 , then we define $n = \frac{dy}{dx} \Big|_{x=x_0}$ to be the

instantaneous rate at which y changes.

$\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

Further, if two variables x and y are varying with respect to another variable t , i.e. if $x = f(t)$ and $y = g(t)$, then by Chain Rule $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$, if $\frac{dx}{dt} \neq 0$. Thus,

the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to x

Examples

1. For the curve $y = x^2 + 1$. Let us find the average rate of change of y with x over the interval $[3, 5]$ and the instantaneous rate of change of y with x at point $x = 3$.

Here $x_0 = 3$ and $x_1 = 5$

$$y_0 = (3)^2 + 1 = 10, y_1 = (5)^2 + 1 = 26$$

So average rate of y over $[3, 5]$ is $\frac{26-10}{5-3} = \frac{16}{2} = 8$. Thus, on the average y increases 8 units for each unit increase in x over the interval $[3, 5]$

$\frac{dy}{dx} = f'(x) = 2x$, so instantaneous rate of change of y at $x = 3$ is $\frac{dy}{dx} \Big|_{x=3} = 2x \Big|_{x=3} = 6$

Thus, at point $x = 3$, y is increasing 6 times as fast as x .

A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10cm , how fast is the enclosed area increasing?

Solution:

The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of

change of area A with respect to t is

$$\frac{dA}{dx} = \frac{d}{dx}(\pi r^2) = \frac{d}{dx}(\pi r^2) \times \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \quad (\text{By chain Rule})$$

It is given that $\frac{dr}{dt} = 4 \text{ cm/s}$

$$\text{Therefore, when } r = 10 \text{ cm, } \frac{dA}{dt} = 2\pi(10)(4) = 80\pi$$

Thus, the enclosed area is increasing at the rate of $80\pi \text{ cm}^2/\text{s}$, when $r = 10 \text{ cm}$

Rolle's Theorem

Suppose that $f(x)$ is a function that satisfies all of the following.

$f(x)$ is continuous on the closed interval $[a, b]$.

- $f(x)$ is differentiable on the open interval (a, b) .
- $f(a) = f(b)$

Then, there is a number c such that $a < c < b$ and $f'(c) = 0$.

Or, in other words $f(x)$ has a critical point on the open interval $]a, b[$.

Example

Consider the function $f(x) = x^2 - 1$ on $[-1, 1]$

This function is continuous on $[-1, 1]$ and differentiable on $] -1, 1[$

Moreover $f(-1) = f(1) = 0$.

Then from Rolle's theorem we must get a number c such that $-1 < c < 1$ and $f'(c) = 0$.

The first derivative is $f'(x) = 2x$ and $f'(x) = 0$ for $x = 0$ and we see that $-1 < 0 < 1$.

Mean Value Theorem

Suppose that $f(x)$ is a function that satisfies both of the following.

- $f(x)$ is continuous on the closed interval $[a, b]$.
- $f(x)$ is differentiable on the open interval $]a, b[$.

Then, there is a number c such that $a < c < b$ and $f'(c) = \frac{f(b) - f(a)}{b - a}$

Or, $f(b) - f(a) = f'(c)(b - a)$

Note that the Mean Value Theorem doesn't tell us what c is. It only tells us that there is at least one number c that will satisfy the conclusion of the theorem.

Also note that if $f(a) = f(b)$ we can think of Rolle's Theorem as a particular case of the Mean Value Theorem.

Geometrical interpretation of the Mean Value Theorem.

First define $A = (a, f(a))$ and $B = (b, f(b))$ and then we know from the Mean

Value theorem that there is a c such that $a < c < b$ and that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

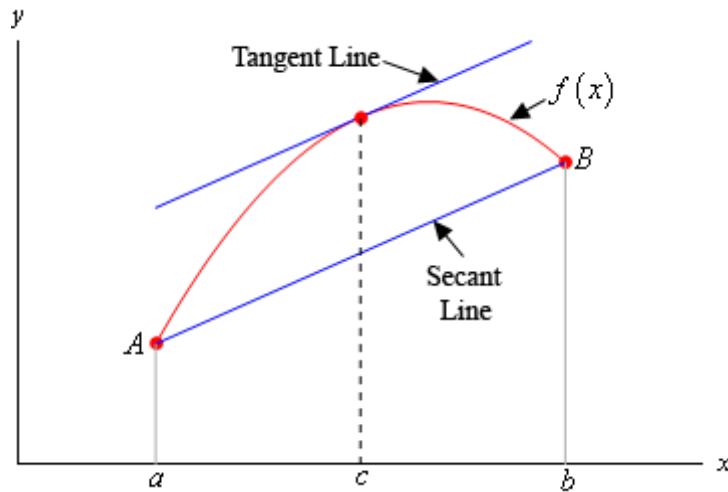
Now, if we draw in the secant line connecting A and B then we can know that the

slope of the secant line is, $\frac{f(b) - f(a)}{b - a}$.

Likewise, if we draw the secant line to $f(x)$ at $x = c$ we know that its slope is $f'(c)$.

What the Mean Value Theorem tells us is that these two slopes must be equal or in other words the secant line connecting A and B and the tangent line at $x = c$ must be parallel.

We can see this in the following sketch.



Example

Let us determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the function $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$

There isn't really a lot with this problem other than to notice that since $f'(x)$ is a polynomial; it is both continuous and differentiable (i.e, the derivative exists) on the given interval.

First derivative, $f'(x) = 3x^2 + 4x - 1$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$\Leftrightarrow 3c^2 + 4c - 1 = \frac{14 - 2}{3} = 4 \Leftrightarrow 3c^2 + 4c - 1 = 4 \Leftrightarrow 3c^2 + 4c - 5 = 0$$

$$\Delta = 16 + 60 = 76$$

$$c = \frac{-4 \pm 2\sqrt{19}}{6} = \frac{-2 \pm \sqrt{19}}{3}$$

Thus, the values of c which satisfy the conclusions of the Mean Value Theorem

for the function $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$ is $\frac{-2 + \sqrt{19}}{3}$. The value $\frac{-2 - \sqrt{19}}{3}$

is excluded since it is not an element of the given interval.

Let us see a couple of facts.

Fact 1

If $f'(x) = 0$ for all x in an interval $]a, b[$ then $f(x)$ is constant on $]a, b[$.

Fact 2

If $f'(x) = g'(x)$ for all x in an interval $]a, b[$ then in this interval we have $f(x) = g(x) + c$ where c is some constant.

Note that in both of these facts we are assuming the functions are continuous and differentiable on the interval $[a, b]$



Application activity 2.10

The length x of a rectangle is decreasing at the rate of 3 cm/min and the width is increasing at the rate of 2 cm/min . When $x = 10 \text{ cm}$ and $y = 6 \text{ cm}$, find the rates of change of :

- The perimeter
- The area of the rectangle



2.11 End of unit assessment

1. By definition calculate the derivative of $f(x) = x - 2$
2. A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at 10 dollars, the average attendance had been 27,000. When ticket prices were lowered to 8 dollars, the average attendance rose to 33,000.
 - a. Find the demand function, assuming that it is linear.
 - b. How should ticket prices be set to maximize revenue?
3. The given $f(x) = 2x^3 - 5x^2 + 4x + 2$
 - a. Calculate extreme points
 - b. Calculate first and second derivative
 - c. Show the interval of increasing and decreasing of the given function
 - d. Show where the concavity is up or down
4. Find the quantity q which will maximize profit if the total revenue and total cost (in dollars) are given by $R(q) = 5q - 0.003q^2$, $C(q) = 300 + 1.1q$ where $0 \leq q \leq 1000$ units.

UNIT 3

MATRICES AND SYSTEM OF LINER EQUATIONS

Key Unit Competence:

Solve problem involving the system of linear equations using matrices



3.0. Introductory activity

A Farmer Kalisa bought 5 Cocks and 4 Rabbits and he paid 35,000 FRW in Ruhango Market, and the following day, he bought 3 Cocks and 6 Rabbits and he paid 30,000 FRW. in the same Market

- a. Arrange what Kalisa bought according to their types in a simple table as follows

Cocks	Rabits	Prics

- b. Discuss and explain in your own words how you can determine the cost of 1 Cocks and 1 Rabbit.

3.1. Definition, size and types of matrices

Activity 3.1



1. A shop sold 20 cell phones and 31 computers in a particular month. Another shop sold 45 cell phones and 23 computers in the same month. Present this information as an array of rows and columns.
2. a) Observe and complete the number of present students in the year three classes by Monday .

	Boys	girls	Total
ECLPE			
SSE			

b) If every class gets new students on Tuesday such that in ECLPE they have 2 boys and 1 girl, in SSE they receive 1 girl and 1 boy, Complete the table for new students.

c) Complete the table for all students in an array of rows and columns.

Content summary

1. A matrix is a rectangular arrangement of numbers, expressions, algebraic symbols which are arranged in rows and columns. A matrix is denoted with a capital letter: A, B, C,...and the elements are enclosed by parenthesis () or square brackets [].

Example 1

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad C = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 12 & 13 \\ 7 & 6 & 0 \end{pmatrix} \quad M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

From matrix B, a_{32} is element of third row and second column

From matrix C, 4 is element of first row and third column

2. The dimension or size or order of matrix

Matrix B is of order 3×3 ("read three by three") because there are 3 rows and 3 columns

Matrix M is of order $m \times n$, ("read **m** by **n**") because there are **m** rows and **n** columns

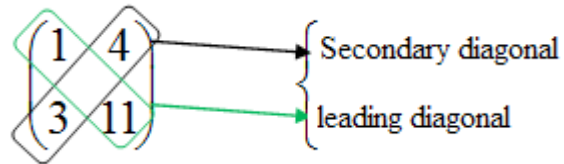
A square matrix is a matrix formed by the same number of rows and columns.

The elements of the form (a_{ij}) , where the two subscripts i and j are equal, constitute the **principal diagonal** (or **leading diagonal** or **main diagonal** or **major diagonal** or **primary diagonal**).

The **secondary diagonal** (or **minor diagonal** or **anti-diagonal** or **counter diagonal**) is formed by the elements with $i + j = n + 1$.

Square matrix of order 2 has the form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

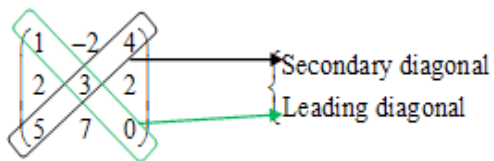
example of Matrix of order two:



Square matrix of order three has the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Example of Matrix of order three:



3. Types of matrices

There are several types of matrices, but the most commonly used are

1) Row matrix: A matrix is said to be a row matrix if it has only one row.

Example: $(2 \ 4 \ 7)$

2) Column matrix or Vector matrix: A matrix is said to be a column matrix if it has only one column.

Example: $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$

3) Zero matrix or null matrix: A matrix is said to be zero matrix or null matrix if all its elements are zero

Example: $[0 \ 0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices. We denote zero

matrix by O .

4) Triangular Matrix: A matrix is said to be a triangular matrix if its elements located below or above the leading diagonal are zeros.

a) Upper Triangular Matrix

In an upper triangular matrix, the elements located below the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & -2 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

b) Lower Triangular Matrix

In a lower triangular matrix, the elements above the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 1 & 0 \\ 18 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 7 & 9 \end{pmatrix}$

5) Diagonal Matrix

In a diagonal matrix, all the elements above and below the leading diagonal are zeros.

Examples: 1) $\begin{pmatrix} 10 & 0 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

6) Scalar Matrix

A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.

Examples: 1) $\begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix}$ 2) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

7) Identity Matrix or Unity matrix

An identity matrix by multiplication of matrices (noted by **I**) is a diagonal matrix in which the leading diagonal elements are equal to 1.

1) Identity matrix of order two $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) Identity matrix of order three $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

8) Rectangular matrix: A matrix is said to be rectangular if the number of rows is not equal to the number of columns

Example: A 5×2 matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 & 5 \\ 2 & 0 & 6 & 7 & -\frac{1}{3} \end{bmatrix}$$

9) Singular matrix: a square matrix that is not invertible

10) Nilpotent matrix: A square matrix A satisfying $A^k = 0$ for some positive integer k

11) Invertible matrix: A square matrix having a multiplicative inverse, that is, a matrix B such that $AB = BA = I$

12) Idempotent matrix or projection matrix: A matrix that is equal to its square i.e. $A^2 = A \times A = A$

13) Involuntary matrix: A square matrix which is its own inverse i. e. $A = A^{-1}$ and $AA = I$

14) Transpose matrix of A: Transpose of matrix is a matrix obtained by changing rows to columns and columns to rows. (Transpose of A is denoted by A^t or A^T).

15) Orthogonal matrix: a matrix whose inverse is equal to its transpose $A^{-1} = A^t$

16) Equality of matrices

Two matrices are equal if the elements of the two matrices that occupy the same position are equal.

$$\text{If } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \text{ then } \begin{cases} a_{11} = b_{11} \\ a_{21} = b_{21} \\ a_{12} = b_{12} \\ a_{22} = b_{22} \end{cases}$$

$$\text{If } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \text{ then } \begin{cases} a_{11} = b_{11}, a_{12} = b_{12}, a_{13} = b_{13} \\ a_{21} = b_{21}, a_{22} = b_{22}, a_{23} = b_{23} \\ a_{31} = b_{31}, a_{32} = b_{32}, a_{33} = b_{33} \end{cases}$$

Example: $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not

equal matrices. Symbolically, if two matrices A and B are equal, we write $A = B$



Application activity 3.1

1. Find the dimension of each matrix

a) $\begin{pmatrix} 5 & 5 & 5 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix}$ b) $(6 \quad -9)$ c) (2) d) $\begin{pmatrix} 0 & 0 & 0 & 2 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 1 \\ 6 & 1 \end{pmatrix}$

2. Name the following matrices

a) $(a \quad b \quad c)$ b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ d) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ e) $\begin{pmatrix} a & b & c \\ 0 & b & d \\ 0 & 0 & e \end{pmatrix}$ f) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

f) $\begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & 0 & c \end{pmatrix}$ g) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

3. If $A = \begin{pmatrix} 3y+2 & 2 \\ 2x+1 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} y-3 & 2 \\ 5 & 6 \end{pmatrix}$ are equal. Find the value of x and y

3.2. Operations on matrices

3.2.1. Addition and subtraction of matrices

Activity 3.2.1



- In a survey of 900 people, the following information was obtained:
200 males thought federal defense spending was too high
150 males thought federal defense spending was too low
45 males had no opinion
315 females thought federal defense spending was too high
125 females thought federal defense spending was too low
65 females had no opinion

Discuss and arrange these data in a rectangular array as follows:

	Too high	Too Low	No opinion
Male			
Female			

Then, form a matrix from the data of this table.

- Consider the matrix A formed by present students in two classes where

students of one class make one row such that $A = \begin{bmatrix} 23 & 2 \\ 20 & 4 \end{bmatrix}$ and B the matrix

formed by students who got absent for they went to participate in a competition

$B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Form one matrix C representing the total number of students.

Content summary

Given two matrices of the same dimension, $A = (a_{ij})$ and $B = (b_{ij})$, the matrix sum is defined as: $A + B = (a_{ij} + b_{ij})$ The sum of two matrices is a matrix

obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order; it means that the addition of two matrices requires both matrices to have the same number of rows and the same number of column.

Again, notice that the subtraction of two matrices requires both matrices to have the same number of rows and the same number of columns.

$$\text{i) If } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \text{ then}$$

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \text{ And}$$

$$A - B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}_W$$

$$\text{ii) If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \text{ then}$$

$$A + B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \\ a_{31} - b_{31} & a_{32} - b_{32} & a_{33} - b_{33} \end{pmatrix}.$$

Example: 1) If $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$, find the sum $A + B$ and the

difference $A - B$

Solution

$$A + B = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 3 & 7 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -5 & -5 \end{pmatrix}$$

2) Consider the matrices $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, find $A+B$ and $A-B$

Solution

$$A+B = \begin{pmatrix} 2+1 & 0+0 & 1+1 \\ 3+1 & 0+2 & 0+1 \\ 5+1 & 1+1 & 1+0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 4 & 2 & 1 \\ 6 & 2 & 1 \end{pmatrix}$$

$$A-B = \begin{pmatrix} 2-1 & 0-0 & 1-1 \\ 3-1 & 0-2 & 0-1 \\ 5-1 & 1-1 & 1-0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & -1 \\ 4 & 0 & 1 \end{pmatrix}$$

Properties

Note: The properties are the same for matrices of order two or three

1. Closure

The sum of two matrices of order two or three is another matrix of order two or three

2. Associative

$$A + (B + C) = (A + B) + C$$

3. Additive identity

$A + 0 = A$, where 0 is the zero-matrix of the same dimension.

4. Additive inverse

$$A + (-A) = O$$

The opposite matrix of A is $-A$.

5. Commutative

$$A + B = B + A$$

6. Scalar multiplication

Given a matrix, $A = (a_{ij})$, and a real number, $k \in \mathbb{R}$, the product of a real number by a matrix is a matrix of the same dimension as \mathbf{A} , and each element is multiplied by \mathbf{k} .

$$k \cdot A = (k a_{ij})$$

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ then } kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$$

Example 1: If $A = \begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix}$, find $2A$

Solution

$$2A = 2 \begin{pmatrix} -3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & 12 \\ 10 & 4 \end{pmatrix}$$

Example 2: Consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$, find $2A$

$$\text{Solution: } 2A = 2 \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 2 \\ 6 & 0 & 0 \\ 10 & 2 & 2 \end{pmatrix}$$

Properties:

1. $\alpha(\beta A) = (\alpha\beta)A$, $A \in M_{m \times n}$, $\alpha, \beta \in \mathbb{R}$
2. $\alpha(A+B) = \alpha A + \alpha B$, $A, B \in M_{m \times n}$, $\alpha \in \mathbb{R}$
3. $(\alpha + \beta)A = \alpha A + \beta A$, $A \in M_{m \times n}$, $\alpha, \beta \in \mathbb{R}$
4. $1A = A$, $A \in M_{m \times n}$



Application activity 3.2.1

1. Consider the matrices $A = \begin{pmatrix} 13 & 4 \\ 6 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix}$, find

a) $A + 3B$ b) $2A - B$

2. If $A = \begin{pmatrix} 1 & -2 & 6 \\ 2 & 2 & -5 \\ 0 & 4 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 8 & 9 \\ 14 & 4 & 6 \\ 0 & 6 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 13 & -2 & 0 \\ 10 & 0 & 3 \\ 9 & 1 & -5 \end{pmatrix}$.
Evaluate

b) $A - B$

c) $A + B - 2C$

d) $2A - B + C$

3.2.2. Multiplication of matrices

Activity 3.2.2



1. A clothing store sells men's shirt for \$40, silk tie for \$20, and wool suit for \$400. Firstly, the store had sales consisting of 150 shirts, 120 ties, and 25 suits. Secondly, the store had sales consisting of 100 shirts, 200 ties, and 50 suits.

1.1 Determine the total revenue due to these sales.

1.2 a) Write down the selling price for items as row matrix

b) Write down the first sales as column matrix

c) Multiply the matrix in (a) with the matrix in (b) in the following order:

$$(a_{11} \ a_{12} \ a_{13}) \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} \text{ to get } a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

d) Find the total revenue due to the sales firstly.

e) Use the same process to find the total revenue due to the sales secondary

f) Compare the solution in (1.2) with the solution in (1.1)

2. Considering that

$$\begin{aligned} A \times B &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix} \end{aligned}$$

Evaluate $A \times B$, given that $A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 2 & 5 \\ 0 & 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 3 \\ -1 & 6 & 4 \end{pmatrix}$ find

Content summary

Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B .

$$A_{m \times n} \times B_{n \times p} = M_{m \times p}$$

The element, c_{ij} , of the product matrix is obtained by multiplying every element in row i of matrix A by each element of column j of matrix B and then adding them together. This multiplication is called ROCO (row, column).

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, then

$$\begin{aligned}
 A \times B &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\
 &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}
 \end{aligned}$$

Example 1

If $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, find the product $A \cdot B$

Solution

$$\begin{aligned}
 A \cdot B &= \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \cdot 2 + 3 \cdot 1 & 1 \cdot 0 + 3 \cdot 1 \\ 2 \cdot 2 + 5 \cdot 1 & 2 \cdot 0 + 5 \cdot 1 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & 3 \\ 9 & 5 \end{pmatrix}
 \end{aligned}$$

Example 2

Consider matrices $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, find $A \times B$

Solution

$$\begin{aligned}
 A \times B &= \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \times 1 + 0 \times 1 + 1 \times 1 & 2 \times 0 + 0 \times 2 + 1 \times 1 & 2 \times 1 + 0 \times 1 + 1 \times 0 \\ 3 \times 1 + 0 \times 1 + 0 \times 1 & 3 \times 0 + 0 \times 2 + 0 \times 1 & 3 \times 1 + 0 \times 1 + 0 \times 0 \\ 5 \times 1 + 1 \times 1 + 1 \times 1 & 5 \times 0 + 1 \times 2 + 1 \times 1 & 5 \times 1 + 1 \times 1 + 1 \times 0 \end{pmatrix}
 \end{aligned}$$

$$= \begin{pmatrix} 3 & 1 & 2 \\ 3 & 0 & 3 \\ 7 & 3 & 6 \end{pmatrix}$$



Application activity 3.2.2

If $A = \begin{pmatrix} 1 & -2 & 6 \\ 2 & 2 & -5 \\ 0 & 4 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 8 & 9 \\ 14 & 4 & 6 \\ 0 & 6 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 13 & -2 & 0 \\ 10 & 0 & 3 \\ 9 & 1 & -5 \end{pmatrix}$. Evaluate

1. $A \times B$
2. $A \times C$
3. $B \times C$

3.2.3. Properties of Multiplication of Matrices

Activity 3.2.3



Consider the matrices $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 1 \\ -2 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

Find

1. $A \times B$ and $B \times A$
2. $A \times (B \times C)$ and $(A \times B) \times C$
3. $A \times (B + C)$ and $A \times B + A \times C$
4. Comment on your results

Content summary

Let A, B, C be matrices of order two or three

1. Associative

$$A \times (B \times C) = (A \times B) \times C$$

2. Multiplicative Identity

$A \times I = A$, where I is the identity matrix with the same order as matrix A

3. Not Commutative

$$A \times B \neq B \times A$$

4. Distributive

$$A \times (B + C) = (A \times B) + (A \times C)$$

Example 1: Given the matrices

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \text{ And } B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Find

- The product $A \times B$
- The product $B \times A$
- Conclude about the commutativity of matrices

Solution

a.

$$\begin{aligned} A \times B &= \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \times 1 + 0 \times 1 + 1 \times 1 & 2 \times 0 + 0 \times 2 + 1 \times 1 & 2 \times 1 + 0 \times 1 + 1 \times 0 \\ 3 \times 1 + 0 \times 1 + 0 \times 1 & 3 \times 0 + 0 \times 2 + 0 \times 1 & 3 \times 1 + 0 \times 1 + 0 \times 0 \\ 5 \times 1 + 1 \times 1 + 1 \times 1 & 5 \times 0 + 1 \times 2 + 1 \times 1 & 5 \times 1 + 1 \times 1 + 1 \times 0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 & 2 \\ 3 & 0 & 3 \\ 7 & 3 & 6 \end{pmatrix} \end{aligned}$$

b.

$$B \times A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 2 + 0 \times 3 + 1 \times 5 & 1 \times 0 + 0 \times 0 + 1 \times 1 & 1 \times 1 + 0 \times 0 + 1 \times 1 \\ 1 \times 2 + 2 \times 3 + 1 \times 5 & 1 \times 0 + 2 \times 0 + 1 \times 1 & 1 \times 1 + 2 \times 0 + 1 \times 1 \\ 1 \times 2 + 1 \times 3 + 0 \times 5 & 1 \times 0 + 1 \times 0 + 0 \times 1 & 1 \times 1 + 1 \times 0 + 0 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 1 & 2 \\ 13 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

a. Since $B \times A \neq A \times B$, then commutativity of multiplication of matrices is not verified.

Example 2: Given matrices $A = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix}$. Find the

product AB . What is your observation?

Solution

$$\begin{aligned} AB &= \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1-2+1 & 2-4+2 & 3-6+3 \\ -3+4-1 & -6+8-2 & -9+12-3 \\ -2+2+0 & -4+4+0 & -6+6+0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Observation: If $AB = 0$, it does not necessarily follow that $A = 0$ or $B = 0$.

Example 3: Given matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix}$. Find the product

AB and BA . What is your observation?

Solution

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & -4 & 2 \end{pmatrix}$$

$$\Rightarrow AB = BA$$

Observation: The given matrices commute in multiplication.

Notice

- If $AB = 0$, it does not necessarily follow that $A = 0$ or $B = 0$.
- **Commuting matrices in multiplication:** In general, the multiplication of matrices is not commutative, i.e., $AB \neq BA$, but we can have the case where two matrices A and B satisfy $AB = BA$. In this case A and/ B are said to be **commuting**.

Trace of matrix

The sum of the entries on the leading diagonal of a square matrix, A , is known as the **trace** of that matrix, noted $tr(A)$.

Example

1. Trace of $\begin{pmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 5 & 7 & 2 \end{pmatrix} = 1 + 3 + 2 = 6$

2. Trace of $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = 1 + 1 = 2$

Properties of trace of matrix

1. $tr(A+B) = tr(A) + tr(B)$
2. $tr(\alpha A) = \alpha tr(A)$
3. $tr(A) = tr(A)^t$
4. $tr(AB) = tr(BA)$
5. $tr(ABC) = tr(BCA) = tr(CAB)$, cyclic property.
6. $tr(ABC) \neq tr(ACB)$, arbitrary permutations are not allowed.



Application activity 3.2.3

1. Consider the matrices $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ and

$$C = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & 0 \\ -1 & 1 & 0 \end{pmatrix} \text{ find}$$

- a. $A \times B$ and $B \times A$
- b. $A \times (B \times C)$ and $(A \times B) \times C$
- c. $A \times (B + C)$ and $A \times B + A \times C$
- d. $tr(AB)$

3.3. Transpose of Matrix

Activity 3.3



Consider the matrices $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 12 & 3 & -1 \\ 3 & -2 & 0 \\ -4 & -1 & 0 \end{pmatrix}$

1. Interchange the rows and columns of matrix A and B
2. Add two matrices obtained in 1
3. Add A and B
4. Interchange the rows and columns of matrix obtained in 3
5. What can you say about result in 2 and 4?
6. Interchange the rows and columns of matrix A twice. What can you conclude?

Content summary

Given matrix A , **the transpose of matrix A** , noted A^t , is another matrix where the elements in the columns and rows have interchanged. In other words, the rows become the columns and the columns become the rows.

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ then } A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Example $A = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 2 & 0 \\ 3 & 5 & 8 \end{pmatrix}$ $A^t = \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & 5 \\ 6 & 0 & 8 \end{pmatrix}$

Properties of transpose of matrices

Let A, B be matrices of order two or three

1. $(A^t)^t = A$
2. $(A + B)^t = A^t + B^t$
3. $(\alpha \times A)^t = \alpha \times A^t, \alpha \in \mathbb{R}$
4. $(A \times B)^t = B^t \times A^t$



Application activity 3.3

1. If $A = \begin{pmatrix} x+30 & 2 \\ 4z+x & 3y+4 \end{pmatrix}$ and $B = \begin{pmatrix} 6x+5 & 2 \\ 1 & 1 \end{pmatrix}$. If $A = B$, find the value of x, y

and z and hence find

- a. A
- b. A^t

2. Consider matrices $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 3 & 6 \\ 3 & -2 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 1 \\ -4 & 0 & 3 \\ 6 & 2 & 5 \end{pmatrix}$. Evaluate

- a. $(A+B)^t$
- b. $3A^t + B$

- c. Find the value of x in $M = \begin{pmatrix} 1 & 2 & x^2 \\ 4 & 1 & 0 \\ 1 & x+3 & 8 \end{pmatrix}$ if $M^t = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 4 & 0 & 8 \end{pmatrix}$

- d. Consider the matrices $A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 12 & 3 & -1 \\ 3 & -2 & 0 \\ -4 & -1 & 0 \end{pmatrix}$

Find $(A \times B)^t$ and $B^t \times A^t$

3.4 Determinants and inverse of matrices of order two and three

3.4.1. Determinant of order two or three

Activity 3.4.1



1. Given that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Determine: a) $\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$ b) $\begin{vmatrix} -2 & -4 \\ 3 & 6 \end{vmatrix}$ c) $\begin{vmatrix} 3 & 1 \\ 6 & 8 \end{vmatrix}$ d) $\begin{vmatrix} 12 & 3 \\ -2 & 9 \end{vmatrix}$

2. Evaluate the following operations by considering the direction of arrows (sum of the blue products minus sum of the red products)

a. $\begin{vmatrix} 1 & -4 & 2 \\ 3 & 6 & 1 \\ 5 & 0 & 1 \end{vmatrix}$

b. $\begin{vmatrix} 10 & 2 & 4 \\ -6 & 5 & 2 \\ 0 & 3 & 1 \end{vmatrix}$

Content summary

Consider two matrices, one of order two and another one of order three:

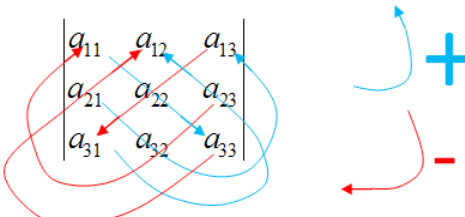
$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \text{ The determinant of } A \text{ is calculated by}$$

SARRUS rule:

The terms with a **positive sign** are formed by the elements of the **principal diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex**.

The terms with a **negative sign** are formed by the elements of the **secondary diagonal** and those of the **parallel diagonals** with its corresponding **opposite vertex**.

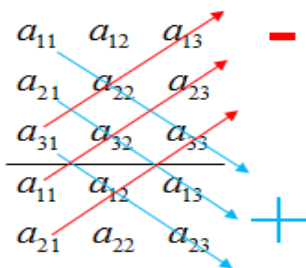
$$\det M = |M| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ or } |M| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$


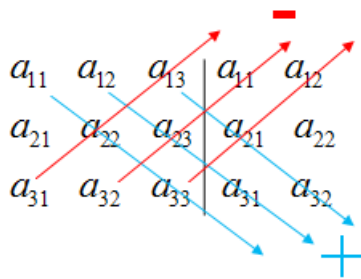
$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{21}a_{12}$$

Or we can work as follow:

To calculate the 3x3 determinant we rewrite the first two rows below the determinant (or first two columns to the right of the determinant).



$$\det = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$



Or

$$\det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

As multiplication of real numbers is commutative, the three are the same.

Remarks:

- For matrix A , $|A|$ is read as determinant of A and not modulus of A
- Only square matrices have determinants

Example 1

Given that $N = \begin{pmatrix} 12 & 6 \\ 5 & 4 \end{pmatrix}$, determine $\det N$.

Solution

$$\det N = |N| = \begin{vmatrix} 12 & 6 \\ 5 & 4 \end{vmatrix} = 12 \times 4 - 5 \times 6 = 48 - 30 = 18$$

Example 2

Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$$

Example 3

Determine $\det Q$ if $Q = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{pmatrix}$

Solution

$$\begin{aligned} \det Q &= \begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3 \times 2 \times 4 + 0 \times 1 \times 1 + (-2) \times (-5) \times 2 - 1 \times 2 \times (-2) - (-5) \times 1 \times 3 - 4 \times 0 \times 2 \\ &= 24 + 0 + 20 + 4 + 15 - 0 \\ &= 63 \end{aligned}$$

Determinant of $n \times n$ matrices by method of minors and cofactors

General method of finding the determinant of matrix with $n \times n$ dimension (2×2 , 3×3 , 4×4 , $5 \times 5, \dots$) is the use of cofactors.

Minor

An element, a_{ij} , to the value of the determinant of order $n-1$, obtained by deleting the row i and the column j in the matrix is called a **minor**.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & [5] & 4 \\ 3 & 6 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$$

Cofactor

For an n by n determinant A , the cofactor of entry a_{ij} , denoted by A_{ij} , is given

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the minor of entry a_{ij} .

The exponent of $(-1)^{i+j}$ is the sum of the row and column of the entry a_{ij} so if

$i+j$ is even, $(-1)^{i+j}$ will equal 1, and if $i+j$ is odd, will equal -1

To find the value of a determinant, multiply each entry in any row or column by its cofactor and sum the results.

This process is referred to as **expanding across a row or column**.

For example, the value of the 3 by 3 determinant; if we choose to expand down column 2, we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2}a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+2}a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2}a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Expand down column 2.

If we choose to expand across row 3, we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{3+1}a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2}a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + (-1)^{3+3}a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Expand across row 3.

It can be shown that the value of a determinant does not depend on the choice of the row or column used in the expansion. However, expanding across a row or column that has an entry equal to 0 reduces the amount of work needed to compute the value of the determinant.

Example

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -5 \\ -2 & 1 & 4 \end{vmatrix} = 3 \begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix} - 2 \begin{vmatrix} 0 & -5 \\ -2 & 4 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 3(8+5) - 2(0-10) + 1(0+4)$$

$$= 39 + 20 + 4$$

$$= 63$$

Note that we choose only one line (row or column).



Application activity 3.4.1

Find the determinants of the following matrices

1. $A = \begin{pmatrix} 1 & 3 & 1 \\ -4 & 5 & -2 \\ -3 & 1 & 3 \end{pmatrix}$

2. $B = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$

$$C = \begin{pmatrix} 1 & 4 & 2 \\ -2 & 0 & 1 \\ -1 & 3 & 0 \end{pmatrix}$$

3.4.2. Properties of the determinant

Activity 3.4.2



Consider the matrices $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 4 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$
and

$$D = \begin{pmatrix} 1 & 3 & -1 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix} \text{ find}$$

1. $|A|$ and $|B|$
2. $|C \cdot D|$ and $|C| \cdot |D|$. What can you conclude?
3. Find the product of the leading diagonal elements of the matrix C
4. Compare the solution in (3) with the determinant $|C|$. What can you conclude on the triangular matrix C ?

Content summary

1. Matrix A and its transpose A' have the same determinant.

$$|A'| = |A|$$

Example

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 7 \\ 2 & 1 & 6 \end{pmatrix}, \quad A' = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 2 & 1 \\ 0 & 7 & 6 \end{pmatrix}, \quad |A| = |A'| = -2$$

2. The determinant $|A| = 0$
 - a. If matrix A has two equal lines

Example

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

- b. All elements of a row or column are zero.

Example

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

- c. The elements of a line are a linear combination of the others. (say line means row or column)

Example

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 4 \\ 3 & 5 & 6 \end{vmatrix} = 0$$

$$\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_2$$

3. A triangular matrix determinant is the product of its leading diagonal elements.

Example

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 2 \times 6 = 24$$

4. If a determinant switches two parallel lines its determinant changes sign.

Example

$$|A| = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 5 & 6 \end{vmatrix}$$

5. If the elements of a line are added to the elements of another parallel line previously multiplied by a real number, the value of the determinant is unchanged.

Example

$$\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 16 \quad c_3 = 2c_1 + c_2 + c_3 \quad \begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 4 \\ 3 & 5 & 17 \end{vmatrix} = 16$$

6. If a determinant is multiplied by a real number, any line can be multiplied by the above-mentioned number, but only one.

Example

$$2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 \times 2 & 1 & 2 \\ 2 \times 1 & 2 & 0 \\ 2 \times 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 2 \\ 2 & 2 & 0 \\ 6 & 5 & 6 \end{vmatrix} = 32$$

$$2 \times \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 5 & 6 \end{vmatrix} = 2 \times 16 = 32$$

7. If all the elements of a line are formed by two addends, the above-mentioned determinant decomposes in the sum of two determinants.

Example

$$\begin{vmatrix} 2 & 1 & 2 \\ a+b & a+c & a+d \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ a & a & a \\ 3 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 2 \\ b & c & d \\ 3 & 5 & 6 \end{vmatrix}$$

8. The determinant of a product equals the product of the determinants.

$$|A \times B| = |A| \times |B|$$

Example

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 8 \\ 3 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 4 & 1 \\ 3 & 2 & 2 \end{pmatrix} \text{ then } A \times B = \begin{pmatrix} 6 & 9 & 5 \\ 30 & 30 & 22 \\ 18 & 11 & 11 \end{pmatrix}$$

$$|A \times B| = \begin{vmatrix} 6 & 9 & 5 \\ 30 & 30 & 22 \\ 18 & 11 & 11 \end{vmatrix} = 72$$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 8 \\ 3 & 2 & 3 \end{vmatrix} = 24, \quad |B| = \begin{vmatrix} 3 & -1 & 1 \\ 0 & 4 & 1 \\ 3 & 2 & 2 \end{vmatrix} = 3$$

$$|A| \times |B| = 24 \times 3 = 72$$



Application activity 3.4.2

Consider the following matrices $A = \begin{pmatrix} 12 & 0 & 1 \\ 34 & 0 & 2 \\ -3 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 2 & 5 \end{pmatrix}$,
 $C = \begin{pmatrix} 6 & 7 & 6 \\ 2 & 4 & 8 \\ 1 & 3 & 9 \end{pmatrix}$,

$$D = \begin{pmatrix} -3 & 5 & 1 \\ 2 & 10 & 1 \\ 1 & 8 & 1 \end{pmatrix} \text{ Find}$$

1. $|A|$, $|B|$, $|C|$ and $|D|$
2. $|BC|$

3.4.3 Inverse of matrices of order two or three

Activity 3.4.3



Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$

1. Calculate the determinant of A , i.e. $|A|$.
2. Replace every element in matrix A by its cofactor (the quantity obtained from a determinant of a square matrix by removal of the row and column containing a specified element) to find a new matrix called cofactor matrix. (Be careful on the signs of the cofactors).
3. Find the transpose of the cofactor matrix.
4. Multiply the inverse value of determinant obtained in 1 by the matrix obtained in 3
5. Multiply matrix A by matrix obtained in 4. Discuss your result.

Content summary

Calculating matrix inverse of matrix, A , is to find matrix A^{-1} such that,

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Where I is identity matrix

The inverse of matrix A is equal to the inverse value of its determinant multiplied by the **adjoint or adjugate matrix**.

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$$

Where $\text{adj}(A)$ is the **adjoint** (also called **adjugate**) matrix which is the transpose of the cofactor matrix. The cofactor matrix is found by replacing every element in matrix A by its cofactor.

Consider the matrix $B = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$; if we delete the row and column that

contains a_1 and find the determinant of the 2 by 2 matrix we are left with, we obtain the minor of a_1

Thus, the minor of $a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the minor of $b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ etc

The cofactor of each element is obtained by multiplying the minor by ± 1 according

+ - +

to the pattern - + - i.e. the cofactor is the minor with an appropriate sign

+ - +

attached to it.

Thus, cofactors of a_1, b_1 and c_1 are $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Example

Find the inverse of the following matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{pmatrix}$$

Solution

We find its inverse as follow: $|A| = 3$

Cofactor of each element:

$$c(2) = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0 \quad c(0) = -\begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = -3 \quad c(1) = \begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = 3$$

$$c(3) = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 \quad c(0) = \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} = -3 \quad c(0) = -\begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix} = -2$$

$$c(5) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \quad c(1) = -\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = 3 \quad c(1) = \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

The cofactor matrix is

$$\begin{pmatrix} 0 & -3 & 3 \\ 1 & -3 & -2 \\ 0 & 3 & 0 \end{pmatrix}, \text{ and then } \text{adj}(A) = \begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 3 \\ 3 & -2 & 0 \end{pmatrix}$$

Therefore, the matrix inverse of A is $A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{3} \begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 3 \\ 3 & -2 & 0 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ -1 & -1 & 1 \\ 1 & \frac{-2}{3} & 0 \end{pmatrix}$$

NB: Let A be a matrix of order two, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the inverse of A is given by

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Find the inverse of $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Solution

$$\det A = 1 - 0 = 1$$

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Notice: If $\det A = 0$ (i.e. the determinant is zero), the matrix has no inverse and is said to be a **singular** matrix.

Example

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$

Solution

$$\det A = 6 - 6 = 0$$

Since the determinant is zero, the given matrix has no inverse.



Application activity 3.4.3

Find the inverse of the following matrices

$$1. \quad A = \begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 2 & 5 \end{pmatrix}$$

$$2. \quad B = \begin{pmatrix} 11 & -8 & 1 \\ 0 & -6 & 2 \\ 3 & 2 & 7 \end{pmatrix}$$

$$3. \quad C = \begin{pmatrix} 6 & 7 & 6 \\ 2 & 4 & 8 \\ 1 & 3 & 9 \end{pmatrix}$$

$$4. \quad D = \begin{pmatrix} -3 & 5 & 1 \\ 2 & 10 & 1 \\ 1 & 8 & 1 \end{pmatrix}$$

3.4.4. Properties of the Inverse Matrix

Activity 3.4.4



Consider the matrices $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ find

1. $(AB)^{-1}$ and $B^{-1}A^{-1}$
2. $(A^{-1})^{-1}$
3. $(4A)^{-1}$ and $\frac{1}{4}A^{-1}$
4. $(A^t)^{-1}$ and $(A^{-1})^t$

What can you conclude for each result?

Content summary

For two invertible matrices A and B

1. $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
2. $(A^{-1})^{-1} = A$

$$3. (\alpha \cdot A)^{-1} = \alpha^{-1} \cdot A^{-1} \text{ with } \alpha \text{ is}$$

$$4. (A^t)^{-1} = (A^{-1})^t$$

Example 1

Consider matrix $A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 3 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, find

a. A^{-1} and B^{-1}

b. $(AB)^{-1}$

c. $(3B)^{-1}$

d. $(B^t)^{-1}$

Solution

a. $|A|=3$, $Adj(A) = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 6 & 1 \\ -3 & 3 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ -1 & 2 & \frac{1}{3} \\ -1 & 1 & \frac{1}{3} \end{pmatrix}$

$|B|=-2$, $Adj(B) = \begin{pmatrix} -2 & -1 & 0 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ -2 & -\frac{1}{2} & 1 \end{pmatrix}$

$$\text{b. } (AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ -2 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ -1 & 2 & \frac{1}{3} \\ -1 & 1 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\text{c. } (3B)^{-1} = \frac{1}{3}B^{-1} = \frac{1}{3} \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ -2 & -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$\text{d. } (B^t)^{-1} = (B^{-1})^t = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ -2 & -\frac{1}{2} & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$



Application activity 3.4.4

1. If $A = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$, find

a. A^{-1}

b. B^{-1}

c. $(AB)^{-1}$

d. $(A^t)^{-1}$

e. $(4B)^{-1}$

2. Consider of the following matrices $A = \begin{pmatrix} 3 & 7 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 & 1 \\ 3 & 1 & 6 \\ -1 & 1 & 1 \end{pmatrix}$

Find:

- a. A^{-1} and B^{-1}
- b. $(A^{-1})^{-1}$
- c. $(10A)^{-1}$
- d. $(A^t)^{-1}$

3.5 Applications of matrices and determinants

3.5.1. Solving System of linear equations using inverse matrix

Activity 3.5.1



1. A Farmer Kalisa bought in Ruhango Market 5 Cocks and 4 Rabbits and he paid 35,000Frw, on the following day, he bought in the same Market 3 Cocks and 6 Rabbits and he paid 30,000Frw.
 - a. Considering x as the cost for one cock and y the cost of one Rabbit, formulate equations that illustrate the activity of Kalisa;
 - b. Make a matrix A indicating the number of cocks and rabbits
 - c. If C is a matrix column made by the money paid by Kalisa, ie $C = \begin{pmatrix} 35,000 \\ 30,000 \end{pmatrix}$, write the equation $A \begin{pmatrix} x \\ y \end{pmatrix} = C$
 - d. Discuss and explain in your own words how you can determine $\begin{pmatrix} x \\ y \end{pmatrix}$ the cost of 1 Cocks and 1 Rabbit.

Content summary

Let us explore how to find the solution of the system of 3 linear equations in 3 unknowns

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases} \quad (1)$$

This equation can be written in the following:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ or } AX = B, \text{ where } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ It is clear that $X = A^{-1}B \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B$, provided that A^{-1} exists.

Notice

- If at least, one of b_i is different to zero the system is said to be **non-homogeneous** and if all b_i are zero the system is said to be **homogeneous**.
- The set of values of x, y, z that satisfy all the equations of system (1) is called **solution of the system**.
- For the homogeneous system, the solution $x = y = z = 0$ is **called trivial solution**. Other solutions are **non-trivial solutions**.

Non-homogeneous system cannot have a trivial solution as at least one of x, y, z is not zero. Example

- **Solve the system of equations**

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ 3x + 2y + z = 10 \end{cases}$$

Solution:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$$

We find the inverse of A.

A is invertible if its determinant is not zero.

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 1 + 4 - 3 - 3 + 2 - 2 = -1 \neq 0, \text{ then A has inverse.}$$

We have seen that the adjugate matrix and determinant of a matrix are used to find its inverse.

Let us use another useful method.

We have $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix}$, to find its inverse, suppose that its inverse is given by

$$A^{-1} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

We know that $AA^{-1} = I$, then

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} a+b+c=1 \\ 2a+b-c=0 \\ 3a+2b+c=0 \end{cases} \quad (1)$$

$$\begin{cases} d+e+f=0 \\ 2d+e-f=1 \\ 3d+2e+f=0 \end{cases} \quad (2)$$

$$\begin{cases} g+h+i=0 \\ 2g+h-i=0 \\ 3g+2h+i=1 \end{cases} \quad (3)$$

We solve these three systems to find value of $a, b, c, d, e, f, g, h,$ and i .

$$\begin{cases} a+b+c=1 \\ 2a+b-c=0 \\ 3a+2b+c=0 \end{cases} \quad (1) \Rightarrow \begin{cases} a=-3 \\ b=5 \\ c=-1 \end{cases}$$

$$\begin{cases} d+e+f=0 \\ 2d+e-f=1 \\ 3d+2e+f=0 \end{cases} \quad (2) \Rightarrow \begin{cases} d=-1 \\ e=2 \\ f=-1 \end{cases}$$

$$\begin{cases} g+h+i=0 \\ 2g+h-i=0 \\ 3g+2h+i=1 \end{cases} \quad (3) \Rightarrow \begin{cases} g=2 \\ h=-3 \\ i=1 \end{cases}$$

Then,

$$A^{-1} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Therefore, $S = \{(1, 2, 3)\}$



Application activity 3.5.1

Solve the following system using the inverse of a matrix

$$\begin{cases} 5x + 15y + 56z = 35 \\ -4x - 11y - 41z = -26 \\ -x - 3y - 11z = -7 \end{cases}$$

3.5.2. Solving System of linear equations using Cramer method

Activity 3.5.2



Consider the following system of 3 linear equations with 3 unknowns.

$$\begin{cases} x + 2y - 3z = 0 \\ 3x + 3y - z = 5 \\ x - 2y + 2z = 1 \end{cases}$$

- Rewrite this system in matrix form $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
- Calculate $\Delta = \det A$, determinant of coefficients of unknowns
- Calculate Δ_x , determinant of coefficients of unknowns where the column of x was replaced by column of independent terms $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
- Calculate Δ_y , determinant of coefficients of unknowns where the column of y was replaced by column of independent terms
- Calculate Δ_z , determinant of coefficients of unknowns where the column of z was replaced by the column of independent terms.
- Find the $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, $z = \frac{\Delta_z}{\Delta}$,
- Compare the obtained values to solutions of the given system, solved by using other methods seen in previous levels.

Content summary

Consider the system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

We use Cramer's rule as follows

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}$$

Notice:

- If $\Delta = 0$, $\Delta_x \neq 0$, $\Delta_y \neq 0$, $\Delta_z \neq 0$, then the system has no solution. In this case, the system is said to be inconsistent.
- If $\Delta = 0$, $\Delta_x = 0$, $\Delta_y = 0$, $\Delta_z = 0$, then the system has infinitely many solutions. In this case, the system is said to be consistent.
- If $\Delta \neq 0$, $\Delta_x \neq 0$, $\Delta_y \neq 0$, $\Delta_z \neq 0$, then the system has a unique solution. In this case, the system is said to be consistent.

Example:

Use Cramer's method to solve following the system

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ 3x + 2y + z = 10 \end{cases}$$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = -1$$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 1 & -1 \\ 10 & 2 & 1 \end{vmatrix} = -1$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 1 & -1 \\ 3 & 10 & 1 \end{vmatrix} = -2$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & 1 \\ 3 & 2 & 10 \end{vmatrix} = -3$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-1}{-1} = 1, \quad y = \frac{\Delta_y}{\Delta} = \frac{-2}{-1} = 2, \quad z = \frac{\Delta_z}{\Delta} = \frac{-3}{-1} = 3$$

Therefore, $S = \{(1, 2, 3)\}$



Application activity 3.5.2

Use matrix inverse method to solve the following systems

$$1. \begin{cases} 3x + y + z = 0 \\ 2x - y + 2z = 0 \\ 7x + y - 3z = 0 \end{cases}$$

$$2. \begin{cases} 4x + y - z = 1 \\ x - 3y + z = 2 \\ 5x - 2y = 4 \end{cases}$$

$$3. \begin{cases} x + y - z = 3 \\ 3x - y + z = 1 \\ -2x + y + z = 0 \end{cases}$$

3.5.3. Solving system of linear equations using Gaussian method (elimination of Gauss)

Activity 3.5.3



Consider the following system of 3 linear equations with 3 unknowns.

$$\begin{cases} x + 2y - 2z = 1 \\ 2x + y - 4z = -1 \\ 4x - 3y + z = 11 \end{cases}$$

equivalent to

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -4 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 11 \end{bmatrix}$$

From the given matrix, complete the augmented matrix of the form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 1 & -4 & -1 \\ 4 & -3 & 1 & 11 \end{array} \right] \quad \begin{matrix} r_1 \\ r_2 = 2r_1 - r_2 \\ r_3 = 4r_1 - r_3 \end{matrix} \sim \left[\begin{array}{ccc|c} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{array} \right]$$

Multiply the equivalent matrix obtained by the matrix of unknowns and determine the values of x, y and z (using the following form).

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Content summary

Resolution of systems of linear equation by Gauss's method

Let system: (1)
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

The system is equivalent to the system:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ b_n \end{pmatrix}$$

A simply $Ax = B$ where $A = (a_{ij})$ and called matrix of coefficients $B = (b_{ij})$ and called matrix of constants $X = (x_{ij})$ and called matrix of unknowns

$$\left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \cdot \\ a_{m1} & \dots & a_{mn} & b_n \end{array} \right)$$

From the matrix called **augmented matrix**

We manipulate the **elimination of Gauss**, for getting the equivalent matrix if we perform one of the following operations:

1. Exchange the rows L_i and L_j
2. Replace the row L_i by L_j
3. Replace the row L_i by $L_i + kL_j$

4. Replace the row L_i by $L_i + \sum_{j \neq i} kL_j$

Example:

$$\text{Solve by elimination method of Gauss } \begin{cases} x + y - z = 0 \\ x + 2y + 3z = 14 \\ 2x + y + 4z = 16 \end{cases}$$

$$\text{Augmented matrix: } \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 2 & 1 & 4 & 6 \end{array} \right) L_2 \sim L_2 - L_1$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 2 & 1 & 4 & 6 \end{array} \right) L_3 \sim L_3 - 2L_1$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 10 & 30 \end{array} \right);$$

from the last matrix we have $10z = 30 \Rightarrow z = 3$

$$\Rightarrow y + 4z = 14 \Rightarrow y + 12 = 14 \Rightarrow y = 2$$

$$\Rightarrow x + y - z = 0 \Rightarrow x + 2 - 3 = 0 \Rightarrow x = 1$$

$$S = \{(1, 2, 3)\}$$

2) Solve by using Gauss's method

$$1) \begin{cases} 3x + 2y + 4z = -1 \\ 2x - y + 2z = -2 \\ -x + y + 2z = 2 \end{cases}$$

$$2) \begin{cases} -x + 2y = 5 \\ 2x + 3y = 4 \\ 3x - 6y = -15 \end{cases}$$

Solutions:

$$1. \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 2 & -1 & 2 & -1 \\ -1 & 1 & 1 & 2 \end{array} \right) L_2 \sim 3L_2 - 2L_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -2 \\ -1 & 1 & 1 & 2 \end{array} \right) L_3 \sim 3L_3 - L_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 5 & 10 & 5 \end{array} \right) L_3 \sim 5L_2 + 7L_3$$

$$\Rightarrow \left(\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 0 & -7 & -2 & -4 \\ 0 & 0 & 60 & 15 \end{array} \right)$$

$$\Rightarrow 60z = 15 \Rightarrow z = \frac{1}{4}$$

$$3x + 2y + 4z = -1 \Rightarrow 3x + 1 + 1 = -1 \Rightarrow 3x = -3 \Rightarrow x = -1$$

$$\Rightarrow -7y - 2z = -4 \Rightarrow -7y - \frac{2}{4} = -4 \Rightarrow y = \frac{1}{2}$$

$$S = \left\{ -1; \frac{1}{2}; \frac{1}{4} \right\}$$

$$2. \begin{cases} -x + 2y = 5 \\ 2x + 3y = 4 \\ 3x - 6y = -15 \end{cases}$$

$$\Rightarrow [A/B] = \left[\begin{array}{cc|c} -1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & -6 & -15 \end{array} \right] L_2 \sim L_2 + 2L_1$$

$$\Rightarrow [A/B] = \left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 7 & 14 \\ 3 & -6 & -15 \end{array} \right] L_3 \sim L_3 + 3L_1$$

$$\Rightarrow \left[\begin{array}{c} A \\ B \end{array} \right] = \left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \end{array} \right] L_2 \sim \frac{1}{7}L_2 \Rightarrow [A/B] = \left[\begin{array}{cc|c} -1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore

$$y = 2 \Rightarrow -x + 4 = 5 \text{ and } x = -1$$

This gives the set of solution $S = \{(-1, 2)\}$.

Note: The method of **GAUSS** helps us to solve the above system where **CRAMER'S** method cannot.



Application activity 3.5.3

Solve by using elimination of Gauss

$$1. \quad \begin{cases} 2x - 2y = 5 \\ 4x + 3y = 1 \end{cases}$$

$$2. \quad \begin{cases} a - 3b = -5 \\ 3a - b + 2c = 7 \\ 5a - 2b + 4c = 10 \end{cases}$$



3.6 End of unit assessment

1. Find the values of x , y and z for $3\begin{pmatrix} x & y-1 \\ 4 & 3z \end{pmatrix} = \begin{pmatrix} 15 & 6 \\ 6z & 3x+y \end{pmatrix}$

2. If $A = \begin{pmatrix} 3 & -1 & 3 \\ 1 & 0 & -6 \\ 0 & -4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 10 & 2 & 3 \\ 1 & -4 & 6 \\ 0 & 6 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 11 & 12 & 0 \\ 1 & 0 & 8 \\ 2 & 0 & 7 \end{pmatrix}$. Evaluate

a. $A - B$

b. $A + B - 2C$

c. $2A - B + C$

d. $A \times B$

3. Find the inverse of $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

4. Use Cramer's method to solve $\begin{cases} x + y + z = 3 \\ 2x - y = 1 \\ 4x + y - z = 4 \end{cases}$

5. Use Gaussian method to solve $\begin{cases} -x + y - z = -4 \\ 3x + 10y + z = 10 \\ x - y - z = 2 \end{cases}$



UNIT 4

LOGARITHM AND EXPONENTIAL FUNCTION

Key Unit Competence:

Extend the use of concepts and definitions of functions to determine the domain and sketch the graphs of logarithmic and exponential functions.



4.0 Introductory activity

An economist created a business which helped him to make money in an interesting way so that the money he/she earns each day doubles what he/she earned the previous day. If he/she had 200USD on the first day and by taking t as the number of days, discuss the money he/she can have at the t^{th} day through answering the following questions:

- Draw the table showing the money this economist will have on each day starting from the first to the 10th day.
- Plot these data in rectangular coordinates
- Based on the results in a), establish the formula for the economist to find out the money he/she can earn on the n^{th} day. Therefore, if t is the time in days, express the money $F(t)$ for the economist.
- Now the economist wants to possess the money F under the same conditions, discuss how he/she can know the number of days necessary to get such money from the beginning of the business.

From the discussion, the function $F(t)$ found in c) and the function $Y(F)$ found in d) are respectively exponential function and logarithmic functions that are needed to be developed for solving related problems. In this unit, we are going to study the behaviour and properties of such essential functions and their application in real life situation.

4.1. Logarithmic functions

4.1.1. Domain and range of logarithmic functions

Activity 4.1



Consider the following real numbers:

50, 100, $1/2$, 0.7, 0.8, -30 , -20 , -5 , 0.9, 10, 20, 40.

- Draw and complete the table of values for $\log_{10}(x)$
- Discuss the value of $\log_{10}(x)$ for $x < 0$
- Discuss the values of $\log_{10}(x)$ for $0 < x < 1$, $x = 1$ and $x > 1$.
- Using the findings in a) plot the graph of $\log_{10}(x)$ for $x > 0$
- Explain in your own words what are the values of x for which $\log_{10}(x)$ is defined (the domain) and what are output values (the range).

Content summary

Given the function $y = \log_a x$, it is proven that if $x > 0$ and a is a constant ($a > 0, a \neq 1$), then $\log_a x$ is a real number called the “logarithm of x to the base a ”

For a positive constant a ($a \neq 1$), we call logarithmic function, the function $f: \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \log_a x$. The domain of the logarithm function is the set of positive real numbers and the range is the line of all real numbers. This means that $\text{dom } f = \{x \in \mathbb{R} : x > 0\} =]0, +\infty[= \mathbb{R}_0^+$ and $\text{range } f = \mathbb{R} =]-\infty, +\infty[$.

The logarithmic function is neither even nor odd. If $u: \mathbb{R} \rightarrow \mathbb{R} : x \mapsto u(x)$ is any other function we can compose u and the logarithmic function as $y = \log_a(u(x))$ defined for x such that $u(x) \geq 0$.

In the expression $y = \log_a x$, y is referred to as **the logarithm**, a is **the base**, and x is **the argument**.

If the base is 10, it is not necessary to write the base, and we say **decimal logarithm or common logarithm or Brigg’s logarithm**. So, the notation will become $y = \log x$.

If the base is (where $e = 2.718281828$), we say "Neperean logarithm or natural logarithm".

The **natural logarithm** is usually written using the shorthand notation $y = \ln x$ instead of $y = \log_e x$.

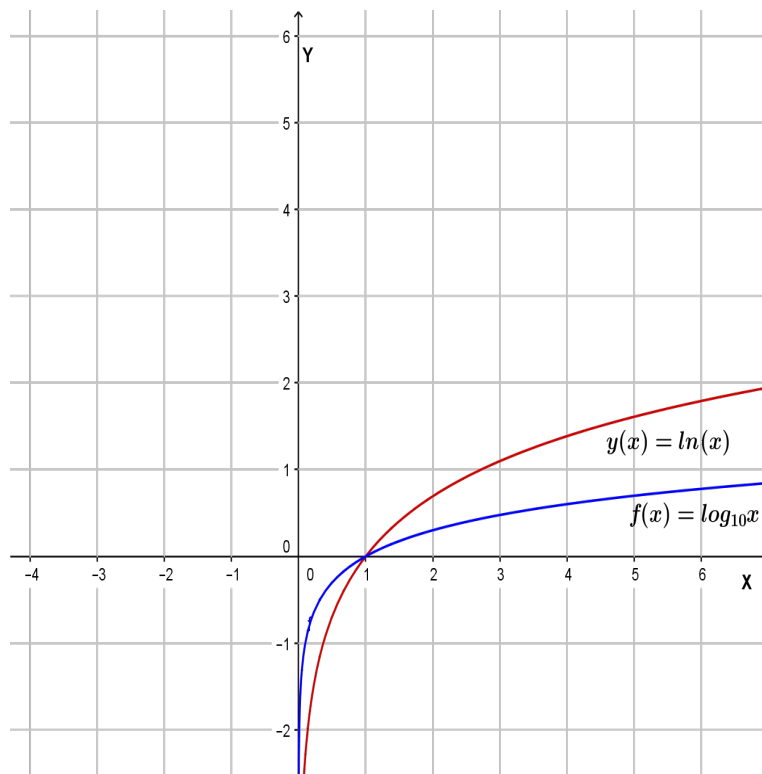
The natural logarithm " $\ln x$ " is defined on positive real numbers, $]0, +\infty[$ and its range is all real numbers.

Particularly,

- If $x = 1$, then $\ln x = \ln 1$. That is, $\ln x = 0$
- If $x > 1$, then $\ln x > \ln 1$ or $\ln x > 0$
- If $0 < x < 1$, then $\ln x < \ln 1$ or $\ln x < 0$

It means that: $\forall x \in]1, +\infty[$, $\ln x > 0$ and $\forall x \in]0, 1[$, $\ln x < 0$ as shown in the **graph 4**.

Figure 4.1: Graphs of Natural logarithmic functions $f(x) = \log_{10}(x)$ and $y(x) = \ln(x)$.



Note that:

- $y = \log_a x$ is equivalent to $y = \frac{\ln x}{\ln a}$, where $x > 0$ and $(a > 0, a \neq 1)$
- $\forall x \in \mathbb{R}_0^+, \log_a x = y \Leftrightarrow x = a^y$
- $y = \log_a (u(x))$ is equivalent to $y = \frac{\ln u(x)}{\ln a}$, where $u(x) > 0$ and $(a > 0, a \neq 1)$

Example 1

Find the domain and range for the function

- $f(x) = \log(x-4)$
- $g(x) = \ln(x+6)$

Solution

- To find the domain and the range of the function $y = \log(x-4)$, recalling that:

- **Domain:** Includes all values of x for which the function is defined
- **Range:** Includes all values y for which there is some such that $y = \log(x-4)$

Because $\log x$ defined only for positive values of x . So, in this problem $y = \log(x-4)$, is defined if and only if $x-4 > 0 \Leftrightarrow x > 4$ and gives that $x \in]4, +\infty[$.

The range of y is still all real number \mathbb{R} or $y \in]-\infty, +\infty[$.

$$\text{Dom } f = \{x \in \mathbb{R} : x - 4 > 0\} = \{x \in \mathbb{R} : x > 4\} =]4, +\infty[. \text{ Range } f = \mathbb{R}.$$

- The function $y = \ln(x+6)$, is defined if and only if $x+6 > 0 \Leftrightarrow x > -6$ and gives that $x \in]-6, +\infty[$ which is the domain. The range is \mathbb{R}

$$\text{Dom } g = \{x \in \mathbb{R} : x + 6 > 0\} = \{x \in \mathbb{R} : x > -6\} =]-6, +\infty[. \text{ Range } g = \mathbb{R}.$$

Example 2

Find the domain of definition of $f(x) = \ln(2x-3)$

Solution

Condition: $2x - 3 > 0$

$$2x - 3 > 0 \Leftrightarrow x > \frac{3}{2}$$

Thus, $Domf = \left] \frac{3}{2}, +\infty \right[$ and $Range f = \mathbb{R}$.

Example 3

Find the domain of definition of $f(x) = \ln(x+3)(x+2)$

Solution

Condition: $(x+3)(x+2) > 0$

$(x+3)(x+2) > 0$ if $x \in]-\infty, -3[\cup]-2, +\infty[$ (sign table can be used)

Thus, $Domf =]-\infty, -3[\cup]-2, +\infty[$ and $Range f = \mathbb{R}$

Example 4

Find the domain of $f(x) = \log_3(1-x) + \log_2 x$

Solution

Conditions: $1-x > 0$ and $x > 0$

$$1-x > 0 \Rightarrow x < 1$$

Domain is the intersection of $x < 1$ and $x > 0$

Thus, $Domf =]0, 1[$

Example 5

Find the domain of $f(x) = \log_2 \left(\frac{\sqrt{x}}{x+1} \right)$

Solution

Conditions: $x > 0$, $x + 1 \neq 0$ and $\frac{\sqrt{x}}{x+1} > 0$

$$x + 1 \neq 0 \Rightarrow x \neq -1$$

x	$-\infty$	-1		0	$+\infty$
\sqrt{x}				0	$+$
$x+1$	$-$	0	$+$	$+$	$+$
$\frac{\sqrt{x}}{x+1}$				0	$+$

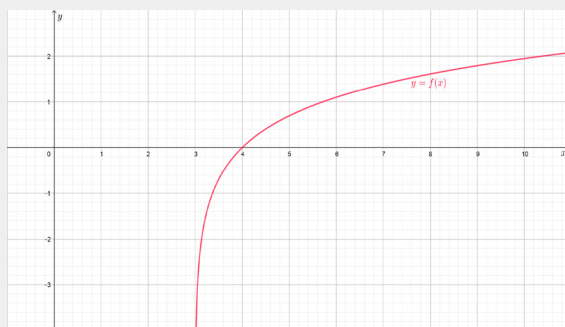
Thus, $Domf =]0, +\infty[$



Application activity 4.1.1

- State the domain and range of the following functions
 - $y = \log_3(x - 2) + 4$
 - $y = \log_5(8 - 2x)$
- Observe the following graph of a given logarithmic function, then state its domain and range.

Justify your answers.



b. $\ln 1 = 0$

c. $\ln(xy) = \ln x + \ln y$

d. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

e. $\ln\left(\frac{1}{y}\right) = -\ln y$

f. $\ln(x)^n = n \ln x$

g. $\ln \sqrt[n]{x} = \frac{1}{n} \ln x$

h. $\ln \sqrt[n]{x^m} = \frac{m}{n} \ln x$

Generally

$\forall x, y \in]0, +\infty[, a \in]0, +\infty[\setminus \{1\} :$

a. $\log_a xy = \log_a x + \log_a y$

b. $\log_a \frac{1}{y} = -\log_a y$

c. $\log_a \frac{x}{y} = \log_a x - \log_a y$

d. $\log_a x^r = r \log_a x$

e. $\log 10^n = n \log 10 = n$

f. $\log_a \sqrt[m]{x^n} = \log_a x^{\frac{n}{m}} = \frac{n}{m} \log_a x$

g. $\log_{a^m} x = \frac{1}{m} \log_a x$

$$h. \log_{a^m} x^n = \frac{n}{m} \log_a x$$

Notice

By changing logarithm from one base to another

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Recall that $\ln a < 0$ for $]0, 1[$ and $\ln a > 0$ for $]1, +\infty[$. Thus,

$f(x) = \log_a x$ is increasing in $]1, +\infty[$ and $f(x) = \log_a x$ is decreasing in $]0, 1[$

- If $x \in]0, 1[$, $\log_a x < \log_a y \Leftrightarrow x > y$
- If $x \in]1, +\infty[$, $\log_a x < \log_a y \Leftrightarrow x < y$
- $\log_a x = \log_a y \Leftrightarrow x = y$

Example 1

Given that $\ln 2 = 0.693$; $\ln 3 = 1.098$ and $\ln 5 = 1.609$. Calculate:

- a) $\ln 6$
- b) $\ln 10$
- c) $\ln 0.045$

Solution

$$\begin{aligned} a) \ln 6 &= \ln 2 \times 3 = \ln 2 + \ln 3 = 0.693 + 1.098 = 1.791 \\ b) \ln 10 &= \ln 2 \times 5 = \ln 2 + \ln 5 = 0.693 + 1.609 = 2.302 \\ c) \ln 0.045 &= \ln 3^2 \times 5 \times 10^{-3} = 2 \ln 3 + \ln 5 - 3 \ln 10 \\ &= 2 \ln 3 + \ln 5 - 3(\ln 2 + \ln 5) = 2(1.098) + 1.609 - 3(2.302) \\ &= 3.805 - 6.906 = -3.101 \end{aligned}$$

Example 2

Given that $y = \log_b x$, express y in function of \log_a

Solution

$$y = \log_b x = \frac{\ln x}{\ln b}$$

$$= \frac{\ln x}{\ln b} \cdot \frac{\ln a}{\ln a}$$

$$= \frac{\ln x}{\ln a} \cdot \frac{\ln a}{\ln b}$$

$$= \frac{\ln x}{\ln a} \cdot \frac{1}{\frac{\ln b}{\ln a}}$$

$$= (\log_a x) \frac{1}{\log_a b}$$

$$= \frac{\log_a x}{\log_a b}$$

Thus, $\log_b x = \frac{\log_a x}{\log_a b}$ (This relation is used to change logarithm from one base to another)

Example 3

Change $\log_4(x+1)$ to base 2

Solution

$$\log_4(x+1) = \frac{\log_2(x+1)}{\log_2 4}$$

$$= \frac{\log_2(x+1)}{\log_2 2^2}$$

$$= \frac{\log_2(x+1)}{2\log_2 2}$$

$$= \frac{1}{2}\log_2(x+1)$$



Application activity 4.1.2

1. Given that $\ln 2 = 0.693$; $\ln 3 = 1.098$ and $\ln 5 = 1.609$. Calculate:

a) $\ln 9$ b) $\ln 0.3$ c) $\ln 15$ d) $\ln 3^5$ e) $\ln 20$

2. Given that $\log 2 = 0.30103$ and $\log 3 = 0.47712$. Calculate

a) $\log_2 10$ b) $\log_2 9$ c) $\log_6 2$ d) $\log_4 5$

3. Show that

i) $\ln \frac{a^3 c}{b} = 3 \ln a - \ln b + \ln c$ ii) $\ln \sqrt[5]{x^2 y^{-5} z^{10}} = \frac{2}{5} \ln x - \ln y + 2 \ln z$

4. Express each of the following as single logarithm

a) $\log_3 x + \log_3 y - \log_3 y^3$ b) $\log(x+1) - \log(x^2 - 1)$

c) $3 \ln x + 2 \ln y - 4 \ln z$ d) $2 \ln 3 - \frac{1}{2} \ln(x^2 + 1)$

e) $\frac{\log 5 - 4 \log 3 + 3 \log 9 + \log 2}{\log 4 - \log 2}$

4.1.3. Domain of validity and Solutions of logarithmic equations

Activity 4.1.3



1. Let three functions defined by:

$$f(x) = \ln x; \quad h(x) = \ln(x+2); \quad g(x) = \ln(x^2 - 5x + 6)$$

For which value(s) of x , each function is defined.

2. Use properties of logarithm and determine the value of x in the following expressions:

a) $\ln x = \ln 10$ b) $\log(100x) = 2 + \log 4$

Content summary

Logarithmic equation in a set of real number \mathbb{R} , is the equation containing the unknown within the logarithmic expression.

To solve logarithmic equations the steps below are proceeded:

- Set existence conditions for solution(s) of equation.
- Express logarithms in the same base
- Use logarithmic properties to obtain:

$\log_a u(x) = \log_a v(x) \Leftrightarrow u(x) = v(x)$; where $u(x)$ and $v(x)$ are the functions in x on another variable given.

- Always check the validity of your solution

Some logarithmic equations can be solved by changing the logarithmic equation to exponential form using the fact that $y = \log_a x \Leftrightarrow a^y = x$.

Noted

- $y = \ln x = \log_e x \Leftrightarrow e^y = x$ (as inverse)
- The equation $\ln x = 1$ has a unique solution; the rational number 2.71828182845904523536... and this number is represented by letter e .

Thus, $e = 2.71828182845904523536...$

Hence, if $\ln x = 1$ then $x = e$.

Example 1

Solve each equation

a) $\ln x - \ln 5 = 0$ b) $3 + 2 \ln x = 7$ c) $\ln 2x + \ln(x+2) = \ln 6$

Solution

a) $\ln x - \ln 5 = 0$

- *Condition of validity* : $x > 0$

Then; $\ln x = \ln 5 \Leftrightarrow x = 5$

- $S = \{5\}$

b) $3 + 4 \ln x = 7$

- *condition of validity* : $x > 0$

Then; $4 \ln x = 7 - 3 \Leftrightarrow \ln x = 1$
 $\Leftrightarrow \ln x = \ln e$
 $\Leftrightarrow x = e$

- $S = \{e\}$

c) $\ln 2x + \ln(x+2) = \ln 6$

- *Conditions of validity* : $2x > 0$ and $x+2 > 0$

$\Leftrightarrow x > 0$ and $\Leftrightarrow x > -2$

- *Domain of validity* : $x \in]-2, +\infty[\cap]0, +\infty[\Leftrightarrow x \in]0, +\infty[$

Then; $\ln 2x + \ln(x+2) = \ln 6 \Leftrightarrow \ln[2x(x+2)] = \ln 6$
 $\Leftrightarrow 2x(x+2) = 6 \quad \Leftrightarrow x^2 + 2x - 3 = 0$
 $\Leftrightarrow 2x^2 + 4x - 6 = 0 \quad \Leftrightarrow (x-1)(x+3) = 0$
 $\Leftrightarrow x = 1$ or $x = -3$

- As $x \in]0, +\infty[$; then $S = \{1\}$

Example 2

Solve each equation

a) $\log_3(x+1) = \log_3 2$ b) $\log_{x-2} 3 = 1$ c) $\log_2(x+14) + \log_2(x+2) = 6$

Solution

a) $\log_3(x+1) = \log_3 2$

Condition

$x+1 > 0 \Leftrightarrow x > -1$

Then, $\log_3(x+1) = \log_3 2 \Leftrightarrow x+1 = 2 \Leftrightarrow x = 1 \therefore S = \{1\}$

$$b) \log_{x-2} 3 = 1$$

Condition

$$x - 2 > 0 \Leftrightarrow x > 2 \text{ and } x - 2 \neq 1 \Leftrightarrow x \neq 3 \text{ this means } x \in]2, 3[\cup]3, +\infty[$$

$$\text{Then, } \log_{x-2} 3 = 1 \Leftrightarrow \log_{x-2} 3 = \log_{x-2} (x-2)$$

$$\Leftrightarrow 3 = x - 2$$

$$\Leftrightarrow x = 5$$

$$\therefore S = \{5\}$$

$$c) \log_2 (x+14) + \log_2 (x+2) = 6$$

Condition

$$x+14 > 0 \text{ and } x+2 > 0 \Leftrightarrow x > -14 \text{ and } x > -2$$

$$\Leftrightarrow x \in]-2, +\infty[$$

$$\log_2 (x+14) + \log_2 (x+2) = 6 \Leftrightarrow \log_2 (x+14)(x+2) = 6 \log_2 2$$

$$\Leftrightarrow \log_2 (x+14)(x+2) = \log_2 2^6$$

$$\Leftrightarrow (x+14)(x+2) = 64$$

$$\Leftrightarrow x^2 + 16x + 28 - 64 = 0$$

$$\Leftrightarrow x^2 + 16x - 36 = 0$$

$$\Leftrightarrow (x+18)(x-2) = 0$$

$$\Leftrightarrow x = 2 \text{ or } x = -18$$

Only the real number 2 verifies the condition.

$$\therefore S = \{2\}$$



Application activity 4.1.3

1. Solve each equation

a) $2 \ln 4x = 7$ b) $\ln(x^2 - 1) = \ln(4x - 1) - 2 \ln 2$

c) $\log_a(x+1) + \log_a(x+2) = \log_a 20$

2. Solve in the following equations in the set of real numbers:

a) $\begin{cases} 2 \ln x + 3 \ln y = -2 \\ 3 \ln x + 5 \ln y = -4 \end{cases}$

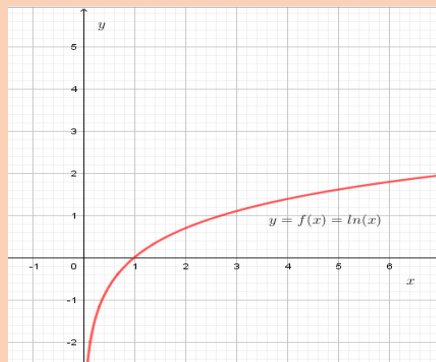
b) $\begin{cases} x - y = -8 \\ \log_2 x - \log_2 y = \log_2 \frac{3}{7} \end{cases}$

4.1.4 Limits of logarithmic functions

Activity 4.1.4



1. The graph below represents natural logarithmic function $f(x) = \ln x$



Considering the form of this graph and the logarithm of the following numbers in the table below,

x	0.5	0.001	0.001	0.0001	2	100	1001	10000
$\ln x$								

a. Discuss the values of $\ln x$ when x takes values closer to 0 from the right and deduce $\lim_{x \rightarrow 0^+} \ln x$.

b. Discuss the values of $\ln x$ when x take greater values and conclude about the $\lim_{x \rightarrow +\infty} \ln x$.

The limit of any logarithmic function can be determined in the same way as the limit of the natural function. If you feel more comfortable with the natural logarithmic function,

use the change base formula: $f(x) = \log_a u(x) = \frac{\ln(u(x))}{\ln a}$ provided $a > 0, a \neq 1$.

Example 1

Determine each of the following limit

a. $\lim_{x \rightarrow e} \ln x$ b. $\lim_{x \rightarrow 2} (1 - \ln x)$ c. $\lim_{x \rightarrow +\infty} \log_3 \left(\frac{x-4}{x+6} \right)$

Solution

a. $\lim_{x \rightarrow e} \ln x = 1$

b. $\lim_{x \rightarrow 2} (1 - \ln x) = 1 - \ln 2$

c. $\lim_{x \rightarrow +\infty} \log_3 \left(\frac{x-4}{2x+6} \right) = \log_3 \frac{1}{2} = -\log_3 2$

$$\text{Since } \lim_{x \rightarrow \infty} \frac{x-4}{2x+6} = \frac{1}{2}$$

Alternatively, using natural logarithmic function, we have

$$\lim_{x \rightarrow +\infty} \log_3 \left(\frac{x-4}{2x+6} \right) = \lim_{x \rightarrow +\infty} \frac{\ln \left(\frac{x-4}{2x+6} \right)}{\ln 3} = \frac{1}{\ln 3} \lim_{x \rightarrow +\infty} \ln \left(\frac{x-4}{2x+6} \right) = \frac{1}{\ln 3} \times \ln \frac{1}{2} = -\frac{\ln 2}{\ln 3} = -\log_3 2 = \log_3 \frac{1}{2}$$



Application activity 4.1.4

I. Evaluate the following limits

- 1) $\lim_{x \rightarrow +\infty} \ln(7x^3 - x^2 + 1)$
- 2) $\lim_{x \rightarrow 1^+} \left(\ln \frac{1}{x-1} \right)$
- 3) $\lim_{x \rightarrow 2^-} \log_5(x^2 - 5x + 6)$
- 4) $\lim_{a \rightarrow 4^+} \ln \frac{a}{\sqrt{a-4}}$
- 5) $\lim_{x \rightarrow +\infty} \ln(x^2 - 4x + 1)$
- 6) $\lim_{x \rightarrow +\infty} \frac{2 + 4 \log x}{x}$

4.1.5 Continuity and asymptotes of logarithmic functions

Activity 4.1.5



Let us consider the logarithmic function $f: \mathbb{R}^+ \rightarrow \mathbb{R}$, $y = f(x) = \log_2(x)$

1. Complete the following table:

$x = x_0$	$y = \log_2 x$	$\lim_{x \rightarrow x_0} \log_2 x$
$\frac{1}{4}$		
$\frac{1}{2}$		
1		
2		
4		

2. Can you conclude that $\lim_{x \rightarrow x_0} \log_2 x = \log_2(x_0)$? What about the continuity of $y = f(x) = \log_2(x)$?
3. By using the information drawn in the above table and the scientific calculator, plot the graph of $y = \log_2(x)$
4. Give any justification that allows you to decide on the continuity of the function.
5. Find equation of asymptotes of $f(x)$ if any.

The graph of the logarithmic function $f: \mathbb{R}^+ \rightarrow \mathbb{R}, f(x) = \log_a(x), a > 1$ has the following characteristics:

- The domain is $]0, +\infty[$ and $f(x)$ is continuous on this interval.
- The range is \mathbb{R}
- The graph intersects the x -axis at $(1, 0)$
- As $x \rightarrow 0^+, y \rightarrow -\infty$, so the line of equation $x = 0$ (the y -axis) is an asymptote to the curve
- As x increases, the graph rises more steeply for $x \in [0, 1]$ and is flatter for $x \in [1, +\infty[$
- The logarithmic function is increasing and takes its values (range) from negative infinity to positive infinity.

Example

Let us consider the logarithmic function $y = \log_2(x - 3)$

- a. What is the equation of the asymptote line?
- b. State the domain and range
- c. Find the x -intercept.
- d. Determine another point through which the graph passes
- e. Sketch the graph

Solution

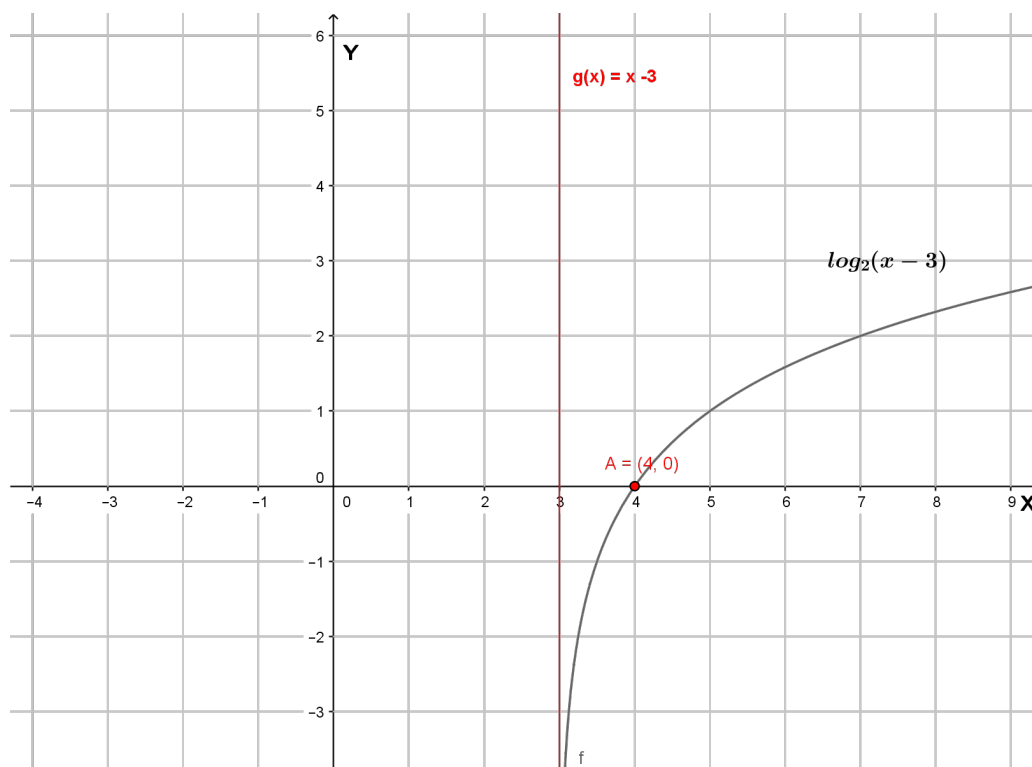
- a. The basic graph of $y = \log_2 x$ has been translated 3 units to the right, so the line $L \equiv x = 3$ is the vertical asymptote.
- b. The function $y = \log_2(x - 3)$ is defined for $x - 3 > 0$

So, the domain is $]3, +\infty[$. The range is \mathbb{R}
- c. The intercept is $(4, 0)$ since $\log_2(x - 3) = 0 \Leftrightarrow x = 4$
- d. Another point through which the graph passes can be found by allocating an arbitrary value to x in the domain then compute y .

For example, when $x = 4, y = \log_2(4 - 3) = \log_2 1 = 0$ which gives the point $(4, 0)$.

Note that the graph does not intercept y -axis because the value 0 for x does not belong to the domain of the function.

The graph of $y = f(x) = \log_2(x - 3)$



Application activity 4.1.5

1. Given the logarithmic function $y = -1 + \ln(x + 1)$,
 - i. find equation of asymptote lines (if any)?
 - ii. State the domain and range
 - iii. Find the x - intercept
 - iv. Find the y - intercept
 - v. Determine another point belonging to the graph
 - vi. Sketch the graph
2. Sketch the graph of the logarithmic function of base a with $0 < a < 1$. Precise the characteristics of the graph.

4.1.6. Differentiation of logarithmic functions

Activity 4.1.6



Let $f(x) = \ln x$

- Find $f(x+h)$ and $f(2+h)$
- Complete the following table

h	$\frac{\ln(2+h) - \ln 2}{h}$
-0.1	
-0.001	
-0.00001	
0.1	
0.001	
0.00001	

From the results found in the above table, approximate the value of $f'(2) = \lim_{h \rightarrow 0} \frac{\ln(2+h) - \ln 2}{h}$

And deduce the expression of $f'(x)$.

Based on your existing knowledge, provide any interpretation of the number $f'(2)$

From the results found in the above table approximate the value of $f'(2) = \lim_{h \rightarrow 0} \frac{\ln(2+h) - \ln 2}{h}$

And deduce the expression of $f'(x)$.

Based on your existing knowledge, provide any interpretation of the number $f'(2)$.

Content summary

The definition of derivative shows that if $y = \ln x$,

$$\begin{aligned}y' &= \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \rightarrow 0} \ln \left(\frac{x+h}{x} \right)^{\frac{1}{h}} \\ &= \lim_{h \rightarrow 0} \ln \left(1 + \frac{h}{x} \right)^{\frac{1}{h}} = \ln \lim_{h \rightarrow 0} \left(1 + \frac{h}{x} \right)^{\frac{1}{h}} = \ln e^{\frac{1}{x}} = \frac{1}{x}\end{aligned}$$

Then, the natural logarithmic function $y = \ln x$ is differentiable on $]0, +\infty[$ and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (x > 0).$$
 Using the formula of base change,

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \cdot \ln a}$$
 for any base a provided the conditions are fulfilled. In the

more general form, if $u(x)$ is any differentiable function such that $u(x) > 0$,

$$\frac{d}{dx}[\ln u(x)] = \frac{1}{u(x)} \times u'(x) = \frac{u'(x)}{u(x)} \text{ and}$$

$$\frac{d}{dx}[\log_a u(x)] = \frac{1}{u(x)} \times \frac{u'(x)}{\ln a} = \frac{u'(x)}{u(x) \cdot \ln a}$$

Example

1. Differentiate each of the following functions with respect to x
 - a) $f(x) = \ln(x^3 + 3x - 4)$
 - b) $f(x) = x^2 \ln x$
 - c) $y = \log_2(5x^3)$
2. Find the slope of the line tangent to the graph of $y = \log_2(3x+1)$ at $x = 1$
3. Find derivative of $\log_a(\ln|\sin x|)$

Solution

1. Differentiation

$$\text{a. } \frac{d}{dx} \ln(x^3 + 3x - 4) = \frac{1}{x^3 + 3x - 4} (x^3 + 3x - 4)' = \frac{3x^2 + 3}{x^3 + 3x - 4}$$

$$\text{b. } \frac{d}{dx} (x^2 \ln x) = \ln x \frac{d}{dx} x^2 + x^2 \frac{d}{dx} \ln x = 2x \ln x + x^2 \left(\frac{1}{x}\right) = 2x \ln x + x$$

$$\text{c. } \frac{d}{dx} (\sin x \ln x) = \ln x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} \ln x = (\cos x) \ln x + \frac{1}{x} \sin x = (\cos x) \ln x + \frac{\sin x}{x}$$

2. To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$

$$\frac{d}{dx} \log_2(3x + 1) = \frac{d}{dx} \left(\frac{\ln(3x + 1)}{\ln 2} \right) = \frac{1}{\ln 2} \frac{d}{dx} (\ln(3x + 1)) = \frac{3}{(3x + 1) \ln 2}$$

By evaluating the derivative at $x = 1$, we see that the tangent line to the curve at the point $(1, \log_2 4) = (1, 2)$ has the slope

$$\left. \frac{dy}{dx} \right|_{x=1} = \frac{3}{4 \ln 2} = \frac{3}{\ln 16}$$

3. Given that $|u|' = \frac{u}{|u|} \cdot u'$, where u is the function,

$$\left[\log_a (\ln |\sin x|) \right]' = \frac{(\ln |\sin x|)'}{(\ln |\sin x|) \ln a} = \frac{\cot x}{(\ln |\sin x|) \ln a}$$



Application activity 4.1.6

1. Differentiate $y = \ln \sqrt{\frac{1+x}{1-x}}$ with respect to x .

2. $h(x) = \log_{\frac{1}{2}} \sqrt{x^3 + 2x - 8}$

3. An airplane takes off from an airport at sea level. If its altitude (in kilometres) at time t (in minutes) is given by $h = 2000 \ln(t + 1)$, find the rate of climb at time $t = 3$ min.

4.1.7. Variation and Graphical representation of logarithmic functions

Activity 4.1.7



Given two functions $f(x) = \ln x$ and $g(x) = \log_{10} x$,

1. Compare $f(2)$ and $f(10)$, $g(2)$ and $g(10)$ to verify whether those functions are increasing or decreasing on $[2,10]$
2. Use the tables of signs for $f'(x)$ and $g'(x)$ to establish the intervals and the variation of those functions
3. Which function f or g is increasing or decreasing faster than another on $[2,10]$
4. Evaluate the second derivative $f''(x)$
5. Use the table of signs of $f''(x)$ and discuss the concavity of the function $f(x) = \ln x$

Content summary

The logarithmic function $f(x) = \log_a x$, $a > 0, a \neq 1$ varies in the following way:

a) For $x > 0$, if, then,

$f'(x) = \frac{1}{x \ln a}$. The sign of $f'(x)$ depends therefore on the value of the base a .

If $a > 1$, $\ln a > 0$ then $f'(x)$ is always positive. Thus $f(x) = \log_a x$ is strictly increasing on \mathbb{R}_0^+

Variation table for $y = f(x) = \log_a x$, $a > 1$

x	0	1	a	$+\infty$
y'		+	$\frac{1}{\ln a}$	+
y				

If $0 < a < 1$, $\ln a < 0$. Therefore $f'(x)$ is always negative.

Thus $f(x) = \log_a x$ is strictly decreasing on \mathbb{R}_0^+ . This implies the absence of extrema (maxima or minima) values.

Table of variation for $y = f(x) = \log_a x$, $0 < a < 1$

x	0	a	1	$+\infty$
y'		-	$\frac{1}{a \ln a}$	-
y				

Example

Discuss variations of the logarithmic function $f(x) = x - \ln x$

Solution

$f(x) = x - \ln x$ is defined for all $x > 0$

$$f'(x) = \frac{d}{dx}(x - \ln x) = 1 - \frac{1}{x}$$

$$f'(x) = 0 \Leftrightarrow 1 - \frac{1}{x} = 0$$

$$\Leftrightarrow \frac{1}{x} = 1$$

$$\Leftrightarrow x = 1$$

If $x = 1$, $y = f(1) = 1 - \ln 1 = 1$, thus $(1,1)$ is a point of the graph.

Variation table of $y = f(x) = x - \ln x$

From the table, one can observe that the function is **decreasing** for values when x lies in $]0,1[$ and **increasing** for x greater than 1. The point $(1,1)$ is **minimum** or equivalently the function takes the minimum value equal for $x = 1$. The minimum value that is equal to 1 is absolute.

There is **no inflection point**, and the graph **curves up/turns up**.

Examples 2:

Derivative can facilitate to remove Limits involving indeterminate forms

Examples

a. Evaluate $\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = 0$

Solution

$\lim_{x \rightarrow +\infty} \frac{\ln x}{x}$ takes indeterminate form $\frac{\infty}{\infty}$. Apply Hospital rule:

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = \lim_{x \rightarrow +\infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

b. Evaluate $\lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x}$

Solution

$$\lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} \text{ (indeterminate form } \frac{0}{0}\text{)}.$$

$$\lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0^+} \frac{1}{1+x} = 1$$

Example 3

For the following function, find relative asymptotes (if any), study the variation,

concavity and sketch the curve: $f(x) = \frac{1+2\ln x}{x}$

Solution

Asymptotes:

First, we need domain of definition:

Condition: $x > 0 \Rightarrow \text{Dom}f = \mathbb{R}_0^+$

$$\lim_{x \rightarrow 0^+} f(x) = \frac{-\infty}{0} = -\infty \Rightarrow x = 0 \text{ is a vertical asymptote}$$

$$\begin{aligned} \lim_{x \rightarrow +\infty} f(x) &= \lim_{x \rightarrow +\infty} \frac{1+2\ln x}{x} = \frac{\infty}{\infty} \quad \text{IF} \\ &= \lim_{x \rightarrow +\infty} \frac{2}{1} \quad \text{[H\^opital rule]} \\ &= 0 \end{aligned}$$

$\Rightarrow y = 0$ is horizontal asymptote.

Since there is horizontal asymptote for $x \rightarrow +\infty$, there is no oblique asymptote for $x \rightarrow +\infty$

Variation:

First derivative

$$\begin{aligned} f'(x) &= \frac{\frac{2}{x}x - (1+2\ln x)}{x^2} \quad \text{with } x > 0 \\ &= \frac{1-2\ln x}{x^2} \end{aligned}$$

Roots of first derivative

$$f'(x) = 0 \Leftrightarrow \frac{1-2\ln x}{x^2} = 0$$

$$\Rightarrow 1 - 2\ln x = 0$$

$$\Leftrightarrow 1 - \ln x^2 = 0$$

$$\Leftrightarrow \ln x^2 = 1$$

$$\Leftrightarrow x^2 = e$$

$$\Rightarrow x = \pm\sqrt{e}$$

As $x > 0$, $x = -\sqrt{e}$ is to be rejected.

The root of $f'(x)$ is $x = \sqrt{e}$

x	0	\sqrt{e}	$+\infty$
$f'(x)$	+		-
$f(x)$	$-\infty$	0	$+\infty$

For $x \in]0, \sqrt{e}[$, $f(x)$ increases while for $x \in]\sqrt{e}, +\infty[$, $f(x)$ decreases

Concavity

$$\begin{aligned} f''(x) &= \left[\frac{1-2\ln x}{x^2} \right]' \\ &= \frac{2x(1-2\ln x) - x^2 \left(-\frac{2}{x} \right)}{x^4} \\ &= \frac{2-4\ln x+2}{x^3} \\ &= \frac{4-4\ln x}{x^3} \end{aligned}$$

Roots of second derivative

$$f''(x) = 0 \Leftrightarrow \frac{4 - 4 \ln x}{x^3} = 0$$

$$\Rightarrow 4 - 4 \ln x = 0$$

$$\Leftrightarrow 4 - \ln x^4 = 0$$

$$\Leftrightarrow \ln x^4 = 4$$

$$\Leftrightarrow x^4 = e^4$$

$$\Rightarrow x = \pm e$$

As $x > 0$, $x = -e$ is to be rejected.

The root of $f''(x)$ is $x = e$

Curve:

Intersection with axes of coordinates:

Intersection with x -axis

$$f(x) = 0 \Leftrightarrow 1 + 2 \ln x = 0$$

$$\ln x = -\frac{1}{2} \Rightarrow e^{\ln x} = e^{-\frac{1}{2}} \Rightarrow x = e^{-\frac{1}{2}}$$

$$\text{Thus, } f(x) \cap \text{ox} = \left\{ \left(e^{-\frac{1}{2}}, 0 \right) \right\}$$

Intersection with y -axis

$$f(0) = \frac{1 + 2 \ln 0}{0} \text{ impossible}$$

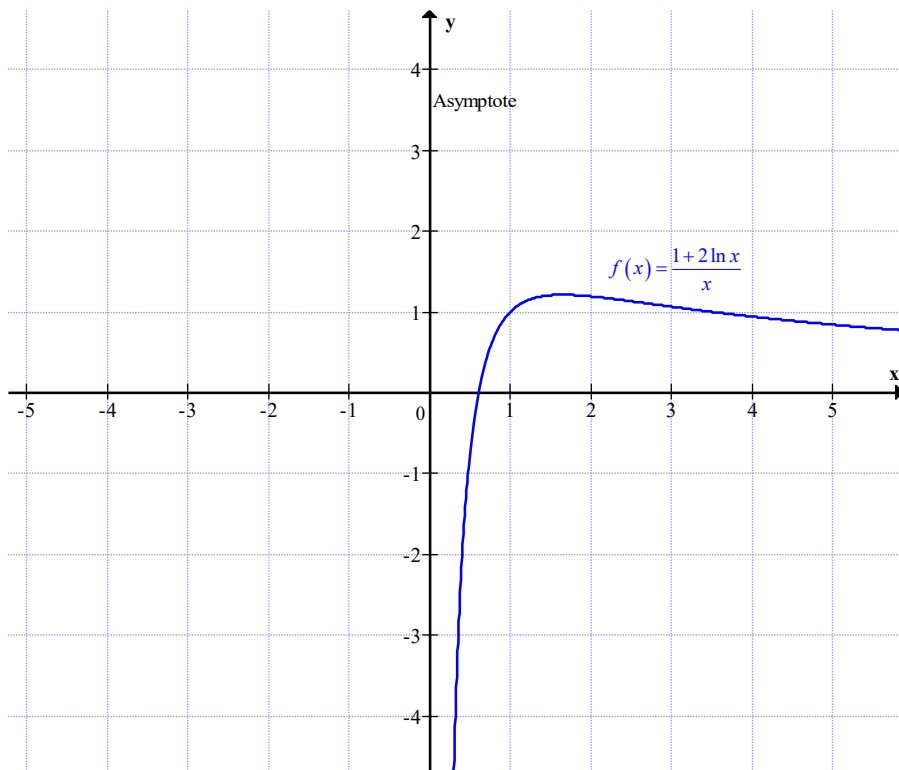
Thus, no intersection with y -axis

Additional points:

x	0.1	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.5	2.8
$f(x)$	-36.05	-2.08	0.40	1.0	1.17	1.21	1.20	1.17	1.13	1.09

x	3.1	3.4	3.7	4.0	4.3	4.6	4.9	5.2	5.5	5.8
$f(x)$	1.05	1.01	0.97	0.94	0.91	0.88	0.85	0.82	0.80	0.77

Graphical representation:



Application activity 4.1.7

- Discuss variations of the function $f(x) = \frac{\ln(x-2)}{x-2}$
- Suppose a satellite has been shot upward with an initial velocity of 20 m/s . Then the equation $h(t) = 100 \ln(t+1)$ gives the height of the satellite in m after t seconds
 - The derivative of the function for the height of the satellite gives the rate of change of the height or the velocity of the satellite, find the velocity function.
 - Find the velocity function after 2 seconds
 - Is the velocity increasing or decreasing?

4.2. Exponential function

4.2.1. Domain and range of exponential functions

Activity 4.2.1



Let $f(x) = \ln x$ and $g(x)$ denotes the inverse function of $f(x)$.

i. Complete the following table:

x	0	1	e	e^2		
$g(x) = f^{-1}(x)$					3	4

ii. Discuss and find out the set of all values of $g(x)$

2. Consider the function $h(x) = 3^x$ and complete the following table

x	-10	-1	0	1	10
$h(x) = 3^x$					

a. Discuss whether $\forall x \in \mathbb{R}, h(x) \in \mathbb{R}$ and deduce the domain of $h(x)$

b. Discuss whether $h(x)$ can be negative or not and deduce the range of $h(x)$.

Content summary

Remember that for $a > 0, a \neq 1$ the logarithmic function is

defined as $\log : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ or $x \rightarrow y = \log_a x$. The inverse of logarithmic function is called exponential function and defined as:

$\exp_a : \mathbb{R} \rightarrow \mathbb{R}_0^+ : x \mapsto y = \exp_a x$. For simplicity we write $\exp_a x = a^x$. Therefore $a^x = y$ if and only if $\log_a y = x$. Obviously, the domain of the exponential function $y = f(x) = a^x$ is $]-\infty + \infty[$ and its range is $]0, +\infty[$.

In the expression $a^x = y$, a is **the base**, x the **exponent** and y **the exponential of x in base a** .

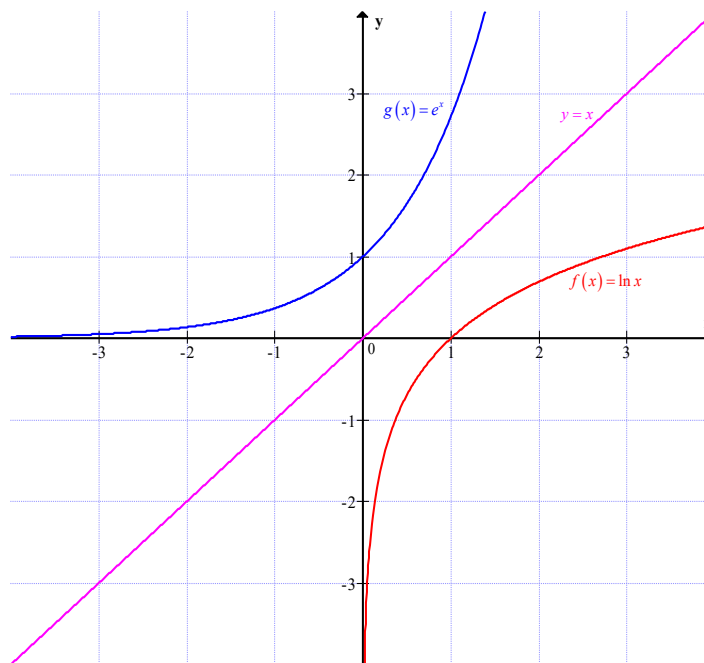
Similarly, to logarithmic function, if the base “ a ” is the number “ e ”, we have

exponential function $y = e^x$ as the inverse of natural logarithm $y = \ln x$.

The domain of definition of $y = e^x$ is $]-\infty, +\infty[$ and its range is $]0, +\infty[$.

Since e^x is the inverse of $\ln x$, the curve of $g(x) = e^x$ is the image of the curve of $f(x) = \ln x$ with respect to the first bisector, $y = x$. Then the coordinates of the points for $f(x) = \ln x$ are reversed to obtain the coordinates of the points for $g(x) = e^x$

The curve of $g(x) = e^x$ is as follows



Generally, if $u(x)$ is a defined function of x , the function $f(x) = a^{u(x)}$ has the range $]0, +\infty[$ and its domain is the domain of $u(x)$.

Example 1

Determine the domain and the range of the function

$$f(x) = 3^{\sqrt{2x}}$$

Solution

Condition for the existence of $\sqrt{2x}$ in \mathbb{R} : $x \geq 0$.

Thus, $Domf = [0, +\infty[$ and the range is $[1, +\infty[$

Example2

Find the domain and the range of $f(x) = 2^{\ln x}$

Solution

Condition: $x > 0$

Thus, $Domf =]0, +\infty[$ and the range is $]0, +\infty[$.

Example 3

Find the domain and the range of $f(x) = 3^{\frac{x+1}{x-2}}$

Solution

Condition: $x - 2 \neq 0 \Rightarrow x \neq 2$

Thus, $Domf = \mathbb{R} \setminus \{2\}$ and the range is $]0, 3[\cup]3, +\infty[$.

Example 4

Find the domain and the range of $f(x) = 4^{\sqrt{x^2-4}}$

Solution

Condition: $x^2 - 4 \geq 0 \Rightarrow x \in]-\infty, -2] \cup [2, +\infty[$

Thus, $Domf =]-\infty, -2] \cup [2, +\infty[$ and the range is $[1, +\infty[$

Example5

Determine the domain and the range of each of the following functions:

1. $g(x) = e^{\frac{x+2}{x-3}}$

2. $h(x) = e^{\sqrt{x^2-4}}$

Solution

1. Condition for the existence of $\frac{x+2}{x-3}$ in \mathbb{R} : $x \neq 3$.

Therefore $Dom\ g = \mathbb{R} \setminus \{3\} =]-\infty, 3[\cup]3, +\infty[$

and range is $]0, e[\cup]e, +\infty[$

2. Condition $x^2 - 4 \geq 0 \Rightarrow x \in]-\infty, -2] \cup [2, +\infty[$.

Thus, $Dom\ h =]-\infty, -2] \cup [2, +\infty[$ and range is $[1, +\infty[$.

Example 6

Find the domain and the range of $f(x) = e^{\sqrt{x}}$

Solution

Condition: $x \geq 0$

Thus, $Dom\ f = [0, +\infty[$ and range is $[1, +\infty[$.

Example 7

Find the domain and the range of $g(x) = e^{\frac{x+1}{x-2}}$

Solution

Condition: $x - 2 \neq 0 \Rightarrow x \neq 2$

Thus, $Dom\ g = \mathbb{R} \setminus \{2\}$ and range is $]0, e[\cup]e, +\infty[$.

Example 8

Find the domain of $h(x) = e^{\sqrt{x^2-1}}$

Solution

Condition: $x^2 - 1 \geq 0 \Rightarrow x \in]-\infty, -1] \cup [1, +\infty[$

Thus, $Domh =]-\infty, -1] \cup [1, +\infty[$ and range is $[1, +\infty[$.

Application activity 4.2.1



Application activity 4.2.1

Discuss and determine the domain and range of the following functions

1) $f(x) = 5e^{2x}$ 2) $h(x) = 2^{\ln x}$ 3) $f(x) = 3^{\frac{x+1}{x-2}}$

4.2.2. Properties and operations on exponential function

Activity 4.2.2



1. Evaluate each expression for $-3 \leq x < 3$

a) e^{3x}

b) e^{x+3}

c) e^{-x}

2. Use calculator to obtain approximations of

a) $2^{-0.6}$

b) $\pi^{0.75}$

c) $(1.56)^{\sqrt{2}}$

Content summary

When working with exponential functions, the properties of exponents, shown below, are useful.

Properties of Exponents

- Let a and b be positive numbers. Then

$$\begin{array}{lll} 1. a^0 = 1 & 5. \frac{a^x}{b^y} = a^{x-y} & 8. \sqrt[n]{a^x} = a^{\frac{x}{n}} \\ 2. a^x a^y = a^{x+y} & 6. (ab)^x = a^x b^x & 9. \left(\sqrt[n]{a^x}\right)^y = a^{\frac{xy}{n}} \\ 3. (a^x)^y = a^{xy} & 7. \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} & 10. \left(\sqrt[n]{a^m}\right)^{-x} = \frac{1}{a^{\frac{mx}{n}}} \\ 4. a^{-x} = \frac{1}{a^x} & & \end{array}$$

- $\forall x \in \mathbb{R}, \ln e^x = x$ and $\forall y \in]0, +\infty[, e^{\ln y} = y$

$$\begin{array}{lll} 1. e^0 = 1 & 4. (e^x)^n = e^{nx} & 7. e^{\ln x} = x \\ 2. e^1 = e & 5. \frac{1}{e^a} = e^{-a} & 8. e^{-\ln x} = \frac{1}{x} \\ 3. e^x e^y = e^{x+y} & 6. \frac{e^a}{e^b} = e^{a-b} & 9. \ln e^{-x} = -x \end{array}$$

Example 1

Evaluate the following expressions by using the properties of exponents

$$a) e^{-\ln x} \quad b) (e^2)^{\ln x} \quad c) e^{1-\ln x}$$

Solution

$$a) e^{-\ln x} = e^{\ln x^{-1}} = e^{\ln \frac{1}{x}} = \frac{1}{x}, \text{ due to } e^{\ln x} = x.$$

$$b) (e^2)^{\ln x} = e^{2 \ln x} = e^{\ln x^2} = x^2$$

$$c) e^{1-\ln x} = e^1 e^{-\ln x} = e^1 e^{\ln x^{-1}} = e^1 e^{\ln \frac{1}{x}} = e^1 \times \frac{1}{x} = \frac{e}{x}$$

Example 2

Simplify each expression using the properties of exponents.

$$a. (0.5)^0 \quad b. 2^x 2^4 \quad c. (3^4)^m$$

Solution

$$a.(0.5)^0 = 1 \quad b.2^x 2^4 = 2^{x+4} \quad c.(3^4)^m = 3^{4m}$$



Application activity 4.2.2

1. Use the properties of exponents to simplify the expression.

$$a) \left(e^{\frac{3}{2}} \right) \left(\frac{1}{e} \right)^{\frac{3}{2}} \quad b) \left[(e^{-1}) \left(e^{\frac{2}{3}} \right) \right]^3$$

2. If $x = -5, y = 3, z = \frac{1}{2}$, simplify each expression

$$a) 140625^z \quad c) (4^x 5^y)^z$$

$$b) 225 \times 15^x \quad d) (3^x 7^z)^y$$

4.2.3. Domain of validity and Solutions of exponential equations

Activity 4.2.3



1. For which value(s), each function $f(x)$ below can be defined. Explain.

$$a) f(x) = e^{(x+2)}$$

$$b) f(x) = e^{x^2-5x+6}$$

2. Detect the value of x , if $2^{1-x} = 6$

CONTENT SUMMARY

Equations that involves powers as terms of their expressions are referred to as **exponential equations**. Such equations can some times be solved by appropriately applying the properties of exponents as shown in the next examples.

Some exponential equations are solved by introducing logarithms within expression.

Example 1

Solve each equation

a) $e^x = 5$ b) $10 + e^{0.1t} = 14$

Solution

a) $e^x = 5$

- Domain of validity: $x \in \mathbb{R}$

$$\ln e^x = \ln 5$$

$$x \ln e = \ln 5; \text{ where } \ln e = 1$$

$$x = \ln 5$$

- $S = \{\ln 5\}$

b) $10 + e^{0.1t} = 14$

- Domain of validity: $t \in \mathbb{R}$

$$e^{0.1t} = 4$$

$$\ln e^{0.1t} = \ln 4$$

$$0.1t = \ln 4$$

$$t = 10 \ln 4$$

- $S = \{10 \ln 4\}$

Example 2

Solve $e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3}$

Solution

$$e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3}$$

- Domain of validity: $x \in \mathbb{R}$ $e^{-x^2} = e^{2x-3}$

$$e^{-x^2} = e^{2x} \cdot e^{-3} = e^{2x-3}$$

$$-x^2 = 2x - 3$$

$$x^2 + 2x - 3 = 0$$

The solution of this equation gives $x = -3$ or $x = 1$

- The solution set is $S = \{-3, 1\}$

Example 3

Solve each equation

a) $2^{x-4} = 8$

b) $5 + 3^{t-4} = 7$

c) $2^{2y} + 3(2^y) = 4$

d) $3(2^{4x}) - 7(2^{2x}) + 4 = 0$

Solution

$$\begin{aligned} a) 2^{x-4} &= 8 & b) 5 + 3^{t-4} &= 7 \\ \Rightarrow 2^{x-4} &= 2^3 & \Rightarrow 3^{t-4} &= 2 \\ \Rightarrow x-4 &= 3 & \Rightarrow \ln 3^{t-4} &= \ln 2 \\ \Rightarrow x &= 7 & \Rightarrow (t-4) \ln 3 &= \ln 2 \\ \therefore S &= \{7\} & \Rightarrow t &= 4 + \frac{\ln 2}{\ln 3} \end{aligned}$$

$$c) 2^{2^y} + 3(2^y) = 4 \Leftrightarrow (2^y)^2 + 3(2^y) = 4$$

Let $2^y = x$, then $x^2 + 3x = 4$

$$\Rightarrow x^2 + 3x - 4 = 0$$

$$\Rightarrow (x+4)(x-1) = 0$$

$$\Rightarrow x = -4 \text{ or } x = 1$$

Replacing the value of x in the equation $2^y = x$

For $x = -4$; $2^y = -4$ doesn't exist

$$2^y = 1$$

For $x = 1$; $\Rightarrow y = 0 \therefore S = \{0\}$

$$d) 3(2^{4x}) - 7(2^{2x}) + 4 = 0 \Leftrightarrow 3(2^{2x})^2 - 7(2^{2x}) + 4 = 0$$

Let $2^{2x} = k$, then $3k^2 - 7k + 4 = 0$

$$3k^2 - 7k + 4 = 0$$

$$\Delta = 1 \Rightarrow k = \frac{4}{3} \text{ or } k = 1$$

Replacing the value of k in the equation $2^{2x} = k$

For $k = \frac{4}{3}$:

$$2^{2x} = \frac{4}{3}$$

$$2x \ln 2 = \ln \frac{4}{3}$$

$$x = \frac{\ln 4 - \ln 3}{2 \ln 2}$$

For $k = 1$: $2^{2x} = 1 \quad \therefore S = \left\{ 0, \frac{\ln 4 - \ln 3}{2 \ln 2} \right\}$
 $\Rightarrow 2x \ln 2 = \ln 1$
 $\Rightarrow x = 0$



Application activity 4.2.3

1. Solve each equation for x or t .

a) $e^x = 6$ b) $5 + e^{0.2t} = 10$ c) $e^{2x} = 3e^x$ d) $e^{2x} = e^x + 12$ e) $e^t = 12 - 32e^{-t}$

2. Solve a) $2e^{-x+1} - 5 = 9$ b) $\frac{50}{1 + 12e^{-0.02x}} = 10.5$ c) $e^{\ln x^2} - 9 = 0$
 d) $e^x - 12 = \frac{-5}{e^{-x}}$

3. Solve in the following equations in the set of real numbers:

$$\frac{e^x + e^{-x}}{2} = 1, (\text{Hint: multiply by } e^x)$$

4. Find the value of marked letter in each equation.

a) $9^t + 3^t = 12$ c) $\frac{2^x}{4} - \frac{3^x}{9} = 0$ e) $4^x - 10 \cdot 2^x + 16 = 0$

b) $2^x + 2^{x-1} = \frac{3}{2}$ d) $\left(\frac{5}{2}\right)^x = 0.16$ f) $5^m \sqrt[m]{8^{m-1}} = 500$

5. Solve in \mathbb{R}

a) $(4)^{1-2x} - \left(\frac{1}{16}\right)^{\frac{x}{2}} = 0$ c) $\begin{cases} 5^{3x} = 25^{2y-2} \\ 9^y = 3^{x+1} \end{cases}$ e) $\begin{cases} 5^x = 3y \\ 3^x = 5y \end{cases}$

b) $\begin{cases} 16 \cdot 2^x = 4^{x+y} \\ 5 \cdot 25^{2x+y} = 5^{x+1} \end{cases}$ d) $\begin{cases} 3^{x+1} = 243 \\ 2^y = 64 \end{cases}$ f) $\begin{cases} 5^{3x-2y} = 1 \\ 11^{6x-7y} = 14641 \end{cases}$

4.2.4 Limits of exponential function

Activity 4.2.4



1. You are familiar with the graph of $f(x) = \ln x$. Explain in your words how you can obtain the graph of its inverse $y = f^{-1}(x) = e^x$.
2. From the graph deduce $\lim_{x \rightarrow -\infty} e^x$ and $\lim_{x \rightarrow +\infty} e^x$.
3. Discuss $\lim_{x \rightarrow -\infty} \left(\frac{1}{2}\right)^x$ and $\lim_{x \rightarrow +\infty} \left(\frac{1}{2}\right)^x$.
4. Generalize above results to $\lim_{x \rightarrow -\infty} a^x$ and $\lim_{x \rightarrow +\infty} a^x$.

Content summary

Drawing on results on logarithmic functions, it is clear that:

$$\lim_{x \rightarrow -\infty} e^x = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} e^x = +\infty$$

In general:

$$\text{If } a > 1, \quad \lim_{x \rightarrow -\infty} a^x = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} a^x = +\infty$$

$$\text{If } 0 < a < 1, \quad \lim_{x \rightarrow -\infty} a^x = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} a^x = 0$$

Example

1. Evaluate

$$\text{a) } \lim_{x \rightarrow \infty} e^{1-4x-5x^2} \quad \text{b) } \lim_{x \rightarrow 1^+} \left(\frac{3}{5}\right)^{\frac{1}{x-1}} \quad \text{c) } \lim_{x \rightarrow -\infty} 3^{\frac{1}{x}} \quad \text{d) } \lim_{x \rightarrow 1} 3^{\frac{1}{x-1}}$$

Solution

$$\text{a) } \lim_{x \rightarrow \infty} e^{1-4x-5x^2}$$

$$\text{We know that } \lim_{x \rightarrow \infty} (1-4x-5x^2) = -\infty$$

Therefore, as the exponent goes to minus infinity in the limit and so the exponential must go to zero in the limit using the ideas from the previous formula.

Hence, $\lim_{x \rightarrow \infty} e^{1-4x-5x^2} = 0$

b) The exponent goes to infinity in the limit and so the exponential will also need to go to zero in the limit since the base is less than 1.

Hence, $\lim_{x \rightarrow 1^+} \left(\frac{3}{5}\right)^{\frac{1}{x-1}} = 0$

c) $\lim_{x \rightarrow -\infty} 3^{\frac{1}{x}} = 3^{\lim_{x \rightarrow -\infty} \frac{1}{x}} = 3^0 = 1$

d) $\lim_{x \rightarrow 1} 3^{\frac{1}{x-1}} = 3^{\lim_{x \rightarrow 1} \frac{1}{x-1}} = 3^{\frac{1}{0}}$

Study one side limit:

x	$\frac{1}{3^{x-1}}$
0	0.33
0.2	0.25
0.4	0.16
0.6	0.06
0.8	0.004
0.9	0.00001

x	$3^{\frac{1}{x-1}}$
2	3
1.8	3.948
1.6	6.24
1.4	15.59
1.2	243
1.1	59049

$\lim_{x \rightarrow 1^-} 3^{\frac{1}{x-1}} = 0$ and $\lim_{x \rightarrow 1^-} 3^{\frac{1}{x-1}} = +\infty$

Hence, $\lim_{x \rightarrow 1} 3^{\frac{1}{x-1}}$ does not exist.

Alternatively:

Since $\lim_{x \rightarrow 1^+} \frac{1}{x-1} = +\infty$ and $\lim_{x \rightarrow 1^+} \frac{1}{x-1} = -\infty$, apply results on $\lim_{x \rightarrow \pm\infty} a^x$ for $a > 1$ to have:

$\lim_{x \rightarrow 1^+} 3^{\frac{1}{x-1}} = 0$ and $\lim_{x \rightarrow 1^+} 3^{\frac{1}{x-1}} = +\infty$

Hence, $\lim_{x \rightarrow 1} 3^{\frac{1}{x-1}}$ does not exist.

2. Consider $f(x) = \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$, evaluate each of the following:

$\lim_{x \rightarrow -\infty} f(x)$ and $\lim_{x \rightarrow +\infty} f(x)$.

Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim_{x \rightarrow -\infty} \frac{e^{-3x}(1 - 2e^{11x})}{e^{-3x}(9e^{11x} - 7)} \\ &= \lim_{x \rightarrow -\infty} \frac{1 - 2e^{11x}}{9e^{11x} - 7} = -\frac{1}{7}\end{aligned}$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}} = \lim_{x \rightarrow +\infty} \frac{e^{8x}(e^{-11x} - 2)}{e^{8x}(9 - 7e^{-11x})} = -\frac{2}{9}$$



Application activity 4.2.4

1. for each given function, evaluate limit at $+\infty$ and $-\infty$

- $f(x) = e^{8+2x-x^3}$
- $f(x) = e^{\frac{6x^2+x}{5+3x}}$
- $f(x) = 2e^{6x} - e^{-7x} - 10e^{4x}$
- $f(x) = 3e^{-x} - 8e^{-5x} - e^{10x}$
- $f(x) = \frac{e^{-3x} - 2e^{8x}}{9e^{8x} - 7e^{-3x}}$

4.2.5. Continuity and asymptotes of exponential function

Activity 4.2.5



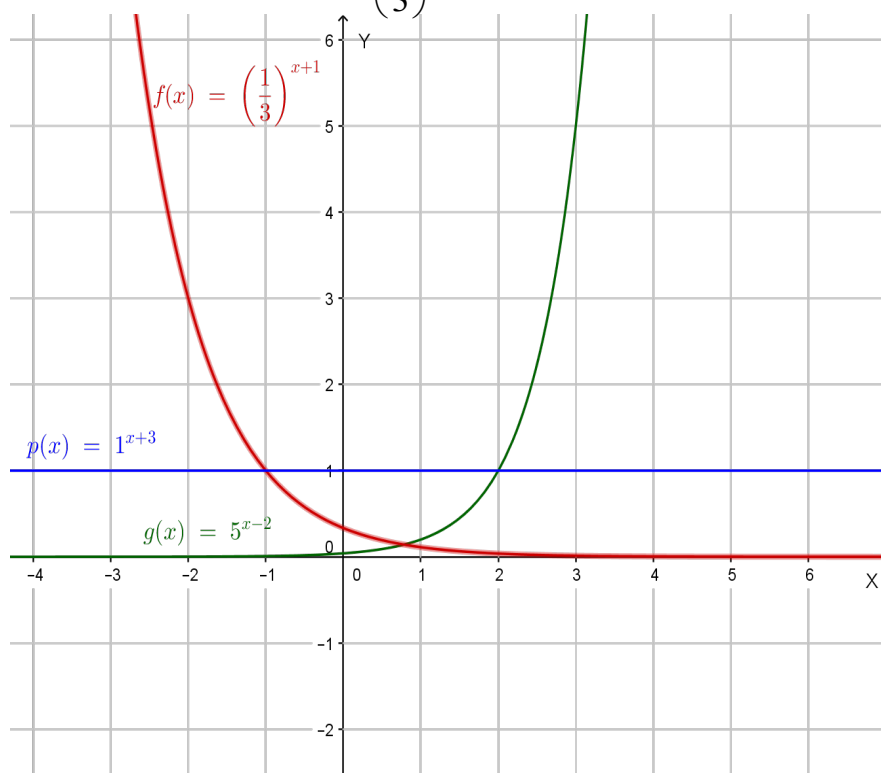
Given the function $f(x) = 2^{(x-2)}$,

- Find the domain and range of f .
- Determine $\lim_{x \rightarrow -\infty} f(x)$ and deduce the equation of horizontal asymptote for the graph.
- Evaluate the value of $f(x)$ for $x = 0$ and deduce y -intercept
- Determine $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} \frac{f(x)}{x}$
- Evaluate $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$. Discuss the continuity of this function at $x = 0$.
- Sketch the graph of $f(x)$

Content Summary

For $a > 0, a \neq 1$, the exponential function $f(x) = a^x$ is continuous on \mathbb{R} and takes always nonnegative values. Its graph admits the line of equation $y = 0$ as horizontal asymptote and intercepts y -axis at $(0, 1)$. The function f is increasing from 0 to $+\infty$ if a is greater than 1 and decreasing from $+\infty$ to 0 if a is smaller than 1. The function is the constant 1 if $a = 1$ and its graph is the horizontal line of equation $y = 1$.

Graphs of $g(x) = 5^{x-2}$, $f(x) = \left(\frac{1}{3}\right)^{x+1}$ and $p(x) = 1^{x+3}$



Example

Let $f(x) = 3^{x+1} - 1$.

Find the domain, range and equation of the horizontal asymptote of the graph of f . Precise intercepts (if any) of the graph with axes.

Solution

The domain of f is the set of all real numbers since the expression $x+1$ is defined for all real values. To find the range of f , we start with the fact that $3^{(x+1)} > 0$ as exponential function.

Then, subtract 1 to both sides to get $3^{x+1} - 1 > -1$.

Therefore, for any value of x , $f(x) > -1$. In other words, the range of f is $]-1, \infty[$.

As x decreases without bound, $f(x) = 3^{x+1} - 1$ approaches -1 , in other words

$\lim_{x \rightarrow -\infty} f(x) = -1$. Thus, the graph of f has horizontal asymptote the line of equation $y = -1$. To find the x intercept we need to solve the equation $f(x) = 0$

This is $3^{(x+1)} - 1 = 0$.

Solving yields to $x = -1$. The x -intercept is the point $(-1, 0)$.

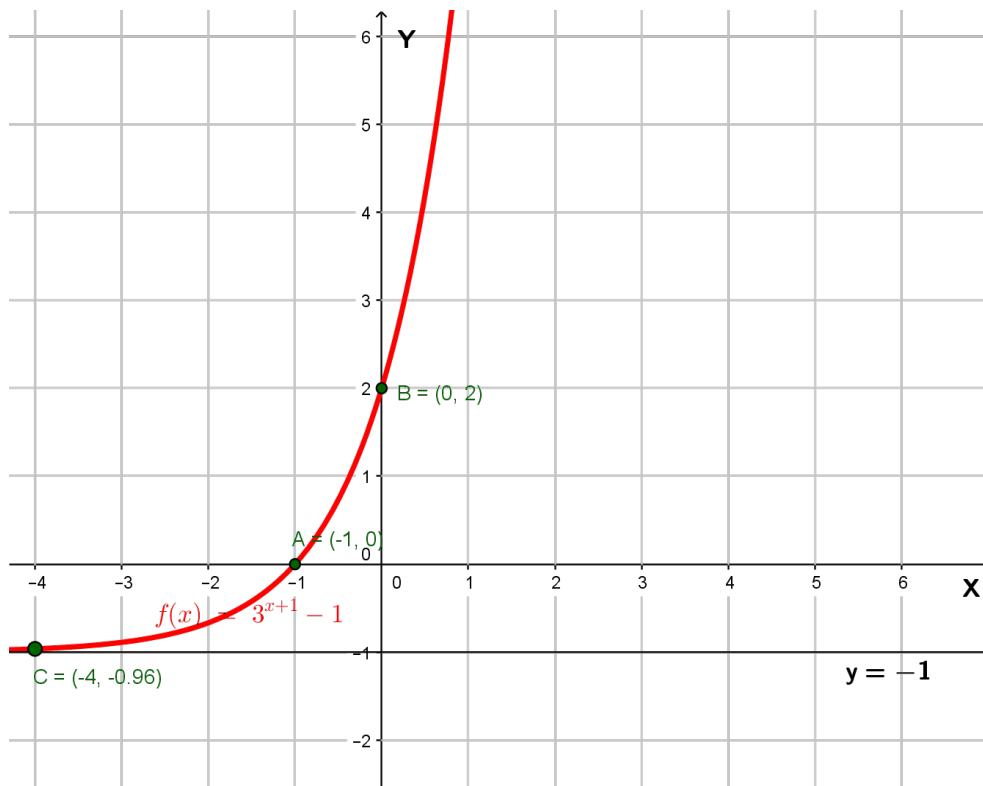
The y -intercept is given by $(0, f(0)) = (0, 3^{(0+1)} - 1) = (0, 2)$.

Extra points:

$$(-2, f(-2)) = (-2, 3^{(-2+1)} - 1) = (-2, -4/3)$$

$$(-4, f(-4)) = (-4, 3^{(-4+1)} - 1) = (-4, -26/27)$$

We can now use all the above information to plot $f(x) = 3^{(x+1)} - 1$:





Application activity 4.2.5

Given the function $f(x) = 2^x + 1$

- Determine domain and range.
- Write the equation of horizontal asymptote of the graph.
- Find the x and y intercepts of the graph if there are any.
- Sketch the graph

4.2.6. Differentiation of Exponential functions

Activity 4.2.6



Given functions $f(x) = e^x$ and $g(x) = 2^x$

- Determine the inverse of $f(x)$ and $g(x)$.
- Use the derivative of logarithmic functions $p(x) = \ln x$ and $k(x) = \log_2 x$, then apply the rule of differentiating inverse functions to find the derivative of $f(x) = e^x$ and $g(x) = 2^x$

Content Summary

The derivative of $f(x) = e^x$ is noted by $\frac{d(e^x)}{dx} = e^x$ or $f'(x) = e^x$. If u is a function

of x , the derivative of $y = e^{u(x)}$ is $y' = d\left(\frac{e^{u(x)}}{dx}\right) = e^{u(x)} \frac{du(x)}{dx} = u'e^{u(x)}$. Thus, $(e^u)' = u'e^u$. The derivative of $g(x) = a^x$ is $g'(x) = a^x \ln a$.

Therefore, if u is a function of x , the derivative of $g(x) = a^{u(x)}$ is $g'(x) = u'(x)a^{u(x)} \ln a$

Example

Find the derivative of the following functions

1) $f(x) = e^{5x^2}$ 2) $y = e^{4x}$

Solution

$$1) f'(x) = (5x^2)' e^{x^2} = 10xe^{5x^2} \quad 2) y' = 4e^{4x}$$



Application activity 4.2.6

1. Given the function $f(x) = 4^x$

Find $f'(x)$ the derivative function of $f(x)$

2. Find the derivative of each of the following function

a) $f(x) = 10^{3x}$ b) $f(x) = xe^{x^2+1}$ c) $f(x) = \frac{3^{4x+2}}{x}$

4.2.7. Variation and graphical representation of exponential functions

Activity 4.2.7



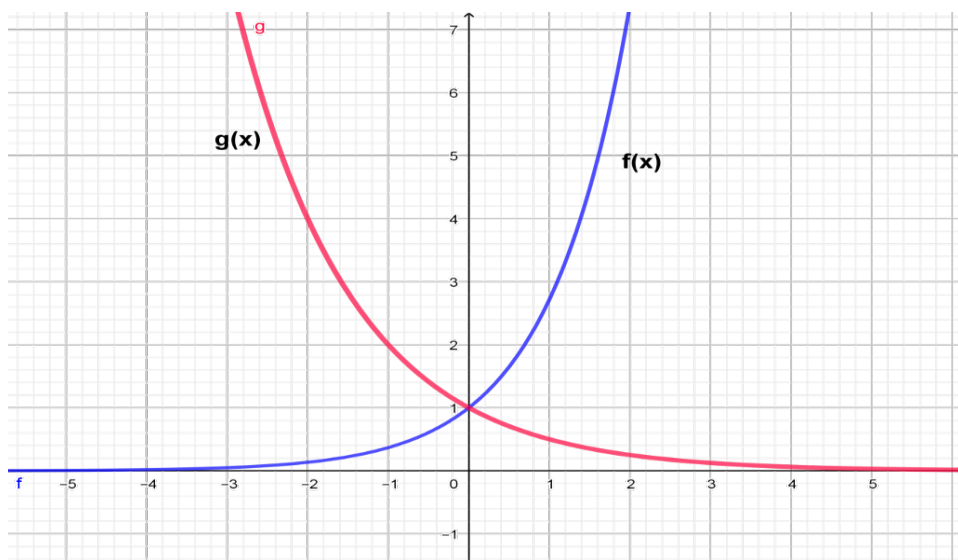
Given two functions $f(x) = 2^x$ and $g(x) = 0.5^x$

1. Compare $f(1)$ and $f(10)$, to verify whether the function $f(x)$ is increasing or decreasing on the interval $[1,10]$
2. Compare $g(1)$ and $g(10)$ to verify whether the function $g(x)$ is increasing or decreasing on the interval $[1,10]$
3. Use derivatives $f'(x)$ and $g'(x)$ to discuss variations of each of the functions
4. Calculate 2nd derivative of $f(x)$ and $g(x)$ as $f''(x)$ and $g''(x)$ respectively
5. Draw table of signs for $f''(x)$ and $g''(x)$ hence discuss the concavity and inflection points if any.
6. Plot the graphs of $f(x)$ and $g(x)$.

Variations of exponential functions

The function $g(x) = a^x$, $a > 1$ defined on \mathbb{R} is always increasing. When $0 < a < 1$, the function $g(x) = a^x$ is always decreasing. This means the exponential functions $g(x) = a^x$ does not have extremum (maximum or minimum); this means that the function increases or decreases “indefinitely”.

Graph of $g(x) = (0.5)^x$ and $f(x) = 2^x$



Example

Given the function $f(x) = xe^x$

- Find the derivative of $f(x) = xe^x$
- Solve $f'(x) = 0$
- Discuss extrema of the function.
- Establish the sign diagram of $f'(x)$ and variations of $f(x)$
- Find the second derivative of $f(x) = xe^x$ and study its signs
- Determine inflexion points of the curve of $f(x) = xe^x$ if any
- Plot the graph of the function.

Solution

i. The domain of the function is \mathbb{R} .

The derivative of $f(x) = xe^x$ is defined by

$$f'(x) = e^x + xe^x = (1+x)e^x$$

ii. $f'(x) = 0$ if $x = -1$

iii. Sign diagram for $f(x)$

There is need to find limit of the function at the boundaries of the domain:

$\lim_{x \rightarrow -\infty} xe^x = 0$ and $\lim_{x \rightarrow +\infty} xe^x = +\infty$. The limit at $-\infty$ tells us that line of equation $y = 0$

is an horizontal asymptote when x is taking "indefinitely" negative values.

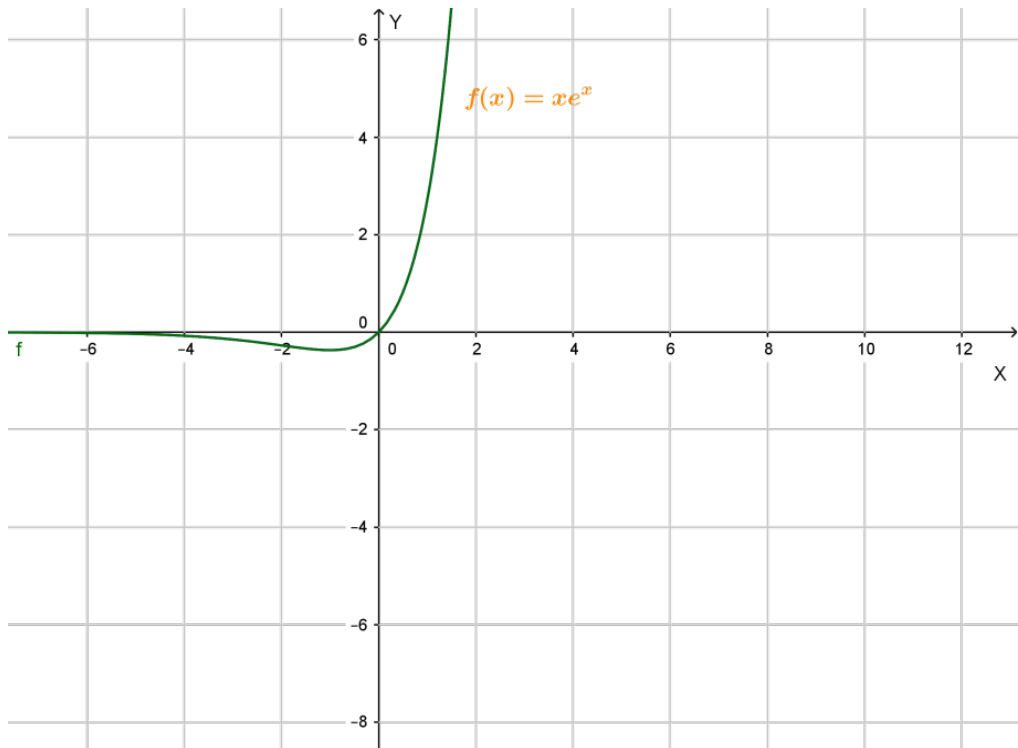
x	$-\infty$	-2	-1	0	$+\infty$
$f'(x)$		$-$	0	$+$	$+$
$f''(x)$		$-$	0	$+$	
$f(x)$	0			0	$+\infty$

iv. Therefore, $f(x)$ is decreasing from 0 to $-\frac{1}{e}$ on the interval $]-\infty, -1]$ and

increases from $-\frac{1}{e}$ to $+\infty$ on $[-1, +\infty[$. The function has minimum (absolute)

equal to $-\frac{1}{e}$ when $x = -1$.

v. Graph of $f(x) = xe^x$



Application of derivatives to remove indeterminate form $0^0, 1^\infty$ and ∞^0

These indeterminate forms are found in functions of the form $y = [f(x)]^{g(x)}$

To remove these indeterminate forms, we change the function in the form

$$y = [f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

$$\text{Also } \lim_{x \rightarrow k} e^{f(x)\ln g(x)} = e^{\lim_{x \rightarrow k} f(x)\ln g(x)}$$

Example

a) Show that $\lim_{x \rightarrow 0^+} x^x = 1$

Solution

$$\lim_{x \rightarrow 0^+} x^x = 0^0 \quad (IF)$$

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x \ln x} = e^{\lim_{x \rightarrow 0^+} x \ln x}$$

$$\lim_{x \rightarrow 0^+} x \ln x \quad (0 \times \infty \text{ IF})$$

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty} \text{ IF}\right)$$

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0 \quad (\text{Hospital rule}).$$

$$\text{Finally, } \lim_{x \rightarrow 0^+} x^x = 1$$

b) Show that $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$

Solution

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x \quad (1^\infty \text{ IF})$$

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow +\infty} e^{x \ln \left(1 + \frac{1}{x}\right)} = e^{\lim_{x \rightarrow +\infty} x \ln \left(1 + \frac{1}{x}\right)}$$

But,

$$\lim_{x \rightarrow +\infty} x \ln \left(1 + \frac{1}{x}\right) = \lim_{x \rightarrow +\infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\frac{0}{0} \text{ IF}\right)$$

$$\lim_{x \rightarrow +\infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow +\infty} \frac{-\frac{1}{x^2}}{\left(1 + \frac{1}{x}\right) \left(-\frac{1}{x^2}\right)} \quad (\text{Hospital rule})$$

$$= \lim_{x \rightarrow +\infty} \frac{1}{1 + \frac{1}{x}} = 1$$

Thus, $\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e^1 = e$

c) Show that $\lim_{x \rightarrow -1} \left(\frac{1}{x+1}\right)^{x+1} = 1$.

Solution

$$\lim_{x \rightarrow -1} \left(\frac{1}{x+1}\right)^{x+1} \quad (\infty^0 \text{IF})$$

$$\Rightarrow \lim_{x \rightarrow -1} \left(\frac{1}{x+1}\right)^{x+1} = \lim_{x \rightarrow -1} e^{(x+1)\ln\left(\frac{1}{x+1}\right)} = e^{\lim_{x \rightarrow -1} (x+1)\ln\left(\frac{1}{x+1}\right)}$$

$$\lim_{x \rightarrow -1} (x+1)\ln\left(\frac{1}{x+1}\right) = \lim_{x \rightarrow -1} \frac{\ln\left(\frac{1}{x+1}\right)}{\frac{1}{x+1}} = \lim_{x \rightarrow -1} \frac{-\frac{1}{x+1}}{\frac{1}{-(x+1)^2}} = 0 \quad (\text{Hospital rule}).$$

Or

$$\text{Finally, } \lim_{x \rightarrow -1} \left(\frac{1}{x+1}\right)^{x+1} = e^0 = 1$$



Application activity 4.2.7

1. Given the function $f(x) = xe^{x^2}$
 - a. Find the first derivative of $f(x)$
 - b. Establish the table of variation of $f(x)$ and verify whether $f(x)$ is increasing or decreasing and write down the interval where the function increasing or decreasing
2. Evaluate the following limit

a. $\lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x - 2}$ b. $\lim_{x \rightarrow -\infty} xe^x$

4.3. Modeling and solving problems involving logarithms and exponential functions

Logarithmic and exponential functions are very essential in pure sciences, social sciences and real-life situations. They are used by bank officers to deal with interests on loans they provide to clients. Economists and demographers use such functions to estimate the number of populations after a certain period and many researchers use them to model certain natural phenomena. In the following paragraphs, we are going to develop some of these applications.

4.3.1. Modeling and solving Interest rates problems

Activity 4.3.1



Given two functions $f(x) = 2^x$ and $g(x) = 0.5^x$

1. Compare $f(1)$ and $f(10)$, to verify whether the function $f(x)$ is increasing or decreasing on the interval $[1,10]$
2. Compare $g(1)$ and $g(10)$ to verify whether the function $g(x)$ is increasing or decreasing on the interval $[1,10]$
3. Use derivatives $f'(x)$ and $g'(x)$ to discuss variations of each of the functions
4. Calculate 2nd derivative of $f(x)$ and $g(x)$ as $f''(x)$ and $g''(x)$ respectively
5. Draw table of signs for $f''(x)$ and $g''(x)$ hence discuss the concavity and inflection points if any.
6. Plot the graphs of $f(x)$ and $g(x)$.

An amount of 2 000 dollars is invested at a bank that pays an interest rate of 10% compounded once annually. Find the total amount at the end of t years by proceeding as follows:

Complete the table below:

At the end of	The total amount
The first year	$2000 + 0.1(2000)$ $= 2000(1 + 0.1)$
The second year	$2000(1 + 0.1) + 0.1[2000(1 + 0.1)]$ $= 2000(1 + 0.1)^2$
The third year	$2000(1 + 0.1) \dots + \dots = 2000(1 + 0.1)^3$
The fourth year	...
The fifth year	...
...	...
The t^{th} year	...

If a principal P (the money you put in) is invested at an interest rate r for a period of t years, then the amount A (how much you make) of the investment can be calculated by the following generalized formula of the interest rate problems:

- $A = P(1 + r)$ Simple interest for one year
- $A = P\left(1 + \frac{r}{n}\right)^n$ Interest compounded n times per year
- $A = Pe^{rt}$ Interest compounded continuously.

Compound interest:

- **Discrete compound interest**

If you put money in a savings account then the bank will pay you interest (a percentage of your account balance) at the end of each time of period, typically one month or one day.

For example, if the time of period is one month this process is called **Monthly compounding**. The term compounding refers to the fact that interest is added to your account each month. If it is one day is called daily compounding. The exponential model that describes this situation is called Discrete compounding interest formula.

$A = P_0 \left(1 + \frac{r}{n}\right)^{nt}$ Where A is total amount at the end of periods of time, P_0 is the principal amount, n is the number of times that the interest is compounded, r is the interest rate per period, t is the time

Examples:

1. If the principal money is \$100, the annual interest rate is 5% and the interest is compounded daily. What will be the balance after ten years?

Solution:

Let $P_0 = 100, r = 5\% = 0.05, n = 1\text{year} = 365\text{days}$ and $t = 10$

$$P(t) = 100 \left(1 + \frac{0.05}{365}\right)^{365 \times 10}$$

$$= 164.87$$

After 10 years balance will be \$164.87

1. An amount of 500 000 FRW is invested at a bank that pays an interest rate of 12% compounded annually.
 - a. How much will the owner have at the end of 10 years, in each of the following cases?

The interest rate is compounded:

- i. once a year.
 - ii. Twice a year
- b. What type of interest rate among the two would the client prefer? Explain why.

Solution

- a. i. For once a year, at the end of 10 years the owner will have

$$A = P(1+r)^t = 500000(1+0.12)^{10}$$

$$= 500000(1.12)^{10} = 1552924.10\text{Frw}$$

- ii. For twice a year, at the end of 10 years the owner will have

$$A = P\left(1 + \frac{r}{2}\right)^{2t} = 500000\left(1 + \frac{0.12}{2}\right)^{2(10)}$$

$$= 500000(1.06)^{20} = 1603567\text{Frw}$$

b. Since $1603567 > 1552924.10$, the client will prefer compounding many times per year as it results in more money.

▪ **Continuous compound interest**

If we start with discrete compound interest formula and let the number of times compounded per year approaches ∞ , then we end up with what is known as **continuous compounding** then the balance at time t years is given by $A = P_0 e^{rt}$ where P_0 is the principal amount, r is annual interest and time t years.

Example

If the principal money is \$10,000 the annual interest rate is 5% and the interest is compounded continuously. What will be the balance after 40 years?

Solution

$$P_0 = 10000, \quad r = 5\% = 0.05, \quad t = 40$$

$$P(40) = 10000e^{0.05 \times 40} = 73890.56$$

The balance after 40 years is \$73890.56



Application activity 4.3.1

Your aunt would like to invests 300 000 FRW at a bank. The Bank I pay an interest rate of 10% compounded once annually. The Bank II pays an interest rate of 9.8% compounded continuously. Your aunt will withdraw the money plus interest after 10 years.

At which bank do you advice your aunt to invest her money so as to get much money at the end of 10 years?

4.3.2. Modeling and solving Mortgage problems

Activity 4.3.2



1. Go to conduct a research in the library, on internet or conduct a conversation with a bank officer to write down the meaning of the following when you get a loan from the bank:
 - i. the periodic payment P
 - ii. the annual interest rate r
 - iii. the mortgage amount M

- iv. the number t of years to cover the mortgage
 - v. the number n of payments per year.
 - vi. Among all these elements/components, what is the most useful for the client to be informed about by the bank once he/she is given the mortgage loan?
2. Your elder brother is newly employed at a company and earns 500 000 FRW per month. He would like to know if he can afford monthly payments on a mortgage of 20 000 000 FRW with an interest rate of 6% that runs for 20 years. Given that the quantities above are governed by the relation

$$P = \frac{\frac{rM}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

Show your brother that he can afford the monthly payments by determining the following:

- i. the monthly payment, that will be retained at the bank
- ii. The balance that your brother can withdraw each month from the bank
- iii. How much interest your brother will pay to the bank by the end of 20 years

Content Summary

When a person gets a loan (mortgage) from the bank, the mortgage amount M , the number of payments or the number t of years to cover the mortgage, the amount of the payment P , how often the payment is made or the number n of payments per year, and the interest rate r , it is proved that all the 5 components are related by the following formula:

$$P = \frac{\frac{rM}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}$$

If we let $i = \frac{r}{n}$, the amount of payment $P = M \left[\frac{i}{1 - (1+i)^{-nt}} \right]$.

Example

A business woman wants to apply for a mortgage of 75 000 US dollars with an interest of 8% that runs for 20 years. How much interest she will pay over the 20 years?

Solution

Substituting for $M=75\,000$, $r=0.08$, $t=20$, $n=12$ in the equation

$$P = \frac{\frac{rM}{n}}{1 - \left(1 + \frac{r}{n}\right)^{-nt}}, \text{ we have } P = \frac{(0.08)(75000)}{12} \frac{1}{1 - \left(1 + \frac{0.08}{12}\right)^{-(12)(20)}} = 627.33$$

Each month she will be paying 627.33 US dollars.

The total amount she will pay is $627.33 \times 12 \times 20$ US dollars = 150559.2 US dollars

The interest will be $(150559.2 - 75000)$ US dollars = 75559.2 US dollars



Application activity 4.3.2

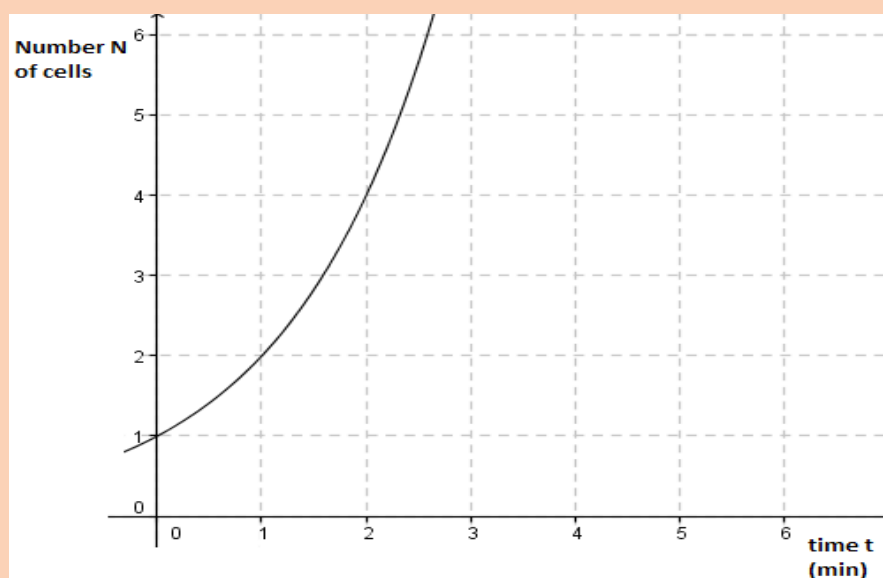
A bank can offer a mortgage at 10 % interest rate to be paid back with monthly payments for 20 years. After analysis, a potential borrower finds that she can afford monthly payment of 200000FRW. How much of mortgage can she ask for?

4.3.3. Population growth problems

Activity 4.3.3



Analyse the graph below showing the number of cells recorded by a student in a biology laboratory of his/her school during an experiment as function of time t .



a) Complete the table below:

Time t(minutes)	0	1	2	3	4			
Number of cells					

b) Given that $N(t) = N_0 e^{kt}$, where $N(t)$ is the quantity at time t , N_0 is the initial quantity and k is a positive constant, what is the value of N_0 ? Predict the number of cells after 5 minutes.

c) What happens to the number of cells as the time becomes larger and larger? Is the number of cells growing or not? Explain your answer.

Content summary

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population for n periods will be $P_n = P_0 (1+r)^n$. This is similar to the final value (F) of an initial investment (A) deposited for t discrete time periods at an interest rate of $i\%$ which is calculated using the formula

$$F = A(1+i)^t$$

To derive a formula that will give the final sum accumulated after a period of continuous growth, we first assume that growth occurs at several discrete time intervals throughout a year. We also assume that A is the initial sum, r is the nominal annual rate of growth, n is the number of times per year that increments are accumulated and y is the final value. This means that after t years of growth the final sum will be:

$$y = A \left(1 + \frac{r}{n} \right)^t$$

Growth becomes continuous as the number of times per year that increments in growth are accumulated increases towards infinity. When $n \rightarrow \infty$, we get

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} A \left(1 + \frac{r}{n} \right)^t = Ae^{rt}$$

This is similar to $N(t) = N_0 e^{kt}$ where A , N_0 , r and k take respectively the same meanings.

Therefore, the final value $A(t)$ of any variable growing continuously at a known annual rate r from a given original value A_0 is given by the following formula $A(t) = A_0 e^{rt}$

Example

1. The number of bacteria in a culture increases according to an equation of the type $N(t) = N_0 e^{kt}$. Given that the number of bacteria triples in 2 hours,
 - a. finds an equation free of N_0 and solve the equation for k
 - b. How long would it take for the number of bacteria to be 5 times the initial number?

Solution

$$\text{a. } N(2) = 3N_0 \Leftrightarrow 3N_0 = N_0 e^{k(2)} \Leftrightarrow e^{2k} = 3 \Rightarrow 2k = \ln 3 \Leftrightarrow k = \frac{\ln 3}{2} = 0.5493$$

$$\text{b. } 5N_0 = N_0 e^{0.5493t} \Leftrightarrow e^{0.5493t} = 5 \Rightarrow t = \frac{\ln 5}{0.5493} \approx 2.93.$$

It will take 2.93 hours for the number of bacteria to be 5 times the initial number.

- Population in a developing country is growing continuously at an annual rate of 3%. If the population is now 4.5 million, what will it be in 15 years' time?

Solution

The final value of the population (in millions) is found by using the formula $y = Ae^{rt}$ and substituting the given numbers: initial value $A = 4.5$; rate of growth $r = 3\% = 0.03$; number of time period's $t = 15$, giving $y = 4.5e^{0.03(15)} = 7.0574048$ million



Application activity 4.3.3

- The population of a city increases according to the law of uninhibited growth. If the population doubles in 5 years and the current population is one million, what will be the size of the population in ten years from now?
- A country economy is forecast to grow continuously at an annual rate of 2.5%. If its Gross National product (GNP) is currently 56 billions of USD, what will the forecast for GNP be after 1.75 years (at the end of the third quarter the year after Next)?
- One town of a given country had a population of 11,000 in 2000 and 13,000 in 2017.

Assuming an exponential growth model, determine the constant rate of growth per year.



4.4 End unit assesment

- Determine the domain and range of the following functions
 - $f(x) = \log_2(3x - 2)$
 - $f(x) = \ln(x^2 - 1)$
 - $f(x) = 2e^{3x+1}$
 - $f(t) = 4^{\sqrt{3t+1}}$
- Evaluate each of the following limits and give the equation of the asymptotes if any
 - $\lim_{x \rightarrow 0^+} \frac{\ln x}{x}$
 - $\lim_{x \rightarrow +\infty} (3 + x^2 \ln x)$
- Determine the domain and range of the following functions
 - $f(x) = \log_2(3x - 2)$
 - $f(x) = \ln(x^2 - 1)$
 - $f(x) = 2e^{3x+1}$
 - $f(t) = 4^{\sqrt{3t+1}}$
- Evaluate each of the following limits and give the equation of the asymptotes if any
 - $\lim_{x \rightarrow 0^+} \frac{\ln x}{x}$
 - $\lim_{x \rightarrow +\infty} (3 + x^2 \ln x)$
- Differentiate with respect to x the following functions
 - $f(x) = \log_2 \sqrt{\frac{x^2 - 4}{x + 2}}$
 - $h(x) = \frac{1}{3}(4^{2x+5})$
- Carry out a research in the library or on internet and explain at least 5 applications of logarithmic or exponential functions in other human sciences.
- The population of the world in 1995 was 5.7 billion, and the estimated relative growth rate is 2% per year. If the population continues to grow at this rate, when will it reach 114 billion?
- Discuss how this unit inspired you in relation to learning other subjects or to your future. If no inspiration at all, explain why.

REFERENCES

1. J. Sadler, D. W. S. Thorning: Understanding Pure Mathematics, Oxford University Press 1987.
2. Arthur Adam Freddy Goossens: Francis Lousberg. Mathématisons 65. DeBoeck, 3^e edition 1991.
3. Charles D. Hodgman, M.S., Samuel M. Selby, Robert C. Weast. Standard Mathematical Table. Chemical Rubber Publishing Company, Cleveland, Ohio 1959.
4. David Rayner, Higher GCSE Mathematics, Oxford University Press 2000
5. Direction des Programmes de l'Enseignement Secondaire. Géométrie de l'Espace 1^{er} Fascule. Kigali, October 1988
6. Direction des Programmes de l'Enseignement Secondaire. Géométrie de l'Espace 2^{ème} Fascule. Kigali, October 1988
7. Frank Ebos, Dennis Hamaguchi, Barbana Morrison & John Klassen, Mathematics Principles & Process, Nelson Canada A Division of International Thomson Limited 1990
8. George B. Thomas, Maurice D. Weir & Joel R. Hass, Thomas' Calculus Twelfth Edition, Pearson Education, Inc. 2010
9. Geoff Mannall & Michael Kenwood, Pure Mathematics 2, Heinemann Educational Publishers 1995
10. H.K. DASS...Engineering Mathematics. New Delhi, S. CHAND & COMPANY LTD, thirteenth revised edition 2007.
11. Hubert Carnec, Genevieve Haye, Monique Nouet, Rene Seroux, Jacqueline Venard. Mathématiques TS Enseignement obligatoire. Bordas Paris 1994.
12. James T. McClave, P. George Benson. Statistics for Business and Economics. USA, Dellen Publishing Company, a division of Macmillan, Inc 1988.
13. J CRAWSHAW, J CHAMBERS: A concise course in A-Level statistics with worked examples, Stanley Thornes (Publishers) LTD, 1984.
14. Jean Paul Beltramonde, Vincent Brun, Claude Felloneau, Lydia Misset, Claude Talamoni. Declic 1^{re} S Mathématiques. Hachette-education, Paris 2005.
15. JF Talber & HH Heing, Additional Mathematics 6th Edition Pure & Applied, Pearson Education South Asia Pte Ltd 1995
16. J.K. Backhouse, SPT Houldsworth B.E.D. Copper and P.J.F. Horril. Pure Mathematics 2. Longman, third edition 1985, fifteenth impression 1998.
17. Mukasonga Solange. Mathématiques 12, Analyse Numérique. KIE, Kigali 2006.

18. N. PISKOUNOV, Calcul Différentiel et Integral tom II gème édition. Editions MIR. Moscou, 1980.
19. Paule Faure- Benjamin Bouchon, Mathématiques Terminales F. Editions Nathan, Paris 1992.
20. Peter Smythe: Mathematics HL & SL with HL options, Revised Edition, Mathematics Publishing Pty. Limited, 2005.
21. Robert A. Adms & Christopher Essex, Calculus A complete course Seventh Edition, Pearson Canada Inc., Toronto, Ontario 2010
22. Seymour Lipschutz. Schaum's outline of Theory and Problems of Finite Mathematics. New York, Schaum Publisher, 1966
23. Seymour Lipschutz. Schaum's outline of Theory and Problems of linear algebra. McGraw-Hill 1968.
24. Shampiyona Aimable : Mathématiques 6. Kigali, Juin 2005.
25. Yves Noiro, Jean-Paul Parisot, Nathalie Brouillet. Cours de Physique Mathématiques pour la Physique. Paris, DUNOD, 1997.
26. Swokowski, E.W. (1994). Pre-calculus: Functions and graphs, Seventh edition. PWS Publishing Company, USA.
27. Allan G. B. (2007). Elementary statistics: a step by step approach, seventh edition, *Von Hoffmann Press*, New York.
28. David R. (2000). Higher GCSE Mathematics, revision and Practice. Oxford University Press, UK.
29. Ngezahayo E.(2016). Subsidiary Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.
30. REB. (2015). Subsidiary Mathematics Syllabus, MINEDUC, Kigali, Rwanda.
31. REB. (2019). Mathematics Syllabus for TTC-Option of LE, MINEDUC, Kigali Rwanda.
32. Peter S. (2005). Mathematics HL&SL with HL options, Revised edition. Mathematics Publishing PTY. Limited.
33. Elliot M. (1998). Schaum's outline series of Calculus. MCGraw-Hill Companies, Inc. USA.
34. Frank E. et All. (1990). Mathematics. Nelson Canada, Scarborough, Ontario (Canada)
35. Gilbert J.C. et all. (2006). Glencoe Advanced mathematical concepts, MCGraw-Hill Companies, Inc. USA.
36. Robert A. A. (2006). Calculus, a complete course, sixth edition. Pearson Education Canada, Toronto, Ontario (Canada).

37. Sadler A. J & Thorning D.W. (1997). Understanding Pure mathematics, Oxford university press, UK.
38. J. CRAWSHAW and J. CHAMBERS 2001. A concise course in Advanced Level Statistics with worked examples 4th Edition. Nelson Thornes Ltd, UK.
39. Ron Larson and David C (2009). Falvo. Brief Calculus, An applied approach. Houghton Mifflin Company.
40. Michael Sullivan, 2012. Algebra and Trigonometry 9th Edition. Pearson Education, Inc
41. Swokowski & Cole. (1992). Preaclaculus, Functions and Graphs. Seventh edition.
42. Glencoe. (2006). Advanced mathematical concepts, Precalculus with Applications.
43. Seymour Lipschutz, PhD. & Marc Lipson, PhD. (2007). Discrete mathematics. 3rd edition.
44. K.A. Stroud. (2001). Engineering mathematics. 5th Edition. Industrial Press, Inc, New York
45. John bird. (2005). Basic engineering mathematics. 4th Edition. Linacre House, Jordan Hill, Oxford OX2 8DP