MATHEMATICS BOOK FOR TTCs

STUDENT'S BOOK

YEAR

1

OPTION:

LANGUAGE EDUCATION (LE)

FOREWORD

Dear Student-Teacher,

Rwanda Education Board (REB) is honored to present Year 1 Mathematics book for Language Education (LE) student teachers. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of tutors' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the Student-Teacher where concepts are mainly introduced by an activity, situation or scenario that helps the student-teacher to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other Students-Teachers through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities: those are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, Teachers and TTC Tutors for their technical support. A word of gratitude goes to Head Teachers and TTCs principals who availed their staff for various activities.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development of this Mathematics book for Year one student teachers in the option of Language Education (LE). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers, Teachers and TTC tutors whose efforts during writing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook Elaboration.

Joan MURUNGI

Head of CTLR Department

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UNIT: 1

ARITHMETICS

Key Unit competence: Use arithmetic operations to solve simple real life problems

1.0. Introductory Activity

The simple interest earned on an investment is $I = prt$ where I is the interest earned, *p* is the principal, r, is the interest rate and t is the time in years. Assume that 50,000Frw is invested at annual interest rate of 8% and that the interest is added to the principal at the end of each year.

a. Discuss the amount of interest that will be earned each year for 5 years.

b. How can you find the total amount of money earned at the end of these 5 years? Classify and explain all Mathematics operations that can be used to find that money.

1.1 Fractions and related problems

Activity 1.1

1) Sam had 120 teddy bears in his toy store. He sold $\frac{2}{3}$ of them at 12 Rwandan francs each. How much did he receive?

2) Simplify the following fractions and explain the method used to simplify.

a) $\frac{8x^2y^3}{2x^3y}$ b) $\frac{2x^2+5x^3}{2x^2+4x^3}$

3) Given that the denominator is different from zero,

4) Explain how to work out $\frac{1}{x+1}-\frac{1}{2x+2}$

5) Do you some times use fractions in your life? Explain your answer.

CONTENT SUMMARY

Consider the expressions $\frac{x}{4}+\frac{3y}{x},\frac{5}{x+4}$ in each of these, the numerator or the denominator or both contain a variable or variables. These are examples of algebraic fractions. Since the letter used in these fractions stand, for real numbers, we deal with algebraic fractions in the same way as we do with fractions in arithmetic. Fractions can be simplified and operations on fractions such as addition, subtraction, multiplication and division can be applied in simple arithmetic for solving related problems.

Adding fractions:

To add fractions there is a simple rule $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ where $(b\neq 0)$

Example:

$$\frac{x}{2}+\frac{y}{5}=\frac{(x)(5)+(2)(y)}{(2)(5)}=\frac{5x+2y}{10};$$

Subtracting fractions

Subtracting fractions is very similar to addition of fractions, except that the sign change

Example:

$$\frac{x+2}{x}-\frac{x}{x-2}=\frac{(x+2)(x-2)-(x)(x)}{x(x-2)}=\frac{-4}{x^2-2x}$$ with $x\neq 0$ and $x\neq 2$

Multiplying fractions

Multiplying fractions is the easiest one of all, just multiply the numerators together, and the denominators together

$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$, where $(b\neq 0, d\neq 0)$

Example:

$$\frac{3x}{x-2}\times\frac{x}{3}=\frac{(3x)(x)}{3(x-2)}=\frac{3x^2}{3(x-2)}=\frac{x^2}{x-2}, \quad (x\neq 2)$$

Dividing fractions

To divide fractions, first flip the fraction we want to divide by, then use the same method as for multiplication:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \text{ where } (b \neq 0, c \neq 0, d \neq 0).$$

Example:

$$\frac{3y^2}{x+1} \div \frac{y}{2} = \frac{3y^2}{x+1} \times \frac{2}{y} = \frac{(3y^2)(2)}{(x+1)(y)} = \frac{6y^2}{(x+1)(y)} = \frac{6y}{x+1} \; ; (x \neq -1), \; y \neq o \; ,$$

As fraction is an expression indicating the division of integers. For example:

$\frac{15}{9}, \frac{3}{8}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator $N(x)$ and the divisor (lower number) is called the denominator $D(x)$ from the operations of fractions, different fractions can be found by taking Low Common Multiple (L.C.M.) and then add all the fractions.

For example: $\frac{1}{x-1} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$, $x \neq 1$, $x \neq -2$ we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions**.

To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution**.

Example

$$\frac{2x + x^2 - 1}{x(x^2 - 1)} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}, \quad x \neq 0 \; , \; x \neq -1 \; , \; x \neq 1$$

$\frac{2x + x^2 - 1}{x(x^2 - 1)}$ is the resultant fraction $\frac{1}{x}, \frac{1}{x-1}$ and $\frac{1}{x+1}$ are its partial fractions.

Rational fraction

We know that $\frac{p}{q}$, $q \neq 0$ is called a rational number. Similarly the quotient of two polynomials $\frac{N(x)}{D(x)}, D(\mathrm{x}) \neq 0$ with no common factors. There are two types of rational fractions such **proper** or **improper** fractions.

Proper fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called a proper fraction if the **degree of numerator is less than the degree of Denominator D(x).**

Example: $\frac{6x+27}{3x^3-9x}$ is a proper fraction , $x \neq 0$, $x \neq \pm 3$

Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called improper fraction if the **degree of numerator is greater than or equal to the degree of Denominator D(x).**

Example: $\frac{6x^3-5x^2-3x-10}{x^2+1}$

An algebraic fraction exists only if the denominator is not equal to zero. The values of the variable that can make the denominator zero are called restrictions on the variable(s). An algebraic fraction can have more than one restriction.

Examples:

1) Identify the restriction on the variable in the fraction $\frac{3xy}{(x+3)(x-2)}$

Solution

In the fraction $\frac{3xy}{(x+3)(x-2)}$, $(x+3)\ (x-2)$ is the denominator.

As the denominator must be different from zero

Thus $x+3 \neq 0$ **or** $x-2 \neq 0$.

Therefore, $x \neq -3$ or $x \neq 2$.

In the fraction $\frac{3xy}{(x+3)(x-2)}$, the restrictions are $x \neq 3$ and $x \neq -2$

2) Simplify $\frac{3x^2y}{4a^2} \div \frac{9xy}{5a}$

Solution: $\frac{3x^2y}{4a^2} \times \frac{5a}{9xy} = \frac{(3x^2y)(5a)}{(4a^2)(9xy)} = \frac{(x)(5)}{(4a)(3)} = \frac{5x}{12a}$, $a \neq 0$, $x \neq 0$, $y \neq 0$

Problems related to fractions

Examples

1) One ninth of the shirts sold at peter's shop are stripped. $5/8$ of the remainder are printed. The rest of the shirts are plain colour shirts. If peter's shop has 81 plain colour shirts, how many more printed shirts than plain colour shirts does the shop have?

Solution

Striped printed plain

3 units equals to 81

1 unit $= 81 \div 3 = 27$

Printed shirts have 2 parts more than plain shirts.

2 units $= 27 \times 2 = 54$

Peter's shop has 54 more printed colour shirts than plain shirts.

2) Oscar sold 2 glasses of milk for every 5 sodas he sold. If he sold 10 glasses of milk, how many sodas did he sell?

Solution

Let x be the number of Sodas that Oscar will sell.

Set up a proportion of milk with soda as $\frac{milk}{soda}$, $\frac{2}{5} = \frac{10}{x}$ $\Rightarrow 2x = 50$,

$x = \frac{50}{2} = 25$

He will sell 25 sodas.

Application activity 1.1

1. A proper fraction is such that its numerator and denominator have a difference of 2. If one is added to the denominator and three subtracted from the numerator, the fraction becomes $\frac{2}{3}$. Find the fraction and explain your colleague how to do it.
2. What is a partial fraction? Express $\frac{x^2+1}{x^3+4x^2+3x}$ in partial fractions.

1.2 Decimals and related problems

Activity 1.2

Refer to the meaning of decimals and fractions learnt in previous years and

1. Calculate 50:100 and write it in the form of

   a) a fraction, b) a decimal number.

2. Express $\frac{1}{3}$ *and* $\frac{22}{7}$ in the form of decimal numbers. Explain the relationship between the set of fractions and the set of decimal numbers.

3. Given that the set D is a set of the limited decimal numbers, discuss the following:

a) D is a subset of $\mathbb{Q}$; b) $D \subset \mathbb{Q}$, c) $\mathbb{Z} \subset D$, d) $\mathbb{N} \subset \mathbb{Z} \subset D \subset \mathbb{Q} \subset \mathbb{R}$.

4. Do you some times use decimal numbers in your life? Explain your answer.

CONTENT SUMMARY

Decimals are just another way of expressing fractions

$0.1 = \frac{1}{10}; 0.01 = \frac{1}{100}; 0.001 = \frac{1}{1000}$

Thus 0.234 is equivalent to 234/1,000. Most of the time you will be able to perform operations involving decimals by applying what was learnt in previous years or by using a calculator.

In mathematics a decimal format is often required for a value that is usually specified as a fraction in everyday usage. For example, $\frac{62}{100} = 0.62$.

Because some fractions cannot be expressed exactly in decimals, one may need to 'round off' an answer for convenience. In many of the economic problems (in various books) there is not much point in taking answers beyond two decimal places. Where this is done, '(to 2 dp)' is normally put after the answer.

For **example**: 1/7 as a decimal is 0.14 (to 2 dp).

Example

1) $1.345 + 0.00041 = 1.34541$

2) $2.463 \times 38 = 93.954$

3) $360.54 \div 0.04 = 9,013.5$

Application activity 1.2

Evaluate the following:

1) $1.345 + 0.00041 + 0.20023 =$

2) $93.954 \div 2.4 =$

1.3 Percentages and related problems

Activity 1.3

Refer to the meaning of decimals and percentage learnt in previous years.

1) Calculate 60:100 and write it in the form of

a) a fraction, b) a percentage and c) a decimal number.

2) Express $\frac{1}{3}$ *and* $\frac{22}{7}$ in the form of decimal numbers . Is it possible to express these numbers in the form of percentage? Is the pecentage obtained an exact number?

3) 24 students in a class took English test. If 18 students passed the test, what percentage of those who did not pass?

4) Are percentages used by bank managers? Explain your answer.

CONTENT SUMMARY

Percentage is defined as the proportion, rate or ratio expressed with a denominator of 100.

For example : $\frac{3}{100}, \frac{25}{100}$ etc.

Fraction can be expressed in the form of percentage as $\frac{1}{4} \times 100 = 25\%$.

Decimal format is often required for a value that is usually specified as a percentage in everyday usage. For example, interest rates are usually specified as percentages. A percentage format is really just another way of specifying a decimal fraction, $62\% = \frac{62}{100} = 0.62$ And so, percentages can easily be converted into decimal fractions by dividing by 100. Because some fractions cannot be expressed exactly in decimals, one may need to 'round off' an answer for convenience. In many of the economic problems (of various books) there is not much point in taking answers beyond *two decimal places (2dp)*. Where this is done then we denote 'to 2 dp' normally put after the answer. For example, 1/7 as a percentage is 14.29% (to 2 dp).

In solving word problems involving percentage, 3 steps can help you:

1) Make sure you understand the question;
2) Sort out the information to make a basic percent problem;
3) Apply the operations to find out what asked.

Example

A town council imposes different taxes on different fixed assets as follows: Commercial property 25% per year, Residential property 15% per year, Industrial property 20% per year.

An investor owns a residential building on a plot all valued at 80,000,000 *frw* an industrial plot worth 75,000,000 *Frw* and a commercial premises worth 12,500,000 *Frw*. How much tax does the investor pay annually?

Solution:

Commercial: $\frac{25}{100} \times 12,500,000 Frw = 3,125,000 Frw$

Residential: $\frac{15}{100} \times 80,000,000 Frw = 12,000,000 Frw$

Industrial: $\frac{20}{100} \times 75,000,000Frw = 15,000,000Frw$

Total tax the investor can pay annually:

$3,125,000Frw + 12,000,000Frw + 15,000,000Frw = 30,125,000Frw$.

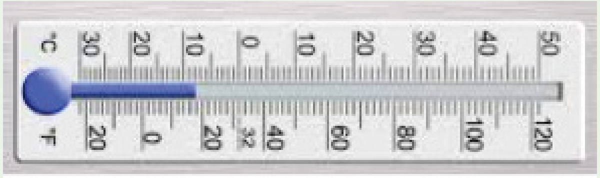
Application activity 1.3

1. In the middle of the first term, the school organize the test and the tutor of mathematics prepared 20 questions for both section A and B. Peter get 80% correct. How many questions did Peter missed?
2. Student earned a grade of 80% on mathematics test that has 20 questions. How many did the student answered correctly? And what percentage of that not answered correctly?
3. John took a mathematics test and got 35 correct answers and 10 incorrect answers. What was the percentage of correct answers?
4. As a future teacher, is it necessary for a student-teacher to know how to determine the percentage? Explain it with supportive examples.

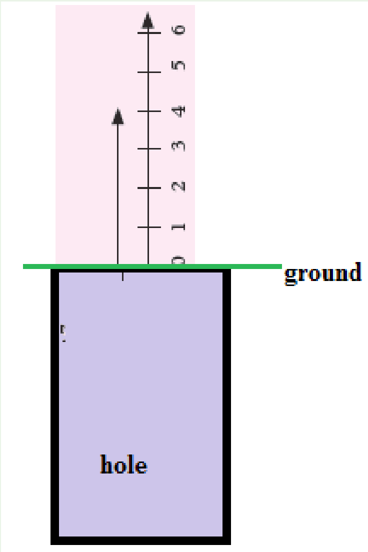
1.4 Negative numbers and related problems

Activity 1.4

1. The temperature of a juice in the bottle was $20^0 C$. If they put this juice in the fridge so that its temperature decreases by $30^0 C$; What is the temperature of this juice? What can you advice the child who wishes to drink that juice?



2. Suppose that you have a long ruler fixed from the hole and graduated such that the point 0 corresponds to the ground level as illustrated on the following figure.



What is the coordinate of the point position for an insect which is at 3 units below the ground level in a hole?

CONTENT SUMMARY

There are numerous instances where one comes across negative quantities, such as temperatures below zero or bank overdrafts. For example, if you have 3500Frw in your bank account and withdraw 6000Frw with an acceptable credit, your bank balance is -2500Frw . There are instances, however, where it is not usually possible to have negative quantities. For example, a firm's production level cannot be negative.

From the above activity, you have learnt that, you can need to use **negative or a positive numbers.**

For example, when measuring temperature, the value of the temperatures of the body or surrounding can be negative or positive. The normal body temperature is about $+37^0C$ and the temperature of the freezing mercury is about -39^0C .

Example

a) Eight students have an overdraft (scholarship advance) of 21,000Frw for each. What is their total bank balance?

Solution:

The total balance in the bank is 8 × (–21,000Frw) = –168,000Frw. The sign negative means that students have the credit to be paid.

b) Calculate $\frac{24}{-5} \div \frac{-32}{-10}$

Solution:

$$\frac{24}{-5} \div \frac{-32}{-10} = \frac{24}{-5} \times \frac{-10}{-32} = \frac{3}{1} \times \frac{2}{-4} = \frac{6}{-4} = -\frac{3}{2}$$

Application activity 1.4

Question1:

a) $\frac{(-10)\times(-5)\times(-6)}{(-3)\times(-2)} = ?$

b) $\frac{(-30)\times(+2)\times(-10)}{(-50\times(+2)} = ?$

c) Where negative numbers are applied in the real life? Do you think that computers of bank managers deal with negative numbers when operating loans for clients?

Question2:

A cylinder is ¼ full of water. After 60ml of water is added the cylinder is 2/3 full. Calculate the total volume of the cylinder.

Question3:

Between 1990 and 1997 the population of an Island fell by 4%. The population in 1997 was 201,600. Find the population in 1990.

1.5. Absolute value

1.5.1 Meaning of absolute value

Activity 1.5.1

1) Draw a number line and state the number of units found between

   a) 0 and -8 b) 0 and 8

   c) 0 and $\frac{1}{2}$ d) 4 and 17

2) Do you think that a distance can be expressed by a negative number?

CONTENT SUMMARY

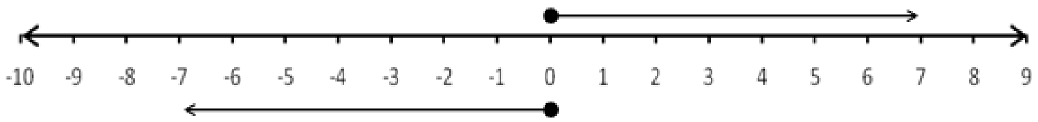
Absolute value of a number is the distance of that number from the origin (zero point) on a number line. The symbol $|\ |$ is used to denote the absolute value.

Example

7 is at 7 units from zero, thus the absolute value of 7 is 7 or $|7|=7$.

Also -7 is at 7 units from zero, thus the absolute value of -7 is 7 or $|-7|=7$.

So $|-7|=|7|=7$ since -7
and 7 are on equal distance from zero on number line.

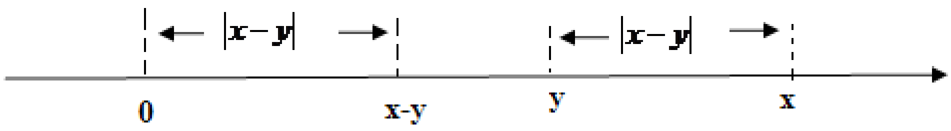


Note:

- The absolute value of zero is zero,
- The absolute value of a non-zero real number is a positive real number.
- Given that $|x|=k$ where k is a positive real number or zero, then $x=-k$ *or* $x=k$.
- $|x|=\begin{cases} x, & if\ x\geq 0 \\ -x, & if\ x<0 \end{cases}$

- Geometrically, $|x|$ represents the nonnegative distance from x to 0 on the real line.
- More generally, $|x-y|$ represents the nonnegative distance between the point x and y on the real line, since this distance is the same as that from the point $x-y$ to 0.(See the following figure)

$$|x-y| = \begin{cases} x-y, \text{ if } x \geq y \\ -(x-y) = y-x, \text{ if } x \leq y. \end{cases}$$



Example

Find x in the following:

a) $|x| = 5$ b) $|x| + 5 = 1$ c) $|x-4| = 10$

Solution

a) $|x| = 5,\ x = -5 \ \ or \ \ x = 5$

b) $|x| + 5 = 1$

$\Leftrightarrow |x| = 1 - 5 \Rightarrow |x| = -4$, This is impossible in the set $\mathbb{R}$ of real numbers.

There is no value of x since the absolute value of x must be a positive real number.

c) $|x-4| = 10$

$x - 4 = -10 \ \ or \ \ x - 4 = 10$

$x = -10 + 4 \ \ or \ \ x = 10 + 4$

$x = -6 \ \ or \ \ x = 14$

Example

Simplify

a) $-|40 - 12|$ b) $|4(-3) - (2)(5)|$ c) $|-4(-2)|$

Solution

a) $-|40-12|=-|28|=-28$

b) $|4(-3)-(2)(5)|=|-12-10|=|-22|=22$

c) $|-4(-2)|=|8|=8$

1.5.2 Properties of the Absolute Value

Activity 1.5.2

Evaluate and compare the following:

1) $|3|$ and $|-3|$ 2) $|3\times 5|$ and $|3|\times|5|$ 3) $|(-8)+5|$ and $|-8|+|5|$

1. Opposite numbers have equal absolute value.

$$|a|=|-a|$$

Example

$|5|=|-5|=5$

2. The absolute value of a product is equal to the product of the absolute values of the factors.

$$|ab|=|a||b|$$

Example

$|4(-6)|=|4||-6|$

$|4(-6)|=|-24|=24$

$|4||-6|=4\times 6=24$

3. The absolute value of a sum is less than or equal to the sum of the absolute values of the ends.

$$|a+b|\le|a|+|b|$$

Example

$$|-3+2| \le |-3|+|2|$$
$$|-1| \le 3+2$$
$$1 \le 5$$

Application activity 1.5

1. Find the value(s) of x: a) $|x|=6$ b) $|x+1|=0$

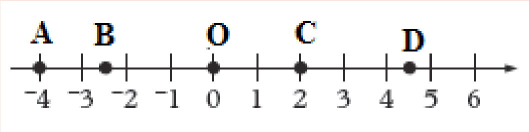
   c) $|x-3|-4=2$ d) $|2x+1|=4$ e) $|x-3|+3=5$

2. Simplify

   1) $|-5|$ 2) $|-4||-5|$ 3) $|-7|+|4|$ 4) $-|4\times 6|$ 5) $-|-6+8|$

3. Let a and b be the coordinates of points A and B, respectively, on a coordinate line. The distance between A and B, denoted by d(A,B) is defined by $d(A,B)=|b-a|$.

a) Refer to the figure below and determine the distance d(A,B), d(C,B) and d(O,A).

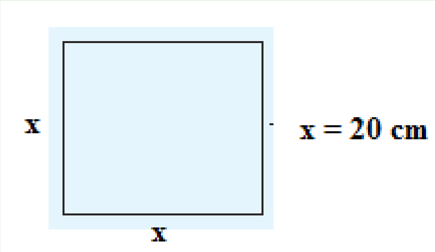


b) Dr. Makoma went from Kabgayi to Muhanga City on foot at the constant speed of 100m/min. If he used 60 minutes to go and 60 minutes to come back, explain to your colleague the distance covered by Dr. Makoma

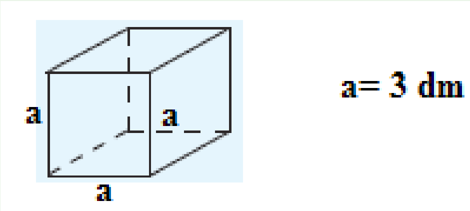
1.6. Powers and related problems

Activity 1.6

1. How can you find the area of the following sheet of paper cut in the form of a square?



2. Given a cube of the following form:



Determine the volume of this cube.

CONTENT SUMMARY

1.6.1 Meaning of power of a number

We call n^{th} power of a real number *a* that we note a^n , the product of *n* factors of *a.* that is

$$a^n = \underbrace{a \cdot a \cdot a \ldots a}_{n\ factors} \quad \begin{cases} n \text{ is an exponent} \\ a \text{ is the base} \end{cases}$$

Example

$$2^4 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{4\ factors} = 16 \qquad 3^3 = \underbrace{3 \cdot 3 \cdot 3}_{3\ factors} = 27$$

Notice

- $a^1 = a$
- $a^0 = 1, a \neq 0$

- If $a=0, a^0$ is not defined.

1.6.2 Properties of powers

Let $a,b\in\mathbb{R}$ and $m,n\in\mathbb{R}$

a) $a^m\cdot a^n=a^{m+n}$

In fact, $a^m\cdot a^n=\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\times\underbrace{a\cdot a\cdot a\cdots a}_{n\ factors}=\underbrace{a\cdot a\cdot a\cdots a}_{m+n\ factors}=a^{m+n}$

b) $\left(a^m\right)^n=a^{mn}$

In fact, $\left(a^m\right)^n=\underbrace{\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\times\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}\cdots\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}}_{n\ factors}=a^{mn}$

c) $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$

In fact,

$$\left(\frac{a}{b}\right)^m=\underbrace{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots\frac{a}{b}}_{m\ factors}=\frac{\underbrace{a\cdot a\cdot a\cdots a}_{m\ factors}}{\underbrace{b\cdot b\cdot b\cdots b}_{m\ factors}}=\frac{a^m}{b^m}$$

d) $\frac{1}{b^m}=b^{-m}$

In fact, $\frac{1}{b^m}=\frac{1}{b^m}^{m}=\left(\frac{1}{b}\right)^m=\left(b^{-1}\right)^m=b^{-m}$

e) $\frac{a^m}{a^n}=a^{m-n}$

In fact,

$$\frac{a^m}{a^n}=a^m\frac{1}{a^n}=a^m a^{-n}=a^{m-n}$$

f) $(ab)^m=a^m b^m$

In fact, $(ab)^m=\underbrace{ab\cdot ab\cdots ab}_{m\ factors}=\underbrace{a\cdot a\cdots a}_{m\ factors}\times\underbrace{b\cdot b\cdots b}_{m\ factors}=a^m b^m$

These properties help us to simplify some powers.

There is no general way to simplify the sum of powers, even when the powers have the same base. For instance, $2^5+2^3=32+8=40$, and 40 is not an integer power of 2. But some products or ratios of powers can be simplified using repeated multiplication models of a^n.

Example

a) $2^4\cdot 2^3\cdot 4=2^4\cdot 2^3\cdot 2^2=2^9=512$

b) $a^4\cdot b^3\cdot a^5\cdot b^8=a^4\cdot a^5\cdot b^3\cdot b^8=a^9\cdot b^{11}$

$a^9\cdot b^{11}$ cannot be simplified further because the bases are different.

c) $\frac{y^9}{y^2}=y^{9-2}=y^7$

1.6.3 The compound interest formula

Considering the case in which P is invested in a bank for *n* interest periods per year at the rate r expressed as a decimal. The accumulated amount of money A after the time t (number of years P is invested) is given by:

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$

where *n* is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

Note: When the interest rate is compounded per year, $A=P(1+r)^t$ where r is expressed as a decimal for example $r=9\%=0.09$.

When the interest rate is compounded monthly, $A=P\left(1+\frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

This is called the compound interest formula. It is conveniently used in solving problems of compound interest especially those involving long periods of investments or payment.

In this method, the accrued compound interest is obtained by subtracting the original principal from the final amount.

Thus, Compound interest = Accumulated amount (A) – Principal (P).

Note that the principal and the interest earned increased after each interest period.

We can also deduce that $I = A - P$

Examples

Question1	Question 2
A trader deposited $63000Frw$ in a fixed deposit account with a local bank which attracted an interest of 8%p.a. compound interest. Find: (a) the total amount after 4 years; (b) The compound interest. **Solution:** $P = 63000Frw; n = 4; r = 8\% p.a.$ $A = P\left(1+\frac{r}{100}\right)^n$ a) $A = 63000\left(1+\frac{8}{100}\right)^4$ $= 63000(1.08)^4 = 63000\times 1.36048896$ $= 85710.80Frw$ b) $I = (85710.80 - 63000)Frw$ $= 22710.80Frw$	Find the compound interest earned on $15000Frw$ invested for 3years, at 20%p.a. compounded quarterly. **Solution:** Here, each year has 4 interest period (quarterly) i.e in 3years, there are 12 interest period $(3\times 4 = 12)$. The rate, $r\% = 20 \div 4 = 5\% p.a.$ $A = 15000\left(1+\frac{5}{100}\right)^{12}$ $A = 15000(1.05)^{12} = 26937.84Frw$ $I = 26937.84 - 15000 = 11937.84Frw$

Application activity 1.6

1) Simplify

a) x^3x^2 b) $(xy^3)^2 + 4x^2y^6$ c) $\frac{6xy^2}{3xy}$ d) $\frac{ab}{a^3} - \frac{a^3b^2}{a^5b}$ e) $\frac{yx}{4xy}$

2) Referring to your real life experience, where powers are used?

1.7 Roots (radicals) and related problems

1.7.1 Meaning of radicals

Activity 1.7.1

1) Evaluate the following powers using a calculator

a) $\sqrt{81}$ b) $(216)^{\frac{1}{3}}$ c) $(-27)^{\frac{1}{3}}$ d) $\sqrt[4]{16}$

2) Using examples, explain whether there is a difference between the square and the square root of a number. How do you calculate a square root of a given number?

CONTENT SUMMARY

The n^{th} root of a real number is $\frac{1}{n}$ power of that real number. It is noted by $\sqrt[n]{b}$, $b \in \mathbb{R}, n \in \mathbb{N} \setminus \{1\}$. $\forall a, b \in \mathbb{R}, \sqrt[n]{b} = a \Leftrightarrow b^{\frac{1}{n}} = a \Leftrightarrow b = a^n$

$$\begin{cases} n \text{ is called the index} \\ b \text{ is called the base or radicand} \\ \sqrt[n]{} \text{ is called the radical sign} \end{cases}$$

Example

a) $\sqrt[3]{27} = (27)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = 3^{3 \times \frac{1}{3}} = 3$

b) $\sqrt[4]{16} = (16)^{\frac{1}{4}} = (2^4)^{\frac{1}{4}} = 2$

If $n = 2$, we say square root and $\sqrt[2]{b}$ is written as $\sqrt{b}$. Here b must be a positive real number or zero.

If $n = 3$, we say cube root noted $\sqrt[3]{b}$. Here b can be any real number.

If $n = 4$, we say 4th root noted $\sqrt[4]{b}$. Here b must be a positive real number or zero.

Generally, we say n^{th} root noted $\sqrt[n]{b}$. Here if n is even, b must be a positive real number or zero and if n is odd b can be any real number.

Example

$\sqrt{-9}$ is not defined in $\mathbb{R}$ as the index in radical is even but

$$\sqrt[3]{-27}=(-27)^{\frac{1}{3}}=\left[(-3)^{3}\right]^{\frac{1}{3}}=-3$$

Properties of radicals

$\forall n\in\mathbb{N}\setminus\{1\}, m\in\mathbb{R}$

a) $\sqrt[n]{a^m}=a^{\frac{m}{n}}$

In fact, $\sqrt[n]{a^m}=\left(a^m\right)^{\frac{1}{n}}=a^{m\times\frac{1}{n}}=a^{\frac{m}{n}}$

b) $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$

In fact, $\sqrt[n]{ab}=(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}b^{\frac{1}{n}}=\sqrt[n]{a}\sqrt[n]{b}$

c) $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

In fact, $\sqrt[n]{\frac{a}{b}}=\left(\frac{a}{b}\right)^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

d) $\sqrt[n]{\sqrt[m]{a}}=\sqrt[nm]{a}=a^{\frac{1}{nm}}$

In fact,

$$\sqrt[n]{\sqrt[m]{a}}=\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}}=a^{\frac{1}{m}\times\frac{1}{m}}=a^{\frac{1}{mn}}=\sqrt[mn]{a}$$

Example

Simplify

a) $\sqrt{46656}$ b) $\sqrt[3]{\sqrt{64}}$ c) $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2}$ d) $\sqrt{\frac{36}{81}}$

Solution

a) $\sqrt{46656}=\sqrt{6^6}=6^3=216$

b) $\sqrt[3]{\sqrt{64}}=\sqrt[6]{64}=\sqrt[6]{2^6}=2$

c) $\sqrt[3]{ab}\times\sqrt[3]{a^2b^2}=\sqrt[3]{a^3b^3}=\sqrt[3]{(ab)^3}=ab$

d) $\sqrt{\frac{36}{81}}=\frac{\sqrt{36}}{\sqrt{81}}=\frac{6}{9}=\frac{2}{3}$

Application activity 1.7.1

Simplify

a) $\sqrt{a}\times\sqrt{ab^3}\times\sqrt{bc^2}$

b) $\sqrt[3]{abc}\times\sqrt[3]{a^2b^2c^2}$

c) $\sqrt[3]{\frac{8}{27}}$

d) $\sqrt{\sqrt[4]{x^8}}$

e) $\sqrt{\frac{x^3y^4}{4x}}$

1.7.1 Operations on radicals

Activity 1.7.2

Simplify the following

1) $\sqrt{18}+\sqrt{2}$ 2) $\sqrt{12}-3\sqrt{3}$ 3) $\sqrt{2}\times\sqrt{3}$ 4) $\frac{\sqrt{6}}{\sqrt{2}}$

Addition and subtraction

When adding or subtracting the radicals we may need to simplify if we have similar radicals. Similar radicals are the radicals with the same indices and same bases.

Example

$$\sqrt{2}+\sqrt{8}=\sqrt{2}+\sqrt{2\times4}=\sqrt{2}+\sqrt{2}\times\sqrt{4}=\sqrt{2}+2\sqrt{2}=3\sqrt{2}$$

$$\sqrt{3}-\sqrt{27}=\sqrt{3}-\sqrt{3\times9}=\sqrt{3}-\sqrt{3}\times\sqrt{9}=\sqrt{3}-3\sqrt{5}=-2\sqrt{3}$$

Application activity 1.7.2

Simplify

1) $\sqrt{20}+\sqrt{5}$ 2) $4\sqrt{3}-\sqrt{12}$ 3) $5\sqrt{7}-\sqrt{28}$

4) $\sqrt{18}\times\sqrt{8}$ 5) $\sqrt{45}+\sqrt{80}+\sqrt{180}$ 6) $\sqrt{108}-\sqrt{48}$

1.7.3 Rationalizing radicals

Activity 1.7.3

Make the denominator of each of the following rational

1) $\frac{1}{\sqrt{2}}$ 2) $\frac{2-\sqrt{3}}{2\sqrt{5}}$ 3) $\frac{2}{1-\sqrt{6}}$ 4) $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$

Rationalizing is to convert a fraction with an irrational denominator to a fraction with rational denominator. To do this,

1) If the denominator involves radicals, we multiply the numerator and denominator by the conjugate of the denominator when the denominator is made by two terms.

Some examples of conjugate are:

The conjugate of $a\pm\sqrt{b}$ is $a\mp\sqrt{b}$.

2) Or we multiply the numerator and the denominator by the same radical when the denominator is formed by one term.

For $\sqrt{a}$ use $\sqrt{a}$

For $a\sqrt{b}$ use $\sqrt{b}$

Remember that $(a+b)(a-b)=a^2-b^2$

Example

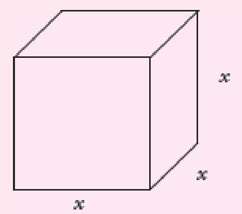
a) $$\frac{1}{1+\sqrt{2}}=\frac{1-\sqrt{2}}{\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)}=\frac{1-\sqrt{2}}{1-2}=\frac{1-\sqrt{2}}{-1}=\sqrt{2}-1$$

b) $$\frac{\sqrt{2}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}=\frac{\sqrt{10}+\sqrt{6}}{5-3}=\frac{\sqrt{10}+\sqrt{6}}{2}$$

c) $\frac{\sqrt{3}+\sqrt{7}}{4\sqrt{2}}=\frac{(\sqrt{3}+\sqrt{7})\sqrt{2}}{(4\sqrt{2})\sqrt{2}}=\frac{\sqrt{6}+\sqrt{14}}{8}$

Application activity 1.7.1

1. On a clear day, the distance d (in meters) that can be seen from the top of a tall building of height h (in meter) can be approximated by $d=1.2\sqrt{h}$. Approximate the distance that can be seen from Kigali Tower which is 30m tall.
2. A cube has a total surface area of 96 square cm. Find the volume of that cube.



3. Rationalize the denominator

a) $\frac{2\sqrt{2}}{4+3\sqrt{3}}$ b) $\frac{a-\sqrt{b}}{\sqrt{d}}$ c) $\frac{3\sqrt{3}+2\sqrt{2}}{1+2\sqrt{2}}$

4. Carry out a research in the library or on internet and discover other real life problems in which you can apply the powers and radicals. Discuss them with your classmates.

5. Discuss orally how to determine the square of a square root of a number. Did you ever need to use a square root in your real life experience? Explain the answer.

1.8 Decimal logarithms and related problems

Activity 1.8

1) What is the real number at which 10 must be raised to obtain:

a)1 b) 10 c) 100 d) 1000 e) 10000 f) 100000

2) Explain your classmate how you can find the number x if $x^3=64$.

CONTENT SUMMARY

The **decimal logarithm** of a positive real number *x* is defined to be a real number *y* for which 10 must be raised to obtain *x*. We write $\forall x > 0, \quad y = \log x$

as $x = 10^y$; $\log x$ is the same as $\log_{10} x$ and is defined for all positive real numbers only. 10 is the base of this logarithm. In general notation we do not write this base for decimal logarithm.

In the notation $y = \log x$, *x* is said to be the **antilogarithm of** *y*.

Example

log (100) = ?

We are required to find the power to which 10 must be raised to obtain 100.

So log(100) = 2 as 100=10^2

$y = \log x$ means $10^y = x$

Be careful! $\log 2x + 1 \neq \log(2x+1)$

$$\log 2x + 1 = (\log 2x) + 1$$

Since logs are defined using exponentials, any "log *x*" has an equivalent "exponent" form, and vice-versa.

Example: 1) $\log 10^5 = 5$ 2) $\log(0.01) = log 10^{-2} = -2$

Properties

$\forall a, b \in]0, +\infty[$

a) $\log ab = \log a + \log b$

b) $\log \frac{1}{b} = -\log b$

c) $\log \frac{a}{b} = \log a - \log b$

d) $\log a^n = n \log a$

e) $\log \sqrt{a} = \frac{1}{2} \log a$

f) $\log \sqrt[n]{a} = \frac{1}{n} \log a$

g) If $a > b, \quad \log a > \log b$, If $a = b, \quad \log a = \log b$

h) $\log \sqrt[n]{a^m} = \frac{m}{n} \log a$

Examples:

1) Calculate in function of $\log a, \log b \; and \; \log c$

a) $\log a^2b^2$

b) $\log \frac{ab}{c}$

c) $\log \frac{ab}{\sqrt{c}}$

Solution

a) $\log a^2b^2 = \log(ab)^2$

$$= 2\log ab$$
$$= 2(\log a + \log b)$$

b) $\log \frac{ab}{c} = \log ab - \log c$

$$= \log a + \log b - \log c$$

c) $\log \frac{ab}{\sqrt{c}} = \log ab - \log \sqrt{c}$

$$= \log a + \log b - \log(c)^{\frac{1}{2}}$$
$$= \log a + \log b - \frac{1}{2}\log c$$

2) Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.7$. Calculate

a) $\log 6$

b) $\log 0.9$

Solution

a) $\log 6 = \log(2 \times 3)$

$$= \log 2 + \log 3$$
$$= 0.30 + 0.48$$
$$= 0.78$$

b) $\log 0.9 = \log\frac{9}{10}$

$= \log 9 - \log 10$

$= \log 3^2 - \log(2\times 5)$

$= 2\log 3 - \log 2 - \log 5$

$= 2(0.48) - 0.30 - 0.7$

$= -0.04$

Co-logarithm

Co-logarithm, sometimes shortened to **colog**, of a number is the logarithm of the reciprocal of that number, equal to the negative of the logarithm of the number itself,

$$\operatorname{colog} x = \log\left(\frac{1}{x}\right) = -\log x$$

Example:

$\operatorname{colog} 200 = -\log 200 = -2.3010$

Change of base formula

If u $(u>0)$ and if a and b are positive real numbers different from 1,

$$\log_b u = \frac{\log_a u}{\log_a b}$$

This means that if you have a logarithm in any other base, you can convert it in the decimal logarithm in the following way where $a = 10$:

$$\log_b u = \frac{\log_{10} u}{\log_{10} b} = \frac{\log u}{\log b}.$$

This is for example: $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} \approx 2.322$

There is another special logarithm called natural logarithm which has the base a number $e \approx 2.71828$. This logarithm is written as : $\log_e x = \ln x$.

Example:

How long will it take10,000Frw to double it on account earning 2% compounded quarterly?

Solution

For this problem, we'll use the compound Interest formula,

$F = P(1+i)^n$ where F is the final value, P the initial value of investment.

Since we want to know how long it will take, let *t* represent the time in years. The number of compounding periods is four times the time or $n = 4t$. The original amount is $P = 10,000$ and the future value is the double or F = 20,000.

The interest rate per period is $i = \frac{0.02}{4} = 0.005$

When these values are substituted into the compound interest formula, we get the exponential equation

$$20000 = 10000(1.005)^{4t}$$

To solve this equation for *t*, isolate the exponential factor by dividing both sides by 10,000 to give

$$2 = (1.005)^{4t}$$

Convert this exponential form to logarithm form and divide by 4

$$4t = \log_{1.005}(2)$$

$$\text{t} = \frac{\log_{1.005}(2)}{4}$$

To find an approximate value, use the Change of Base Formula to convert to a natural logarithm (or a common logarithm)

$$t = \frac{\frac{\ln(2)}{\ln(1.005)}}{4} \approx 34.7\,years$$ or $$t = \frac{\frac{\log(2)}{\log(1.005)}}{4} \approx 34.7\,\text{years}.$$

It is interesting to note that the starting amount is irrelevant when doubling. If we started with P dollars and wanted to accumulate 2P at the same interest's rate and compounding periods, we would need to solve

$$2P = P(1.005)^{4t}.$$

This reduces to the same equation as above $2 = (1.005)^{4t}$ when both sides are divided by P. This means it takes about 37.4 years to double any amount of money at an interest rate of 2% compounded quarterly.

Application activity 1.8

1. Without using calculator, compare the numbers a and b.

   d) $a = 3\log 2$ and $b = \log 7$

   e) $a = \log 2 + \log 40$ and $b = 4\log 2 + \log 5$

   f) $a = 2\log 2$ and $b = \log 16 - \log 3$

2. Given that $\log 2 = 0.30$, $\log 3 = 0.48$ and $\log 5 = 0.69$. Calculate

   a) $\log 150$ b) $\log \frac{9}{2}$ c) $\log 0.2 + \log 10$

3. Find co-logarithm of

   a) 100 b) 42 c)15

4. How long will 70,000Frw take to accumulate to 100,000Frw if it is invested at 11%?

5. Mr. Mateso was astonished when he was told that his money \$200 in the Bank was raised to a certain power and became \$8,000,000 after 3 years. Explain how he can discover that t power.

1.9 Important applications of arithmetic

Activity 1.9

Make a research in the library or on internet and categorize problems of Economics and Finance that are easily solved with the use of arithmetic.

Focus on the following: Elasticity of demand, Arc of elasticity for demand, Simple interest and compound interest, Final value of investment.

CONTENT SUMMARY

1. Elasticity' of demand and Arc of elasticity for demand

Elasticity' of demand:

Price elasticity of demand is a measure of the responsiveness of demand to changes in price. It is

usually defined as

$$e = (-1)\frac{\%change\,in\,quantity\,demand}{\%change\,in\,price}$$

Arc of elasticity for demand

The (–1) in this definition ensures a positive value for elasticity as either the change in price or the change in quantity will be negative. When there are relatively large changes in price and quantity it is best to use the concept of 'arc elasticity' to measure elasticity along a section of a demand schedule.

This takes the changes in quantity and price as percentages of the averages of their values before and after the change.

Thus arc elasticity is usually defined as

$$arc\,e = (-1)\frac{\dfrac{change\,in\,quantity}{0.5\,(1st\,quantity + 2nd\,quantity)}.100}{\dfrac{change\,in\,price}{0.5\,(1st\,price + 2nd\,price)}.100}$$

Although a positive price change usually corresponds to a negative quantity change, and vice versa, it is easier to treat the changes in both price and quantity as positive quantities. This allows the (–1) to be dropped from the formula. The 0.5 and the 100 will always cancel top and bottom in arc elasticity calculations.

Thus we are left with

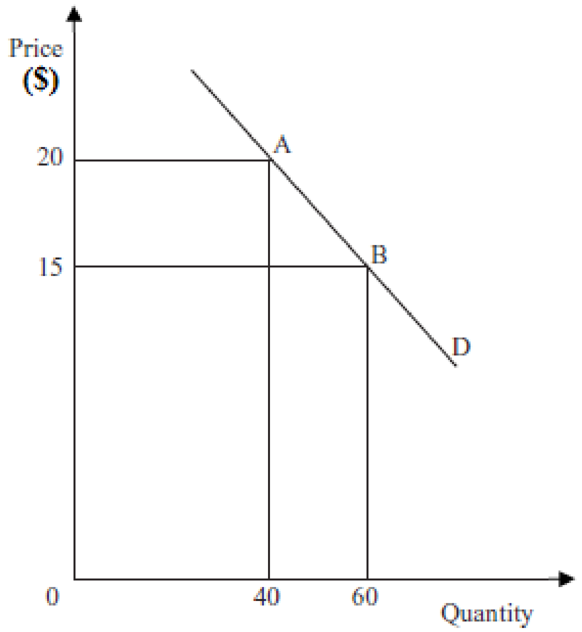
$$arc\,e = (-1)\frac{\dfrac{change\,in\,quantity}{0.5\,(1st\,quantity + 2nd\,quantity)}}{\dfrac{change\,in\,price}{0.5\,(1st\,price + 2nd\,price)}}$$

as the formula actually used for calculating price arc elasticity of demand

Example:

Calculate the arc elasticity of demand between points A and B on the demand schedule shown

In the following figure:



Solution:

Between points A and B price falls by 5 from 20 to 15 and quantity rises by 20 from 40 to 60. Using the formula defined above:

$$arce = \frac{\frac{20}{40+60}}{\frac{5}{20+15}} = \frac{7}{5}$$

2. Simple interest

Simple interest is the amount charged when one borrows money or loan from a financial institution which accrue yearly.

This interest is a fixed percentage charged on money/loan that is not yet paid.

This interest is calculated based on the original principal or loan and is paid at regular intervals

From, above example, we see that when the principal (P), rate in percentage (R) and time in year (T) are given, then simple interest (I) for the given period is given by

$$I = P.\frac{R}{100}.T = \frac{PRT}{100}$$

The total amount (A) paid back by the borrower or the financial institution on the expiry of the interest period is the sum of the principal and the interest earned (I).

Thus, $A = P + I$

Example:

1. Find the simple interest earned from 3 400 *FRW* borrowed for 3years at the rate of 10%p.a.

Solution: $Interst, I = \frac{PRT}{100} = \frac{3400 \times 10 \times 3FRW}{100} = 1020FRW$

2. Gatete borrowed 32 000FRW from a lending institution to start a business. If the institution charged interest at a rate of 8%p.a., calculate the simple interest and the total amount she eventually paid back after 4years.

Solution: $Interest, I = \frac{PRT}{100} = \frac{32000 \times 8 \times 4FRW}{100} = 10240FRW$

$Amount, A = (32000 + 10240)FRW = 42\,240FRW$

3. The compound interest *(developed above)*

Considering the case in which P is invested in a bank for *n* interest periods per year at the rate r expressed as a decimal. The accumulated amount A after the time t (number of years P is invested) is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where *n* is the number of interest periods per year, r is the interest rate expressed as decimal, A is the amount after t years.

Note: When the interest rate is compounded per year, $A = P(1+r)^n$ where r is expressed as a decimal for example $r = 9\% = 0.09$. When the interest rate is compounded monthly, $A = P\left(1 + \frac{r}{12}\right)^{12.t}$ where t is the number of years and r expressed as a decimal.

4. Calculating the final value of an investment

Consider an investment at compound interest where:

P is the initial sum invested, A is the final value of the investment, r is the interest rate per time period (as a decimal fraction) and *n* is the number of time periods.

The value of the investment at the end of each year will be $1+r$ times the sum invested at the start of the year.

Thus, for any investment,

The value after one year $= P(1+r)$

Value after 2 years $= P(1+r)(1+\mathrm{r}) = \mathrm{P}(1+\mathrm{r})^2$

Value after 3 years $= P(1+r)(1+\mathrm{r})(1+r) = \mathrm{P}(1+r)^3 \quad etc$

We can see that each value is multiplied by $(1+r)$ to the power of number of years that the sum is invested. Thus, after n years the initial sum P is multiplied by $(1+r)^n$ to find the The formula for the final value A. Thus, after an investment of P money, for n time periods at interest rate r , the final value is therefore $A = P(1+r)^n$.

Examples:

1) If $600 is invested for 3 years at 8% then the known values for the formula are

$P = \$600; \quad n = 3; \quad r = 8\% = 0.08$. Thus the final sum will be

$$A = P(1+r)^n = 600(1.08)^3 = \$755.83$$

2) If $4,000 is invested for 10 years at an interest rate of 11% per annum what will the final value of the investment be?

Solution:

P = $4,000, n = 10   r = 11% = 0.11

$$A = P(1+r)^n = 4000(1.11)^{10} = 11,253.68$$

The final value of the investment be $\$11,253.68$

5. Time periods, initial amounts and interest rates

The formula for the final sum of an investment contains the four variables *F*, *A*, *i* and *n*.

So far we have only calculated *F* for given values of *A*, *i* and *n*. However, if the values of any three of the variables in this equation are given then one can usually calculate the fourth.

Initial amount

A formula to calculate *A*, when values for *F*, *i* and *n* are given, can be derived as follows.

Since the final sum formula is

$$A = P(1+r)^n,$$

then, dividing through by $(1+r)^n$ (we get the initial sum formula

$$P = \frac{A}{(1+r)^n} \quad or\ P = A(1+r)^{-n}$$

Example

How much money needs to be invested now in order to accumulate a final sum of $12,000 in 4 years' time at an annual rate of interest of 10%?

Solution:

Using the formula derived above, the initial amount is

$$P = A(1+r)^{-n} = 12,000(1.1)^{-4} = \$8,196.16$$

What we have actually done in the above example is find the sum of money that is equivalent to $12,000 in 4 years' time if interest rates are 10%. An investor would therefore be indifferent between (a) $8,196.16 now and (b) $12,000 in 4 years' time. The $8,196.16 is therefore known as the 'present value' (PV) of the $12,000 in 4 years' time. We shall come back to this concept in the next few sections when methods of appraising different types of investment project are explained.

Time period

Calculating the time period is rather more tricky than the calculation of the initial amount.

From the final sum formula.

$A = P(1+r)^n$, then $\frac{A}{P} = (1+r)^n$

If the values of A, P and *r* are given and one is trying to find *n* this means that one has to work out to what power (1 + r) has to be raised to equal $\frac{A}{P}$. One way of doing this is via logarithms.

Example

For how many years must $1,000 be invested at 10% in order to accumulate $1,600?

Solution

P= $1,000, *A* = $1,600, r = 10% = 0.1

Substituting these values into the formula

$\frac{A}{P} = (1+r)^n$ then, $\frac{1600}{1000} = (1+0.1)^n$

We get $1.6 = (1.1)^n$

Since to find the n^{th} power of a number its logarithm must be multiplied by *n*. Finding logs, this means that our equation becomes

$\log 1.6 = n \log(1.1)$

And $n = \frac{\log(1.6)}{\log(1.1)} = 4.93$. Given that 4,93 years is approximately 5years,

If investments must be made for whole years then the answer is 5 years.

This answer can be checked using the final sum formula $\mathrm{A} = \mathrm{P}(1+r)^n = 1000(1.1)^5 = 1{,}610.51 \approx 1600$

If the $1,000 is invested for a full 5 years then it accumulates to just over $1,600, which checks out with the answer above.

A general formula to solve for *n* can be derived as follows from the final sum formula:

$A = P(1+r)^n$, $\frac{A}{P} = (1+r)^n$ and $n = \frac{\log(A/P)}{\log(1+r)}$

An alternative approach is to use the iterative method and plot different values on a spreadsheet. To find the value of *n* for which $1.6 = (1.1)^n$.

this entails setting up a formula to calculate the function $y = (1.1)^n$ and then computing it for different values of *n* until the answer 1.6 is reached. Although some students who find it difficult to use logarithms will prefer to use a spreadsheet, logarithms are used in the other examples in this section. Logarithms are needed to analyze other concepts related to investment and so you really need to understand how to use them.

Example:

1) How many years will \$2,000 invested at 5% take to accumulate to \$3,000?

Solution:

$P = 2,000$; $A = 3,000$; $r = 5\% = 0.05$

Using these given values in the time period formula derived above gives

$$n = \frac{\log(A/P)}{\log(1+r)} = \frac{\log 1.5}{\log 1.05} = 8.34$$

This money will need to be invested in 8.34 years.

2) How long will any sum of money take to double its value if it is invested at 12.5%?

Solution

Let the initial sum be *A*. Therefore the final sum is

$A = 2P$ and $r = 12.5\% = 0.125$

Substituting these value for *A* and r into the final sum formula $A = P(1+r)^n$, we find $2A = A(1.125)^n$

Or $2 = (1.125)^n$ which gives $n = \dfrac{\log 2}{\log 1.125} = 5.9$

For any sum of money, it takes 5.9 years to double its value if it is invested at 12.5%.

Interest rates

A method of calculating the interest rate on an investment is explained in the following example.

If \$4,000 invested for 10 years is projected to accumulate to \$6,000, what interest rate is used to derive this forecast?

Solution

$P = 4{,}000$ A = 6,000 and $n = 10$

Substituting these values into the final sum formula

$A = P(1+r)^n$ gives $6000 = 4000(1+r)^{10}$

$1.5 = (1+r)^{10}$

$1 + r = \sqrt[10]{1.5}$

$r = 4.14\%$

A general formula for calculating the interest rate can be derived. Starting with the familiar final sum formula

$$A = P(1+r)^n \Leftrightarrow \frac{A}{P} = (1+r)^n \Leftrightarrow r = \left(\frac{A}{P}\right)^{1/n} - 1$$

Therefore, $r = \left(\frac{A}{P}\right)^{1/n} - 1$

Example:

1) At what interest rate will 3000Frw accumulate to 10,000Frw after 15 years?

Solution:

$$r = \left(\frac{A}{P}\right)^{1/n} - 1 = \left(\frac{10000}{3000}\right)^{1/15} - 1 = 0.083574 = 8.36\%$$

2) An initial investment of 50,000Frw increases to 56,711.25Frw after 2 years. What interest rate has been applied?

Solution

Given that P = 50,000 A = 56,711.25 n = 2 and $r = \sqrt[n]{\frac{A}{P}} - 1$

We have $r = \sqrt[2]{\frac{56{,}711.25}{50{,}000}} - 1 = 0.065 = 6.5\%$

Application activity 1.9

1. An initial investment of £50,000 increases to £56,711.25 after 2 years. What interest rate has been applied?

2. Consider the formula for $A = P\left(1+\frac{r}{n}\right)^{nt}$ and the case of the continuously compounded interest where n (the number of interest periods per year) increases without bound (ie $n \to \infty$) and prove that the amount after t years is $A = Pe^{rt}$ where P is the principal, r is an interest rate expressed as decimal, t the number of years P is invested.

1.10. END UNIT ASSESSMENT

1. Why is it necessary for a student teacher to study arithmetic? Explain your answer on one page and be ready to defend your arguments in a classroom discussion;

2. The price of a house was 2000000Frw in the year 2000. At the end of each year the price has increased by 6%.

a) Find the price of the house after one year

b) Find the price of that house after 3 years

c) Find the price that such a house should have in this year.

UNIT: 2

EQUATIONS AND INEQUALITIES

Key Unit competence: Model and solve daily life problems using linear, quadratic equations or inequalities

2.O. Introductory Activity

1) By the use of library and computer lab, do the research and explain the linear equation.
2) If x is the number of pens for a learner, the teacher decides to give him/her two more pens. What is the number of pens will have a learner with one pen?
   a) Complete the following table called table of value to indicate the number $y = f(x) = x + 2$ of pens for a learner who had x pens for $x \geq 0$.

x	-2	-1	0	1	2	3	4
$y = f(x) = x + 2$			2				
(x,y)			(0,2)				

   b) Use the coordinates of points obtained in the table and complete them in the Cartesian plan.
   c) Join all points obtained. What is the form of the graph obtained?
   d) Suppose that instead of writing $f(x) = x + 2$ you write the equation $y = x + 2$. Is this equation a linear equation or a quadratic equation? What is the type of the inequality "$x + 2 \geq 0$"?
3) Find out an example of problem from the real life situation that can be solved by the use of linear equation in one unknown.

2.1 Linear equations in one unknown and related problems

Activity 2.1

1) Assume that in a competitive market, the supply schedule is $p = 60 + 0.4q$ where p is a price function and q is the quantity supplied. Is the price increasing or decreasing.
   a) Find the price for q = 600 units.
   b) What is the value of p for $q = 0$. What does it mean for an industry which is producing a certain good and has just fixed a price p_0 at the beginning ($q = 0$). Can you justify this price p_0.
2) Solve the equation $3 - x = 3$
3) Solve the following inequality. $2x - 1 > x + 3$ Express the solution set in terms of interval and present solution on graph.

CONTENT SUMMARY

An equation is a mathematical statement expressing the equality of two quantities or expressions. Equations are used in every field that uses real numbers. A linear equation is an equation of a straight line.

Consider the statement $x - 1 = 0$, this statement is equation as there is two quantities to be equal and is true for the value $x = 1$. The value $x = 1$ is called the solution of the statement $x - 1 = 0$ the number 1 is called the root of the equation. Thus, to find a solution to the given equation is to find the value that satisfies that equation.

To do this, rearrange the given equation such that variables will be in the same side and constants in the other side and then find the value of the variable.

Example:

Solve in set of real numbers the following equations

a) $x + 2 = 10$

b) $3 + x = 9 - 2x$

c) $x = 16 - x$

Solutions:

a) $x + 2 = 10$

$x = 10 - 2$

$x = 8$

b) $3 + x = 9 - 2x$

$x + 2x = 9 - 3$

$3x = 6$

$x = \frac{6}{3}$

$x = 2$

$S = \{2\}$

c) $x = 16 - x$

$x + x = 16$

$2x = 16$

$x = \frac{16}{2}$

$x = 8$

$S = \{8\}$

Real life problems involving linear equations

Problems which are expressed in words are known as problems or applied problems. A word or applied problem involving unknown number or quantity can be translated into linear equation consisting of one unknown number or quantity. The equation is formed by using conditions of the problem. By solving the resulting equation, the unknown quantity can be found.

In solving problem by using linear equation in one unknown the following steps can be used:

i) Read the statement of the word problems

ii) Represent the unknown quantity by a variable

iii) Use conditions given in the problem to form an equation in the unknown variable

iv) Verify if the value of the unknown variable satisfies the conditions of the problem.

Examples

1) The sum of two numbers is 80. The greater number exceeds the smaller number by twice the smaller number. Find the numbers.

Solution

Let the smaller number be x

Therefore the greater number be $80 - x$

According to the problem,

$(80 - x) - x = 2x$

$80 - x - x = 2x$

$80 - 2x = 2x$

$80 = 2x + 2x$

$80 = 4x$

$4x = 80$

$x = \frac{80}{4}$

$x = 20$

Now substitute the value of $x = 20$ in $80 - x$ we get $80 - 20 = 60$

Therefore, the smaller number is 20 and the greater number is 60

2) A boat covers a certain distance downstream in 2 hours and it covers the same distance upstream in 3 hours. If the speed of the stream is $^{2km}/_{hr}$ Find the speed of the boat.

Solution

Let the speed of the boat be $^{xkm}/_{hr}$

Speed of the stream=$2km/hr$

Speed of the boat downstream = $(x + 2)\, km/hr$

Speed of the boat upstream = $(x - 2)\, km/hr$

Distance covered in both the cases in same.

$2(x + 2) = 3(x - 2)$

$2x + 4 = 3x - 6$

$2x - 3x = -6 - 4$

$-x = -10$

$x = 10$

Therefore, the speed of the boat is $10km/hr$

Product equation

The equation in the form $(ax + b)(cx + d) = 0$ is product equation since the product of factors is null (zero) either one of them is zero. To solve this we proceed as follows:

$(ax + b)(cx + d) = 0$
$ax + b = 0 \; or \; cx + d = 0$

$x = \frac{-b}{a} \qquad or \;\; x = \frac{-d}{c}$

Example

Solve the equation $(2x + 4)(x - 1) = 0$

Solution:

$2x + 4 = 0$ or $x - 1 = 0$

$x = -2 \qquad or \;\; x = 1$

Fractional equation of the first degree

The general form $\frac{ax+b}{cx+d} = 0$ to find solution of this equation we need to have the condition of existence $cx + d \neq 0$ and $\frac{ax+b}{cx+d} = 0 \Rightarrow ax + b = 0$

We solve $ax + b$ and we take the value(s) which verify the condition of existence

Example

Solve $\frac{2x-6}{x+1}=0$

Solution:

The existence condition is $x+1\neq 0$

$2x-6=0$

$2x=6$

$x=3$

$S=\{3\}$

Application activity 2.1

1) Solve in set of real numbers:

   a) $4x + 5 = 20 + x$

   b) $x-31=50-8x$

   c) $\frac{2x+5}{x-6}=4$

2) Mr. Peter wants to fence his garden of rectangle form where the length of that rectangle is twice its breadth. If the perimeter is 72 metres, help Peter to find the length and breadth of the rectangle.

3) The sum of two numbers is 25. One of the numbers exceeds the other by 9. Explain how you can determine these numbers.

4) Find the number whose one fifth is less than the one fourth by 3.

2.2 Linear inequalities in one unknown and related real life problems

Activity 2.2

1) Find the value(s) of x such that the following statements are true

   a) $x < 5$ b) $x > 0$ c) $-4 < x < 12$

2) Use library and computer lab to do the following:

   a) Discuss the difference between linear equations and linear inequalities

   b) Find out example of linear inequalities and try to solve them.

   c) Examples on how linear inequalities are applied when solving real word problems

CONTENT SUMMARY

2.2.1 Meaning of an inequality

The statement $x+3=10$ is true only when $x=7$. If *x* is replaced by 5, we have a statement $5+3=10$ which is **false**. To be true we may say that 5 + 3 is less than 10 or in symbol $5+3<10$. If *x* is replaced by 8, the statement 8+3=10 is also **false**. In those two cases we no longer have equality but inequality.

Suppose that we have the inequality $x+3<10$, in this case we have an inequality with one unknown. Here the real value of *x* satisfies this inequality is not unique. For example 1 is a solution but 3 is also a solution. In general all real numbers less than 7 are solutions. In this case we will have many solutions combined in an interval.

Now, the solution set of $x+3<10$ is an open interval containing all real numbers less than 7 whereby 7 is excluded. How?

We solve this inequality as follow

$x+3<10$

$\Leftrightarrow x<10-3$

$\Leftrightarrow x<7$

And then $S=]-\infty,7[$

Note that:

- When the same real number is added or subtracted from each side of inequality the direction of inequality is not **changed**.
- The direction of the inequality is **not changed** if both sides are multiplied or divided by the same **positive real number** and is **reversed** if both sides are multiplied or divided by the **same negative real** number.

2.2.2 Intervals

A subset of real line is called an interval if it contains at least two numbers and also contains all real numbers between any two of its elements. For example, the set of real numbers x such that $x > 6$ is an interval, but the set of real numbers y such that $y \neq 0$ is not an interval.

If a and b are real numbers and $a < b$, we often refer to:

i) The open interval from a to b, denoted by (a, b) or $]a,b[$, consisting of all real numbers x satisfying $a < x < b$

ii) The closed interval from a to b, denoted by $[a, b]$, consisting of all real numbers x satisfying $a \leq x \leq b$

iii) The half-open interval $[a, b[$, consisting of all real numbers x satisfying the inequalities $a \leq x < b$

iv) Half-open interval $]a, b]$ consisting of all real numbers x satisfying the inequalities $a < x \leq b$

Examples

Solve the following inequalities and express the solution in terms of intervals.

a) $2x - 1 > x + 3$

b) $-\frac{x}{3} \geq 2x - 1$

c) $2(x+5) > 2x - 8$

d) $2x + 5 \leq 2x + 4$

Solution:

a) $2x - 1 > x + 3$

$2x > x + 3 + 1$

$2x - x > 4$

$x > 4$

The solution set is the interval $]4, \infty[$

b) $-\frac{x}{3} \geq 2x - 1$

$-x \geq 3(2x-1)$

$x \leq -6x + 3$

$7x \leq 3$

$x \leq \frac{3}{7}$

The solution set is the interval $\left]-\infty, \frac{3}{7}\right]$

c) $2(x+5) > 2x-8$
$2x+10 > 2x-8$

$0x > -18$ which is impossible in the set of real numbers.

Then,
$S = \varnothing$.

d) $2x+5 \leq 2x+4$

$2x - 2x \leq 4 - 5$

$0x \leq -1$

Since any real number times zero is zero and zero is not less or equal to -1 then the solution set is the empty set. $S = \varnothing$

2.2.3 Inequalities products / quotients

Activity 2.2.3

Explain the method you can use to solve the following inequalities:

1) $(x+1)(x-1) < 0$

2) $\frac{2x-3}{x} < 0$

Suppose that we need to solve the inequality of the form $(ax+b)(cx+d) < 0$ For this inequality we need the set of all real numbers that make the left hand side to be negative. Suppose also that we need to solve the inequality of the form $(ax+b)(cx+d)>0$. For this inequality we need the set of all real numbers that make the left hand side to be positive.

We follow the following steps:

a) First we solve for $(ax+b)(cx+d)=0$

b) We construct the table called **sign table**, find the sign of each factor and then the sign of the product or quotient if we are given a quotient.

   For the quotient the value that makes the denominator to be zero is always excluded in the solution. For that value we use the symbol **II** in the row of quotient sign.

c) Write the interval considering the given inequality sign.

Example

Solve in the set of real numbers the following inequalities

a) $(3x+7)(x-2)<0$

b) $\dfrac{x+4}{2x-1}\geq 0$

Solution

a) $(3x+7)(x-2)<0$

Start by solving $(3x+7)(x-2)=0$

$3x+7=0$

$\Leftrightarrow x=-\dfrac{7}{3}$ or $x-2=0$

$\Leftrightarrow x=2$

Then, we find the sign table.

x	$-\infty$	$-\frac{7}{3}$		2	$+\infty$
$3x+7$	-	0	+		+
$x-2$	-		-	0	+
$(3x+7)(x-2)$	+	0	-	0	+

Since the inequality is $(3x+7)(x-2)<0$ we will take the interval where the product is negative. Thus, $S=\left]-\frac{7}{3},2\right[$

b) $\frac{x+4}{2x-1}\geq 0$

$x+4=0 \Rightarrow x=-4$

$2x-1=0 \Rightarrow x=\frac{1}{2}$

x	$-\infty$		-4		$\frac{1}{2}$		$+\infty$
$x+4$		-	0	+		+	
$2x-1$		-		-	0	+	
$\frac{x+4}{2x-1}$		+	0	-	||	+	

$S=]-\infty,-4]\cup\left]\frac{1}{2},+\infty\right[$

2.2.4 Inequalities involving absolute value

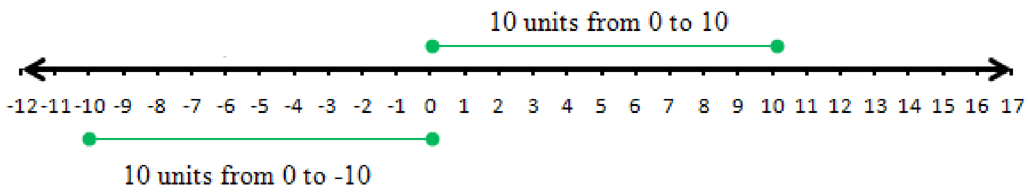
Activity 2.2.4

State the set of all real numbers whose number of units from zero, on a number line, are

1) greater than 4
2) less than 6

The inequality $|x-a|<k$ says that the distance from x to a is less than k, so x must lie between $a-k$ and $a+k$ or equivalently a must lie between $x-k$ and $x+k$ if k is a positive number.

Recall that absolute value of a number is the number of units from zero to a number line. That is, $|x|=k$ means k units from zero (k is a positive real number or zero).



For all real number x and $k \geq 0$

a) $|x| < k \Leftrightarrow -k < x < k$
b) $|x-a| < k \Leftrightarrow a-k < x < a+k$
c) $|x| > k \Leftrightarrow x > k \ or \ x < -k$
d) $|x-a| > k \Leftrightarrow x > a+k \ or \ x < a-k$

Example

1) Solve $|3x - 2| \leq 1$

Solution:

$-1 \leq 3x - 2 \leq 1$, we solve this pair of inequalities

$-1 \leq 3x - 2$ and $3x - 2 \leq 1$

$-1 + 2 \leq 3x$ $3x \leq 1 + 2$

${}^1/_3 \leq x {}^1/_3 \leq x$ $x \leq 1$

Thus the solution lies in the interval $[{}^1/_3, 1]$

2) Find the solution set of the inequality $|3x-15| < 3$

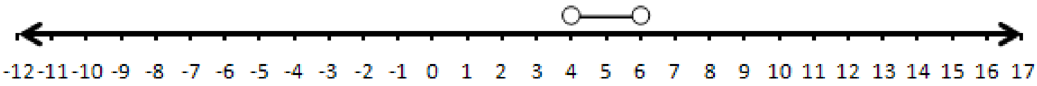
Solution

$|3x-15| < 3 \Leftrightarrow -3 < 3x-15 < 3$

$-3+15 < 3x < 3+15 \Leftrightarrow 12 < 3x < 18$ or $\frac{12}{3} < x < \frac{18}{3} \Leftrightarrow 4 < x < 6$

The solution set is $S = \{x \in IR : 4 < x < 6\}$ or S=]4,6[

Number line:



2.2.5 Real life problems involving linear inequalities

Activity 2.2.5

Sam and Alex play in the same team at their school. Last Saturday their team played with another team from other school in the same district, Alex scored 3 more goals than Sam. But together they scored less than 9 goals.

What are the possible number of goals Alex scored?

Inequalities can be used to model a number of real life situations. When converting such word problems into inequalities, begin by identifying how the quantities are relate to each other, and then pick the inequality symbol that is appropriate for that situation. When solving these problems, the solution will be a range of possibilities. Absolute value inequalities can be used to model situations where margin of error is a concern.

Examples

1) The width of a rectangle is 20 meters. What must the length be if the perimeter is at least 180 meters?

Solution:

Let x be length of rectangle

$perimeter = 2length + 2width$

$2x + 2(20) \geq 180$

$2x \geq 180 - 40$

$x \geq 70$

The length must be at least 70 meters.

2) John has 1 260 000 Rwandan Francs in an account with his bank. If he deposits 30 000 Rwanda Francs each week into the account, how many weeks will he need to have more than 1 820 000 Rwandan Francs on his account?

Solution:

Let *x* be the number of weeks

We have total amount of deposits to be made plus the current balance is greater than the total amount wanted:

That is $30000x + 1260000 > 1820000$

$$30000x > 1820000 - 1260000$$

$$30000x > 560000$$

$$x > \frac{560000}{30000} \approx 19$$

Thus, John needs at least 19 weeks to have more than 1 820 000 Rwandan Francs on his account.

Application activity 2.2

1) Joe enters a race where he has to cycle and run. He cycles a distance of 25 km, and then runs for 20 km. His average running speed is half of his average cycling speed. Joe completes the race in less than 2½ hours, what can we say about his average speeds?
2) Explain your colleague whether or not a solution set for an inequality can have one element.

2.3 Simultaneous linear equations in two unknowns (Solving by equating two same variables)

Activity 2.3

In each of the following systems find the value of one variable from one equation and equalize it with the same value from the second equation. Calculate the values of those variables.

1) $\begin{cases} 3x - 2y = -6 \\ x + y = -2 \end{cases}$ 2) $\begin{cases} x + 2y = 10 \\ -3x + 2y = 12 \end{cases}$ 3) $\begin{cases} x + 4y = 8 \\ y - x = 2 \end{cases}$

CONTENT SUMMARY

To find the value of unknown from simultaneous equation by equating the same variable in terms of another, we do the following steps:

i) Find out the value of one variable in first equation,

ii) Find out the value of that variable in the second equation,

iii) Equating the two same values obtained

iv) Solve the equation obtained to find out the unknown variables.

Example

1) Algebraically, solve the simultaneous linear equation by equating the same variables.

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

Solution:

$$\begin{cases} 4x + 5y = 2 \\ x + 2y = -1 \end{cases}$$

From equation (1) $4x + 5y = 2 \Rightarrow x = \frac{2-5y}{4}$, from equation (2) $x + 2y = -1 \Rightarrow x = -1 - 2y$

Equalize the values of x from equation (1) and (2)

$$\frac{2-5y}{4} = -1 - 2y$$

$$2 - 5y = 4(-1 - 2y)$$

$$2 - 5y = -4 - 8y$$

$$-5y + 8y = -4 - 2$$

$$y = -2$$

$$x = \frac{2-5y}{4}, x = \frac{2-5(-2)}{4} = \frac{12}{4} = 3 \quad \text{then, } S = \{(3, -2)\}$$

2) Solve algebraically the following system by equating two same variables

$$\begin{cases} x + y = 1 \\ 2x + 3y = 2 \end{cases}$$

Solution

From equation $x + y = 1 \Rightarrow y = 1 - x$

From equation $2x + 3y = 2 \Rightarrow y = \frac{2-2x}{3}$

By equating those values of y, $1 - x = \frac{2-2x}{3}$

$$3(1-x) = 2 - 2x$$
$$3 - 3x = 2 - 2x$$
$$-3x + 2x = 2 - 3$$
$$x = 1,\ y = 1 - x \Rightarrow y = 1 - 1 = 0$$
$$s = \{(1,0)\}$$

3) Solve the following simultaneous equation

$$\begin{cases} x + 4y = 8 \\ y - x = 2 \end{cases}$$

Solution

From equation (1), $x + 4y = 8 \Rightarrow x = 8 - 4y$

From equation (2), $y - x = 2 \Rightarrow x = -2 + y$

By equating values from (1) and (2)

$$8 - 4y = -2 + y$$
$$-4y - y = -2 - 8$$
$$-5y = -10$$
$$y = 2,\quad x = -2 + 2 = 0 \qquad \text{then, } s = \{(0,2)\}$$

Application activity 2.3

Solve the simultaneous linear equations by equating two same variables

1) $\begin{cases} x - y = 3 \\ 2x - 2y = 6 \end{cases}$ 2) $\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$ 3) $\begin{cases} 3x - 5y = 10 \\ 2x + y = 12 \end{cases}$

4) $\begin{cases} x - 4y = 1 \\ x - y = 2 \end{cases}$ 5) $\begin{cases} -x + y = 0 \\ x + 2y = 3 \end{cases}$ 6) $\begin{cases} 5y + 3x = 2 \\ 10x + 6y = 0 \end{cases}$

2.4 Simultaneous linear equations in two unknowns (solving by row operations or elimination method)

Activity 2.4

For each of the following, find two numbers to be multiplied to the equations such that one variable will be eliminated when making the addition or subtraction of the two equations.

$$\begin{cases} -2x+5y=-7 \\ 7x-3y=-19 \end{cases}$$

CONTENT SUMMARY

To eliminate one of the variables from either of equations to obtain an equation in just one unknown, make one pair of coefficients of the same variable in both equations negatives of one another by multiplying both sides of an equation by the same number. Upon adding the equations, that unknown will be eliminated.

Example

1) Solve the system of equations using elimination method.

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases}$$

Solution

$$\begin{cases} x+y=1 \\ 2x+3y=2 \end{cases} \left| \begin{matrix} -2 \\ 1 \end{matrix} \right. \Leftrightarrow \begin{cases} -2x-2y=-2 \\ \;\;2x+3y=2 \end{cases} +$$

$$y=0$$

$$x+y=1 \Leftrightarrow x=1-y=1$$

$$S=\{(1,0)\}$$

2) Solve the system of equation by using elimination method

$$\begin{cases} -2x+5y=-7 \\ 7x-3y=-19 \end{cases}$$

Solution

$$\begin{cases}-2x+5y=-7\\7x-3y=-19\end{cases}\Leftrightarrow\begin{matrix}7(-2x+5y)=7(-7)\\2(7x-3y)=2(-19)\end{matrix}$$

$$\begin{matrix}-14x+35y=-49\\14x-6y=-38\end{matrix}\Leftrightarrow\begin{array}{r}-14x+35y=-49\\14x-6y=-38\\\hline 29y=-87\end{array}$$

$$29y=-87\Leftrightarrow y=-3$$

$$x=\frac{-7-5y}{-2}=\frac{-7-5(-3)}{-2}=-4$$

Then, $s=\{(-4,-3)\}$

Application activity 2.4

1) Solve the following system of equation by using elimination method.

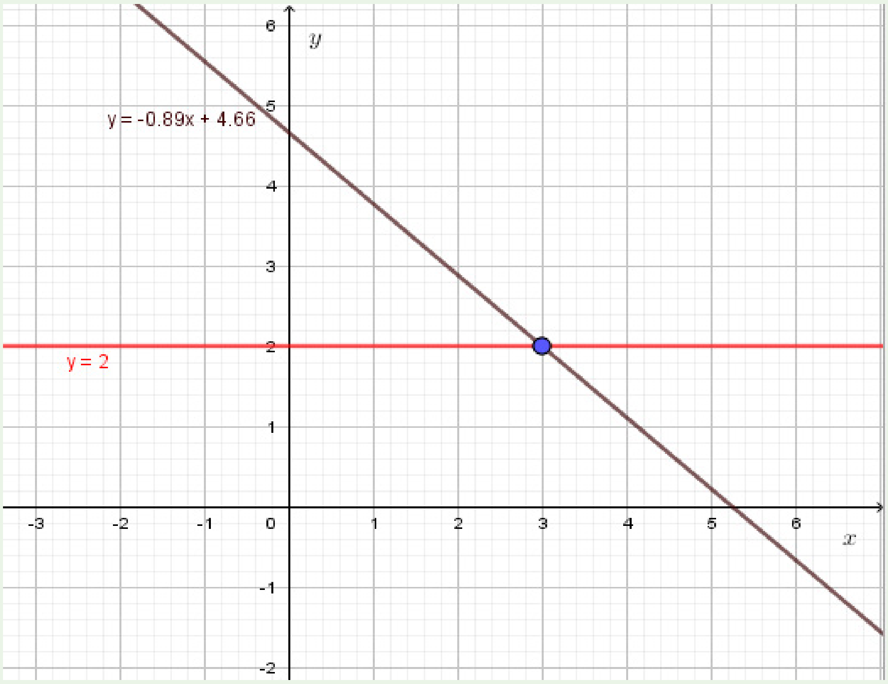
a) $\begin{cases}3x-4y=1\\x-3y=2\end{cases}$ b) $\begin{cases}x-4y=1\\x-y=2\end{cases}$ c) $\begin{cases}3x-2y=-6\\x+y=-2\end{cases}$ d) $\begin{cases}2x+3y=8\\x-y=2\end{cases}$

2) Use your own words to explain how to solve algebraically simultaneous linear equations.

2.5 Solving graphically simultaneous linear equations in two unknowns

Activity 2.5

1) Discuss how you can find the coordinate of the point intercept of two lines whose equations are known.



2) Given the system of equations

$$\begin{cases} 3x-2y=-6 \\ x+y=-2 \end{cases}$$

a) For each equation from the system, choose any two values of x and use them to find values of y. this gives you two points in the form

b) Plot the obtained points in XY plane and join these points to obtain the lines.

c) What is the point of intersection for two lines?

d) Write the obtained point as solution of the system.

CONTENT SUMMARY

One way to solve a system of linear equations is by graphing. The intersection of the graphs represents the point at which the equations have the same x -value and the same y-value. Thus, this ordered pair represents the solution common to both equations. This ordered pair is called the solution to the system of equations.

The following steps can be applied in solving system of linear equation graphically:

1) Find at least two points for each equation.
2) Plot the obtained points in *XY* plane and join these points to obtain the lines. Two points for each equation give one line.
3) The point of intersection for two lines is the solution for the given system

Examples

1) Solve the following system by graphical method.

$$\begin{cases} x+y=4 \\ x-y=2 \end{cases}$$

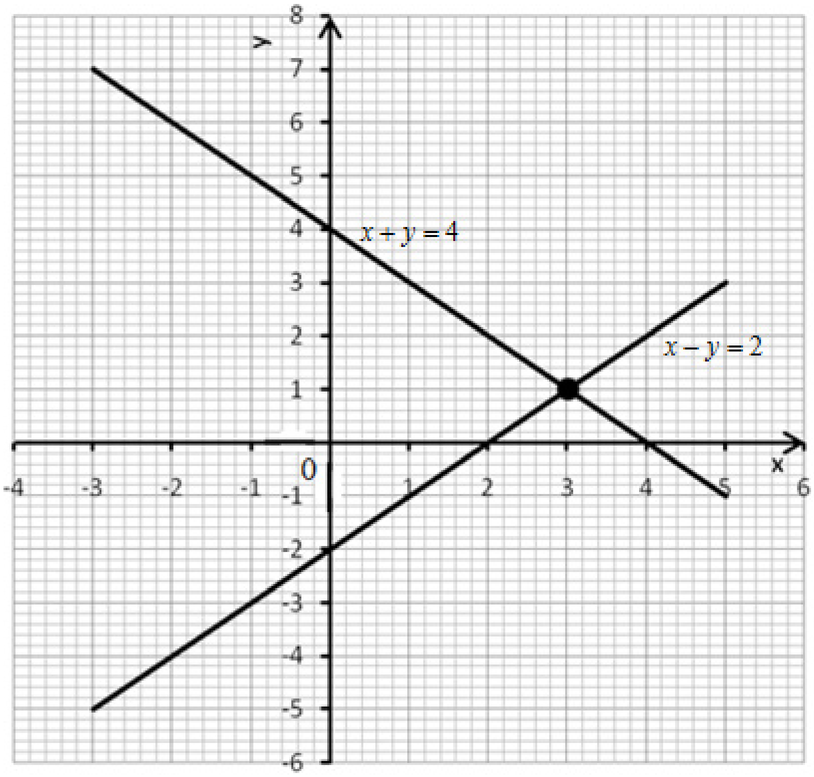
Solution For $x+y=4$

x	-3	5
y	7	-1

For $x-y=2$

x	-3	5
y	-5	3

Graph



The two lines intersect at point $(3,1)$. Therefore the solution is $S=\{(3,1)\}$.

2) Solve graphically the following system of linear equations

Solve the following equations graphically

$$\begin{cases} x+y=2 \\ 2y=4-2x \end{cases}$$

Solution

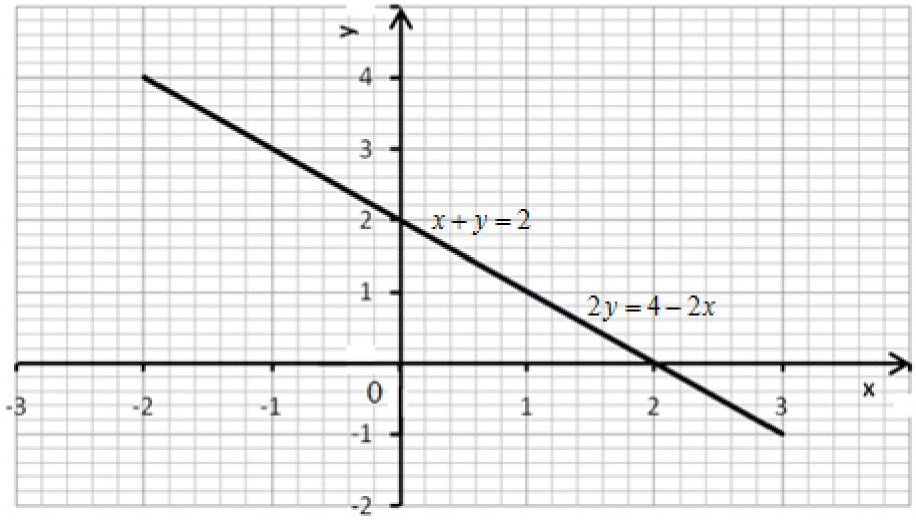
For $x+y=2$

x	-2	3
y	4	-1

For $2y=4-2x$

x	-2	3
y	4	-1

Graph



We see that the two lines coincide as a single line. In such case there is infinite number of solutions.

Application activity 2.5

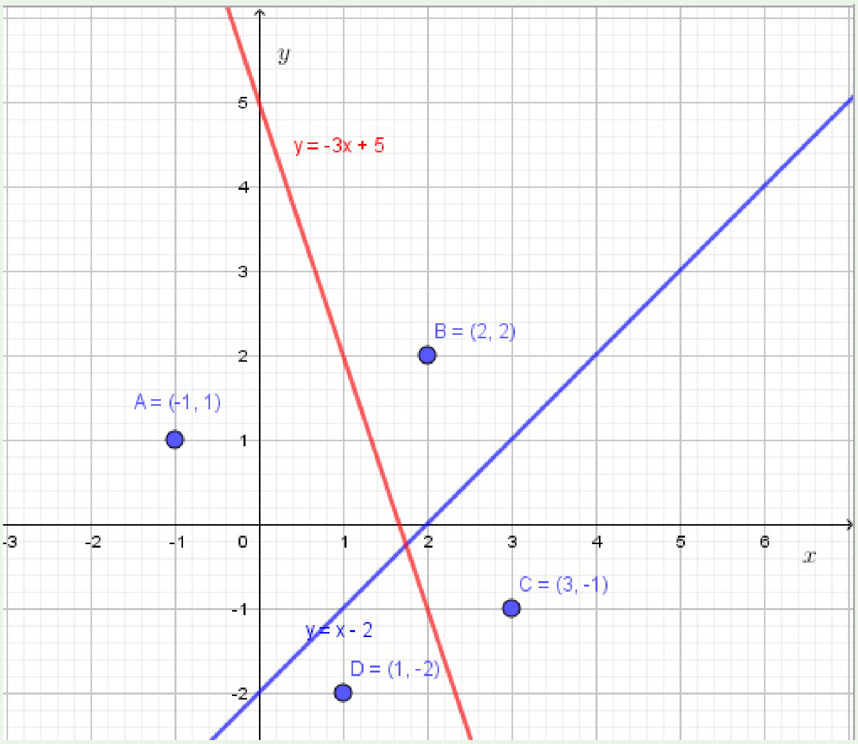
1) Solve the graphically the following system of linear equations.

a) $\begin{cases} 4y + x = 8 \\ -x + y = 2 \end{cases}$ b) $\begin{cases} 3x + y = 1 \\ x - y = -1 \end{cases}$ c) $\begin{cases} 2x + y = 3 \\ 6x + 3y = 9 \end{cases}$

2.6 Solving algebraically and graphically simultaneous linear inequalities in two unknowns

Activity 2.6

The following graph illustrates two lines and their equations



For each point A, B, C, and D, replace its coordinate in the two inequalities to verify which one satisfies the following system:

$$\begin{cases} y < x - 2 \\ y > -3x + 5 \end{cases}$$

CONTENT SUMMARY

A system of inequalities consits of a set of two inequalities with the same variables. The inequalities define the conditions that are to be considered simultaneously.

Each inequality in the set contains infinetely many ordered pair solutions defined by a region in rectangular coordinate plane. When considering two of these inequalities together, the intersection of these sets will define the set of simultaneous ordered pair solutions.

Linear inequalities with two unknowns are solved to find a range of values of the two unknowns which make the inequalities true at the same time. The solution is represented graphically by a shaded region or unshaded region.

In finding solution, first, graph the "equals" line, then shade in the correct area.

The following steps can be used to find the solution of simultaneous inequalities graphically:

1) Rearrange the equation so that "y" is on the left and everything else on the right.
2) Plot the y line (make it a solid line for $y \leq$ $or\ y \geq$ and a dashed line for $y < or\ y >$
3) Shade above the line for a greater than $y > or\ y \geq$ or below the line for a less than $y < or\ y \leq$
4) The intersection will define the set of ordered pairs as a solution.

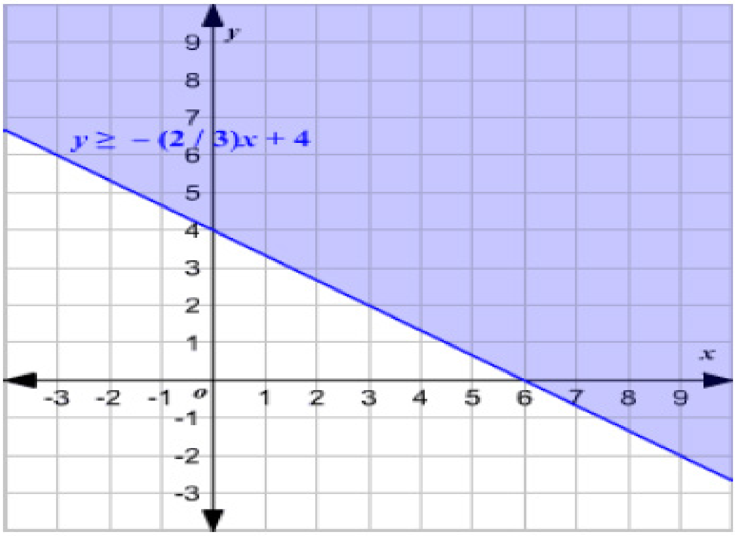
Example

Solve the system of inequalities by graphing (shaded wanted region):

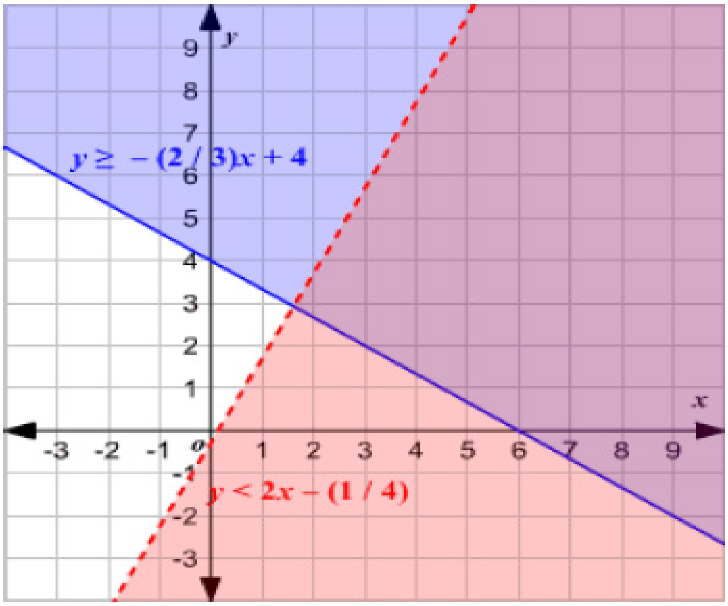
$$\begin{cases} 2x + 3y \geq 12 \\ 8x - 4y > 1 \\ x < 4 \end{cases}$$

Solution

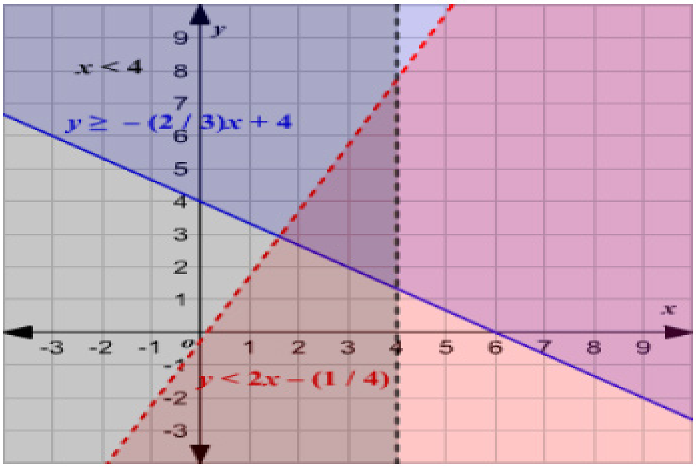
From Each inequality from the system we have $2x + 3y \geq 12 \Rightarrow y \geq -\frac{2}{3}x + 4$, the related equation to this is $y = -\frac{2}{3}x + 4$ since the inequality is $\geq$, not a strict one, the border line is solid. Graph the line $y = -\frac{2}{3}x + 4$



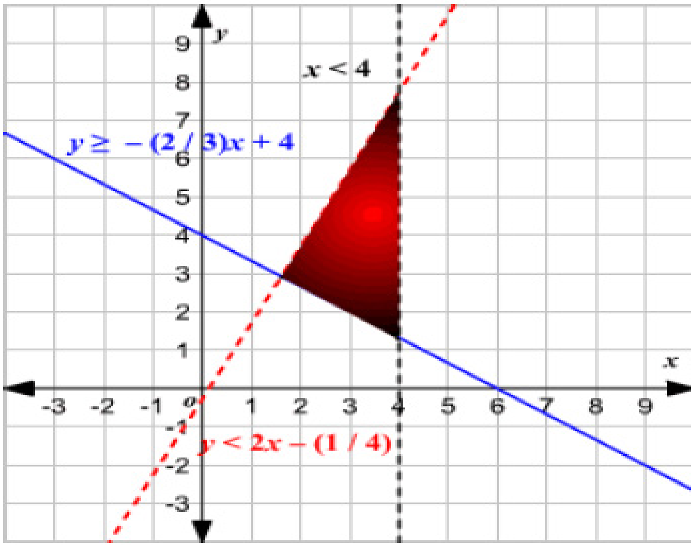
Similarly, draw a dashed line of related equation of the second inequality $y < 2x - \frac{1}{4}$ which has a strict inequality.



Draw the dashed vertical line $x = 4$ which is the related to the equation of the third inequality.



The solution of the system of inequalities is the intersection region of the solutions of the three inequalities as it is done in the following figure.



Application activity 2.6

1) Algebraically and graphically, solve the following simultaneous inequalities.

a) $\begin{cases} y - 2x \leq 1 \\ x + y \leq 10 \\ x \geq 0 \end{cases}$　　b) $\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases}$　　c) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \leq 4 \end{cases}$

2) Use your own words to explain how to solve graphically the simultaneous inequalities.

2.7 Solving quadratic equations by the use of factorization and discriminant

Activity 2.7

Smoke jumpers are fire fighters who parachute into areas near forest fires. Jumpers are in free fall from the time they jump from a plane until they open their parachutes. The function $y=-16t^2+1600$ gives a jumper's height y in metre after t seconds for a jump from $1600m$.

a) How long is free fall if the parachute opens at $1000m$?

b) Complete a table of values for t = 0, 1, 2, 3, 4, 5 and 6.

CONTENT SUMMARY

Equations which are written in the form of $ax^2+bx+c=0$ $(a\neq 0)$ are called quadratic equations. To find solution of this equation the two main ways can be used in solving such equation

a) Use of factorization or finding square roots

Grouping terms or decomposition can be used to factorize the quadratic equations and later help us to find the solution of equation. By having the product of ac and the sum of those two integers which gives b, it helps you to decompose into a product of factors.

Example

1) Solve in $\mathbb{R}$: $6y^2+5y-25=0$

Solution:

To obtain a common factor, terms that have a common factor are grouped together, and the common factor of each group is divided as follows

$$6y^2+5y-25=0$$

Guess two integers whose sum is 5 and product is -150

$$6y^2+15y-10y-25=0$$

$$(6y^2+15y)-(10y+25)=0$$

$$3y(2y+5)-5(2y+5)=0$$

$(2y+5)(3y-5)=0$

$(2y+5)=0$ or $(3y-5)=0$

$y=\frac{-5}{2}$ or $y=\frac{5}{3}$

2) Solve the equation $x^2-4y-45=0$

Solution:

$x^2-4x-45=0$

$(x-9)(x+5)=0$

$x=9$ *or* $x=-5$. Then, $S=\{-5,9\}$

3) Find the solution set of $(3x-1)(2x+3)=-5$

Solution:

We expand the product as the equation is different from zero

$6x^2+7x-3=-5$

$6x^2+7x-3+5=0$

$(3x+2)(2x+1)=0$

$3x+2=0$ *or* $2x+1=0$

$x=-\frac{2}{3}$ or $x=-\frac{1}{2}$ Thus, the solution set is $\left\{-\frac{2}{3},-\frac{1}{2}\right\}$

b) Use of discriminant method

In quadratic equation: $ax^2+bx+c=0$, Let $\Delta=b^2-4ac$ be, a discriminant,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(There are two values of x which can help us to write down factor form of a quadratic expression).

Examples

1) Solve in $\mathbb{R}$: $6x^2+5x-25=0$

Solution:

$$a=6,\ b=5,\ c=-25$$

$$\Delta=b^2-4ac,\ \Delta=b^2-4ac$$

$$\Delta=(5)^2-4(6)(-25)=625,\ \sqrt{\Delta}=\sqrt{625}=\pm 25$$

$$x_1=\frac{-5+25}{12}=\frac{20}{12}=\frac{5}{3} \qquad x_2=\frac{-5-25}{12}=\frac{-30}{12}=\frac{-5}{2} \text{ then } S=\left\{\frac{-5}{2},\frac{5}{3}\right\}$$

2) For what value of k will the equation $:x^2+2x+k=0$ have one double root? Find that root.

Solution:

For one double root $\Delta=0$.

$$\Delta=4-4k$$

$$4-4k=0\Rightarrow k=1$$

Thus, the value of *k* is 1.

That root is x = -1 $S=\{-1\}$

Application activity 2.7

a) Use factorization and discriminant to solve the following equations

1) $3x^2=10-x$ 2) $x^2-3x=-11$

3) $x^2-12x+11=0$ 4) $x^2+16=8x$

b) The area of a rectangular garden is 30 meter square. If the length is 7 meter longer than the width, find the dimensions of the rectangle.

c) Does a quadratic equation have more than one solution? Explain your answer.

2.8 Applications of linear and quadratic equations in economics and finance: Problems about supply and demand (equilibrium price)

Learning activities 2.8

Assume that a firm can sell as many units of its product as it can manufacture in a month at

180 Rwandan francs each. It has to pay out 2400 Rwandan francs fixed costs plus a marginal cost of 140 Rwandan francs for each unit produced. How much does it need to produce to break even (where total revenue equals to total cost)?

CONTENT SUMMARY

When only two or single variables and equations are involved, a simultaneous equation system can be related to familiar graphical solutions, such as supply and demand analysis.

For **example,** assume that in a competitive market the demand schedule is given by $p = 420 - 0.2q$ and the supply schedule is given by $p = 60 + 0.4q$,

If this market is in equilibrium, the equilibrium price and quantity will be where the demand and supply schedules intersect. This requires you to solve the system formed by the two simultaneous equations. Its solution will correspond to a point which is on both the demand schedule and the supply schedule. Therefore, the equilibrium values of p and q will be such that both equations (1) and (2) hold.

Example

1) In a competitive market the demand schedule is given by $p = 420 - 0.2q$ and the supply schedule is given by $p = 60 + 0.4q$, Solve for p and q *the* simultaneous equation and determine the point at which the market is in equilibrium.

Solution:

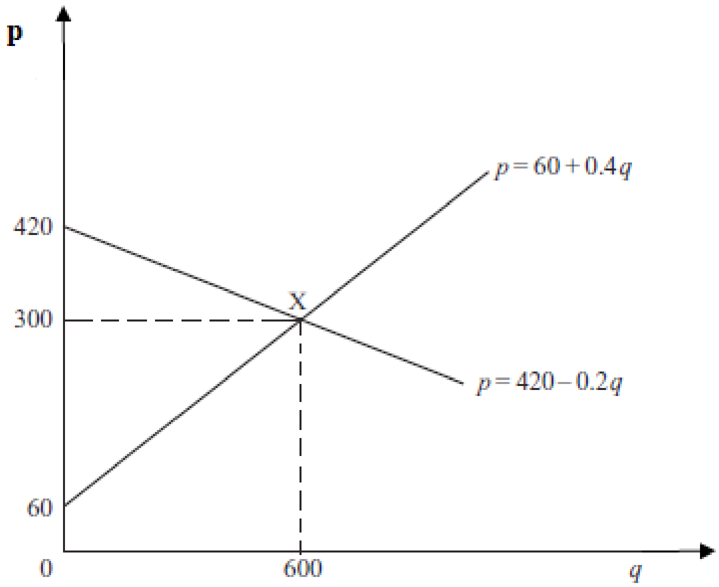
Let us solve the system

$$\begin{cases} p = 420 - 0.2q \\ P = 60 + 0.4q \end{cases}$$

Equalizing the value of p, we find $420 - 0.2q = 60 + 0.4q$.

Which gives $q = 600$. Replacing this value in the given two equations, we find $p = 300$. The market is in equilibrium at the point $q = 600$.

These two functional relationships are plotted in the figure and both hold at the intersection point $X(600,300)$.



2) Calculate the equilibrium values of *p* and *q* in a competitive market where the demand schedule is $p = 200q^{-1}$ and the supply is *p* =30+2*q*

Solution

In equilibrium, demand price equals supply price. Therefore

$$200q^{-1} = 30 + 2q$$

Multiplying through by *q*, $200 = 30q + 2q^2$

$$0 = 30q + 2q^2 - 200$$

$$0 = (2q - 10)(q + 20)$$

Therefore $2q - 10 = 0$ and $q + 20 = 0$

$$q = 5 \quad \text{or} \quad q = -20$$

We can ignore the second solution as negative quantities cannot exist.

Thus the equilibrium quantity is 5.

Substituting this value into the supply function gives equilibrium price

$$p = 30 + 2 \times 5 = 40$$

3) A firm makes two goods A and B which require two inputs K and L. One unit of A requires 6 units of K plus 3 units of L and one unit of B requires 4 units of K plus 5 units of L. The firm has 420 units of K and 300 units of L at its disposal. How much of A and B should it produce if it wishes to exhaust its supplies of K and L totally?

Solution:

This question requires you to use the economic information given to set up a mathematical problem in a format that can be used to derive the desired solution.

The total requirements of input K are 6 for every unit of A and 4 for each unit of B, which

Can be written as $K = 6A + 4B$

Similarly, the total requirements of input L can be specified as $L = 3A + 5B$

As we know that $K = 420$ and $L = 300$ because all resources are used up

Then, $420 = 6A + 4B$ and $300 = 3A + 5B$

Solve the system of linear equations to find values of A and B

$$\begin{cases} 420 = 6A + 4B \\ 300 = 3A + 5B \end{cases}$$

From first equation $420 = 6A + 4B \Longrightarrow A = \frac{420-4B}{6}$

$$420 = 6A + 4B \Longrightarrow A = \frac{420-4B}{6}$$

From second equation $300 = 3A + 5B \Longrightarrow A = \frac{300-5B}{3}$

$$300 = 3A + 5B \Longrightarrow A = \frac{300-5B}{3}$$

$$\frac{420 - 4B}{6} = \frac{300 - 5B}{3}$$

$$1260 - 12B = 1800 - 30B$$

$$-12B + 30B = 1800 - 1260$$

$$18B = 540$$

$$B = 30 \quad \text{or } A = \frac{420-4(30)}{6} = 50$$

The firm should therefore produce 50 units of A and 30 units of B.

4) Two student teachers were driving at constant speeds to the same direction during their holidays. The first travelling at 40 km/hr left Kigali at 8:30 a.m. The second travelling at 60 km/hr followed him after 1 hour.

   a) What is the distance between the two divers one hour after the departure of the first driver?

   b) If t is the same time used by the two drivers after the departure of the second driver and y the entire distance covered,

      i) Establish the functions $y = f(t)$ of the first;

      ii) Establish the functions $y = g(t)$ of the second driver

      iii) When did the second bus over take the first bus?

      iv) Illustrate the two functions on the Cartesian plan from $t = 0$ to $t = 5\,hrs$ and show the point where the second met the first driver.

Solution:

The speed of the first driver is 40 km/hour. $v_1 = 40km/hr$

The second driver has a speed of 60km/hr and left after the departure of the first. $v_2 = 60km/hr$.

a) Distance covered by the first driver is $y_0 = v_1.t = (40km/hr).1hr = 40km$

b) i) The distance covered by the first driver is the initial distance plus the distance covered after the departure of the second driver.

This is $y = y_0 + v_1.t = 40 + 40.t$.

Then $y = f(t) = 40 + 40.t$

ii) The distance y covered by the second driver is $y = v_2.t = 60.t$.

Therefore, $y = g(t) = 60.t$

iii) After the departure of the second driver, the two drivers will be using the same time t and when this second driver will meet the first, the two will be covered the same distance

$$d = y = g(t) = f(t)$$

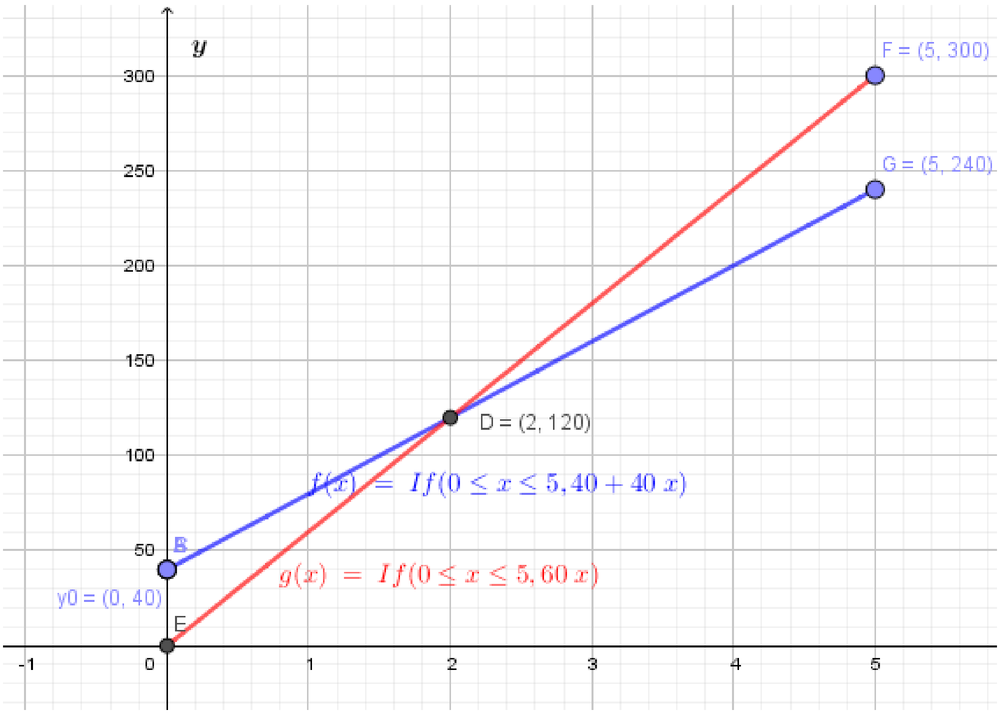
This means: $40 + 40.t = 60.t$

$$40 + 40.t = 60.t$$
$$60t - 40t = 40$$
$$20t = 40$$
$$t = 2$$

The second driver will meet the first after 2 hours. It will be at 11h30 because 9h30+2hs =11h30.

iv) The table of values can help to draw the graphs of the function f and g.

T	0	1	2	3	4	5
$y = f(t) = 40 + 40.t$ **(in km)**	40	80	120	160	200	240
$y = g(t) = 60.t$ **(in km)**	0	60	120	180	240	300



Application activity 2.8

1) Today a house-worker has 3000 Frw. His boss pays him/her 400Frw per day and he saves all the money received.

   a) What is the money the house-worker will have after 2 days, 5 days and after 10 days ?

   b) Discuss the function $y = f(t)$ for the money saved by the house-worker where t represents the time in days.

   c) Illustrate the function $y = f(t)$ in the Cartesian plan on a manila paper;

   d) Discuss the money the house-worker will have after 2 months.

   e) Is it possible for a house-worker to save the money? Explain your answer.

2) The national income in the basic Keynesian macroeconomic model is given by $Y = C + I$. If C is given by $40 + 0.5Y$ and $I = 200$. Calculate the national income in the basic Keynesian macroeconomic model.

2.9. END UNIT ASSESSMENT

1) Solve by factorization the quadratic equation $2x^2 - 6x - 20 = 0$

2) Solve the inequalities $-4x - 3 > -2x - 11$

3) Algebraically and graphically, solve the simultaneous inequalities.

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + y \leq 4 \end{cases}$$

4) The length of a rectangular garden is $5cm$ more than its width and the area is $50cm^2$. Calculate the length and width of the garden.

5) A ball is thrown upwards from a rooftop, 80m above the ground. It will reach a maximum vertical height and then fall back to the ground. The height of the ball from the ground at time t is h, which is given by, $h = -16t^2 + 64t + 80$

   a) What is the height reached by the ball after 1 second?

   b) What is the maximum height reached by the ball?

   c) How long will it take before hitting the ground?

6) Two cyclists move away from a town along two perpendicular paths at $20km/hr$ and $40km/hr$ respectively. The second cyclist starts the journey an hour later than the first one. Find the time taken for them to be $100km$ apart.

UNIT: 3

DESCRIPTIVE STATISTICS

Key Unit competence: Analyse and interpret statistical data from daily life situations

3.0. Introductory Activity

1) 1. At the market a fruit-seller has the following daily sales Rwandan francs for five consecutive days: 1000Frw, 1200Frw, 125Frw, 1000Frw, and 1300Frw. Help her to determine the money she could get if the sales are equally distributed per day to get the same total amount of money.

2) During the welcome test of Mathematics for the first term 10 student-teachers of year one language education scored the following marks out of 10 : 3, 5,6,3,8,7,8,4,8 and 6.

   a) What is the mean mark of the class?

   b) Chose the mark that was obtained by many students.

   c) Compare the differences between the mean of the group and the mark for every student teacher.

3.1 Definition and type of data

Learning activities 3.1

Carry out research on statistics to determine the meanings of statistics and types of data. Use your findings to select qualitative and quantitative data from this list: Male, female, tall, age, 20 sticks, 45 student-teachers, and 20 meters, 4 pieces of chalk.

CONTENT SUMMARY

Statistics is the branch of mathematics that **deals with** data collection, data organization, summarization, analysis and draws conclusions from data.

The use of graphs, charts, and tables and the calculation of various statistical measures to organize and summarize information is called descriptive statistics. Descriptive statistics helps to reduce our information to a manageable size and put it into focus.

Every day, we come across a wide variety of information in form of facts, numerical figures or table groups. A **variable** is a characteristic or attribute that can assume different values. **Data** are the values (measurements or observations) that the variables can assume.

Variables whose values are determined by chance are called **random variables.** A collection of data values forms a **data set.** Each value in the data set is called a **data value** or a **datum.**

For example information related to profit/ loss of the school, attendance of students and tutors, used materials, school expenditure in term or year, etc. These facts or figure which is numerical or otherwise, collected with a definite purpose is called data. This is the word derived from Latin word Datum which means pieces of information.

Qualitative variable

The qualitative variables are variables that cannot be expressed using a number. A qualitative data is determined when the description of the characteristic of interest results is a non-numerical value. A qualitative variable may be classified into two or more categories. Data obtained by observing values of a qualitative variable are referred to as **qualitative data**.

Examples

Qualitative variable	Possible categories of data
Marital status	Single, married, divorced
Gender	Male, Female
Pain level	None, moderated, severe
Colour	Red, black, green, yellow

The possible categories for qualitative variables are often coded for the purpose of performing computerized statistical analysis so that we have qualitative data.

Quantitative variable

Quantitative variables are variables that are expressed in numerical terms, counted or compared on a scale. A quantitative data is determined when the description of the characteristic of interest results in numerical value. When measurement is required to describe the characteristic of interest or when it is necessary to perform a count to describe the characteristic, a quantitative variable is defined.

Discrete data is a quantitative data whose values are countable. Discrete data usually result from counting.

Continuous data is a quantitative data that can assume any numerical value over an interval or over several intervals. Continuous data usually results from making a measurement of some type. Data obtained by observing values of a quantitative variable are referred to as **quantitative data**. Quantitative data obtained from a discrete variable are also referred to as discrete data and quantitative data obtained from a continuous variable are called continuous data.

Examples

Discrete variable	Possible values of data
The number of defective needles in boxes of 100 diabetic syringes	0,1,2,……………..100
The number of individuals in groups of 30 with a type **A** personality	0,1,2,3,……………30
Number of student-teachers in classroom	0,1,2,………………45

Continuous variable	Possible values of data
The household income for households with incomes less than or equal to 200,000 Rwandan francs	All the real numbers between A and 200,000 where A is the smallest household income in the population
Height	All values for the length in length measurements.

Data can be used in different ways. The body of knowledge called statistics is sometimes divided into two main areas, depending on how data are used. The two areas are descriptive statistics and inferential statistics.

Descriptive statistics consists of the collection, organization, summarization, and presentation of data.

Inferential statistics consists of generalizing from samples to populations, performing estimations and hypothesis tests, determining relationships among variables, and making predictions.

A **population** consists of all subjects (human or otherwise) that are being studied.

A **sample** is a group of subjects selected from a population.

Application activity 3.1

1) Select qualitative and quantitative data from the list below:

   Product rating, basketball team classification, number of student-teachers in the classroom, weight, age, number of rooms in a house, number of tutors in school.

2) The table below gives the fasting blood sugar reading for five patients at a small medical clinic. What is the variable? Are they continuous? Compare this data and give your observations.

Patient	Fasting blood sugar level
1st Patient	125
2nd patient	175
3rd patient	160
4th patient	110

3.2 Data presentation or organization

Activitiy 3.2

1) At the beginning of the school year student-teachers came to school with all requirements materials and others don't attend the school on first day. The table below shows the number of student-teachers who attended the school in 5 classrooms on first day.

Number of student-teachers	4	5	6	7	8
Classroom	A	B	C	D	E

Present the above data on bar chart and discuss what you can say when interpreting the chart.

2) One tutor of TTC wants to check the level of how his/her student-teachers like different subjects taught in Language Education option. The survey is done on 60 student-teachers. Here is the number of participants of the survey.

Subject	Number of students
English	12
Mathematics	24
French	6
Entrepreneurship	10
Kinyarwanda	8

Present these data on a pie chart. Using that chart, explain the data

CONTENT SUMMARY

After the collection of data, the researcher needs to organize and present them in order to help those who will benefit from the research and lead him or her to the conclusion. When the data are in original form, they are called **raw data** and are listed next.

Raw data

After the collection of data, one can present them in the raw form presentation.

Example

The following are numbers of notebooks for 54 student-teachers.

3 4 5 6 6 7 3 2 5 4 5 7 4 3 2 3 6 5

8 5 6 4 2 6 5 3 4 7 9 8 6 7 4 5 4 6

3 4 3 6 7 8 2 7 6 5 4 6 4 7 8 9 9 5

Since little information can be obtained from looking at raw data, the researcher organizes the data into what is called a *frequency distribution.* A frequency distribution consists of the number of times each values appears in the raw data and sometimes the corresponding percentage vis-à-vis the total number in the sample.

1. Frequency distribution

Data can be presented in various forms depending on the type of data collected. A **frequency distribution** is the organization of raw data in the table form by the use of frequencies.

Example

The following data represent marks obtained by 12 student-teachers out of 20 in mathematics test of a certain TTC.

13	10	15	17	17	18
17	17	11	10	17	10

The set of outcomes is displayed in a frequency table, as illustrated below:

Marks	Tallies	Frequencies (fi)
10	||	2
11	|	1
13	|	1
15	||	2
17	~~||||~~	5
18	|	1
Total		**12**

The total frequency is the number of items in the population. In the survey about the marks of student-teachers above, the total frequency is the number of all student-teachers. We find this by adding up the numbers from the third column of the table: $2+1+1+2+5+1$ which is equal to 12.

Grouped data

When the number of data is too large, a simple distribution is not appropriate. In this case we come up with a grouped frequency distribution. A grouped frequency distribution organizes data into **groups or classes.**

Definitions related to grouped frequency distribution

a) **Class limits:** The class limits are the lower and upper values of the class

b) **Lower class limit:** Lower class limit represents the smallest data value that can be included in the class.

c) **Upper class limit:** Upper class limit represents the largest data value that can be included in the class.

d) **Class mark or class midpoint:**

$$class\,mid\;po\text{int} = \frac{Lower\ class + upper\ class}{2}$$

Example:

The following data represent the marks obtained by 40 students in Mathematics test. Organize the data in the frequency table; grouping the values into classes, stating from 41-50:

54 83 67 71 80 65 70 73 45 60 72 82 79 78 65 54 67 64 54 76 45 63 49 52 60 70 81 67 45 58 v69 53 65 43 55 68 49 61 75 52.

Solution:

Classes	Class midpoint x	Frequency f
41-50	45.5	6
51-60	55.5	10
61-70	65.5	13
71-80	75.5	8
81-90	85.5	3

The classes: 41-50, 51-60, 71-80, 81-90

Lower class limits: 41, 51, 61, 71, and 81

Upper class limits: 50, 60, 70, 80, and 90

$$Class\ midpoint\ for\ the\ first\ class = \frac{41 + 50}{2} = 45.5$$

Class boundaries

Class boundaries are the midpoints between the **upper class limit** of a class and **the lower class limit of the next class.**

Therefore each class has a lower and an upper class boundary.

Example

Classes	Class midpoint	Frequency
[5,10[	7.5	2
[10,15[	12.5	6
[15,20[	17.5	4
[2,25[	22.5	3
[25,30[	27.5	4
[30,35[	32.5	1

For [10,15[the lower class boundary is 10, The upper class boundary is 15

$$Class\ width = 15 - 10 = 5$$

We have **Categorical Frequency Distributions and Grouped Frequency Distributions**

The **categorical frequency distribution** is used for data that can be placed in specific categories, such as nominal- or ordinal-level data.

Example:

Data such as political affiliation, religious affiliation, or major field of study would use categorical frequency distributions.

Example:

Twenty-five army inductees were given a blood test to determine their blood type.

The data set is:

A	B	B	AB	O
O	O	B	AB	B
B	B	O	A	O
A	O	O	O	AB
AB	A	O	B	A

A Class	B Tally	C Frequency	D Percent
A	𝍸	5	20
B	𝍸 //	7	28
O	𝍸 ////	9	36
AB	////	4	16
		Total 25	100

Grouped Frequency Distributions

When the range of the data is large, the data must be grouped into classes that are more than one unit in width, in what is called a **grouped frequency distribution.** For example, a distribution of the number of hours that boat batteries lasted is the following

Class limits	Class boundaries	Tally	Frequency
24–30	23.5–30.5	///	3
31–37	30.5–37.5	/	1
38–44	37.5–44.5	𝍸	5
45–51	44.5–51.5	𝍸 ////	9
52–58	51.5–58.5	𝍸 /	6
59–65	58.5–65.5	/	1
			25

The procedure for constructing a grouped frequency distribution, i.e., when the classes contain more than one data value

Example

These data represent the record high temperatures in degrees Fahrenheit (*F) for each of the 50 states. Construct a grouped frequency distribution for the data using 7 classes.

112	100	127	120	134	118	105	110	109	112
110	118	117	116	118	122	114	114	105	109
107	112	114	115	118	117	118	122	106	110
116	108	110	121	113	120	119	111	104	111
120	113	120	117	105	110	118	112	114	114

Source: *The World Almanac and Book of Facts.*

1) Determine the classes

 – Find the highest value and lowest value: H =134 and L=100.

 – Find the range: R = highest value-lowest value H -L, so

 R=134 -100 = 34

- Select the number of classes desired (usually between 5 and 20). In this case, 7 is arbitrarily chosen.
- Find the class width by dividing the range by the number of classes

$$width = \frac{R}{Number\ of\ classes} = \frac{34}{7} = 4.9$$

- Select a starting point for the lowest class limit. This can be the smallest data value or any convenient number less than the smallest data value. In this case, 100 is used. Add the width to the lowest score taken as the starting point to get the lower limit of the next class. Keep adding until there are 7 classes, as shown, 100, 105, 110, etc.
- Subtract one unit from the lower limit of the second class to get the upper limit of the first class. Then add the width to each upper limit to get all the upper limits. 105-1=104.

  The first class is 100–104, the second class is 105–109, etc.

- Find the class boundaries by subtracting 0.5 from each lower class limit and adding 0.5 to each upper class limit:

  99.5–104.5, 104.5–109.5, etc.

2) Tally the data
3) Find the numerical frequencies from the tallies.

   The completed frequency distribution is

Class limits	Class boundaries	Tally	Frequency
100–104	99.5–104.5	//	2
105–109	104.5–109.5	~~////~~ ///	8
110–114	109.5–114.5	~~////~~ ~~////~~ ~~////~~ ///	18
115–119	114.5–119.5	~~////~~ ~~////~~ ///	13
120–124	119.5–124.5	~~////~~ //	7
125–129	124.5–129.5	/	1
130–134	129.5–134.5	/	1
			$n = \Sigma f = 50$

2. Cumulative frequency

The cumulative frequency corresponding to a particular value is the sum of all frequencies up to the last value including the first value. Cumulative frequency can also defined as the sum of all previous frequencies up to the current point.

Example:

The set of data below shows marks obtained by student-teachers in Mathematics. Draw a cumulative table for the data.

11	15	18	15	10	16	11	10	17
13	17	11	17	16	17	15	13	16

Solution:

The cumulative frequency at a certain point is found by adding the frequency at the present point to the cumulative frequency of the previous point.

Marks	Frequencies (fi)	Cumulative frequencies (cufi)
10	2	2
11	3	$2+3=5$
13	2	$5+2=7$
15	3	$7+3=10$
16	3	$10+3=13$
17	4	$13+4=17$
18	1	$17+1=18$

3. Relative frequency and percentage

The relative frequency is obtained by dividing the frequency for every data by the sum of all the frequencies. The percentage for a category is obtained by multiplying the relative frequency for that category by 100.

Example:

Marks	Frequencies (fi)	Relative frequency	Percentage
10	2	$2/18=0.111$	11.1
11	3	$3/18=0.167$	16.7
13	2	$2/18=0.111$	11.1
15	3	$3/18=0.167$	16.7
16	3	$3/18=0.167$	16.7
17	4	$4/18=0.222$	22.2
18	1	$1/18=0.55$	5.5

4. Histograms, Frequency Polygons, and Ogives

After you have organized the data into a frequency distribution, you can present them in graphical form. The purpose of graphs in statistics is to convey the data to the viewers in pictorial form. It is easier for most people to comprehend the meaning of data presented graphically than data presented numerically in tables or frequency distributions.

The three most commonly used graphs in research are

1) The histogram.
2) The frequency polygon.
3) The cumulative frequency graph or ogive (pronounced o-jive).

a) The Histogram

The **histogram** is a graph that displays the data by using contiguous vertical bars (unless the frequency of a class is 0) of various heights to represent the frequencies of the classes.

Example:

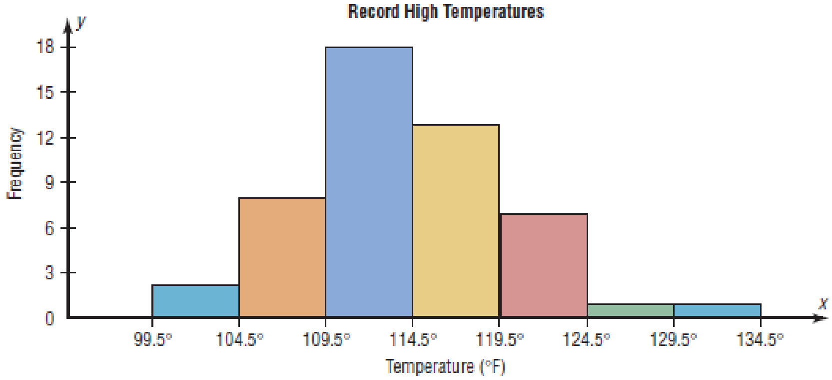
Construct a histogram to represent the data shown for the record high temperatures for each of the 50 states.

Class boundaries	Frequency
99.5–104.5	2
104.5–109.5	8
109.5–114.5	18
114.5–119.5	13
119.5–124.5	7
124.5–129.5	1
129.5–134.5	1

Step 1: Draw and label the *x* and *y* axes. The *x* axis is always the horizontal axis, and the *y* axis is always the vertical axis.

Step 2: Represent the frequency on the *y* axis and the class boundaries on the *x* axis.

Step 3: Using the frequencies as the heights, draw vertical bars for each class. See Figure below



b) The Frequency Polygon

The **frequency polygon** is a graph that displays the data by using lines that connect points plotted for the frequencies at the midpoints of the classes. The frequencies are represented by the heights of the points.

Example:

Using the frequency distribution given in Example 2–4, construct a frequency polygon

Step 1 Find the midpoints of each class. Recall that midpoints are found by adding the upper and lower boundaries and dividing by 2:

$$\frac{99.5+104.5}{2}=102\,,\quad \frac{104.5+109.5}{2}=107$$

and so on.

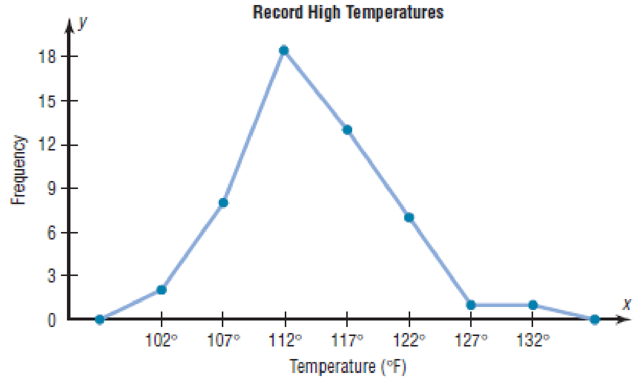
The midpoints are:

Class boundaries	Midpoints	Frequency
99.5–104.5	102	2
104.5–109.5	107	8
109.5–114.5	112	18
114.5–119.5	117	13
119.5–124.5	122	7
124.5–129.5	127	1
129.5–134.5	132	1

Step 2 Draw the x and y axes. Label the x axis with the midpoint of each class, and then use a suitable scale on the y axis for the frequencies.

Step 3 Using the midpoints for the x values and the frequencies as the y values, plot the points.

Step 4 Connect adjacent points with line segments. Draw a line back to the *x* axis at the beginning and end of the graph, at the same distance that the previous and next midpoints would be located, as shown in Figure 2–3.



The frequency polygon and the histogram are two different ways to represent the same data set. The choice of which one to use is left to the discretion of the researcher.

c) The Ogive

The **ogive** is a graph that represents the cumulative frequencies for the classes in a frequency distribution.

Step 1: Find the cumulative frequency for each class.

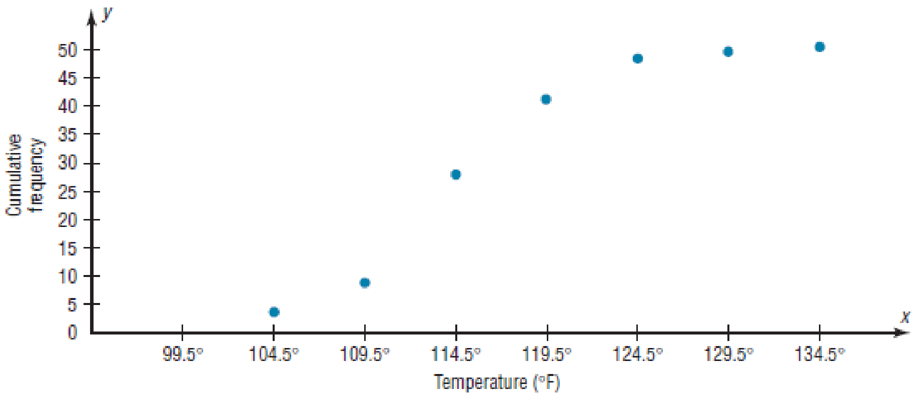
	Cumulative frequency
Less than 99.5	0
Less than 104.5	2
Less than 109.5	10
Less than 114.5	28
Less than 119.5	41
Less than 124.5	48
Less than 129.5	49
Less than 134.5	50

Step 2: Draw the *x* and *y* axes. Label the *x* axis with the class boundaries. Use an appropriate scale for the *y* axis to represent the cumulative frequencies.

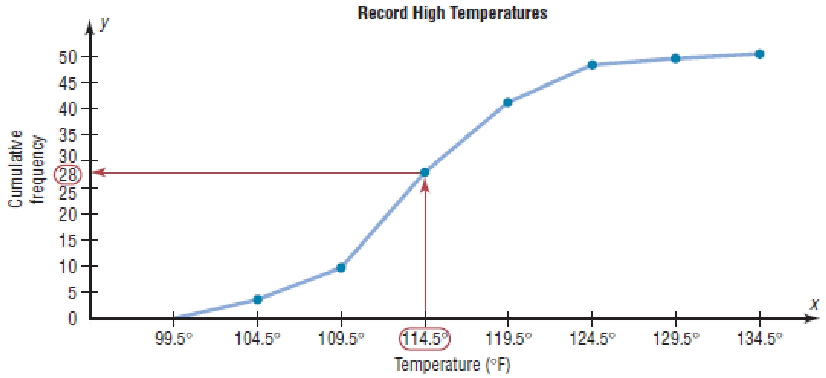
(Depending on the numbers in the cumulative frequency columns, scales such as 0, 1, 2, 3, . . . , or 5, 10, 15, 20, . . . , or 1000, 2000, 3000, . . . can be used.

Do *not* label the *y* axis with the numbers in the cumulative frequency column.) In this example, a scale of 0, 5, 10, 15, . . . will be used.

Step 3 Plot the cumulative frequency at each upper class boundary, as shown in Figure below. Upper boundaries are used since the cumulative frequencies represent the number of data values accumulated up to the upper boundary of each class.



Step 4 Starting with the first upper class boundary, 104.5, connect adjacent points with line segments, as shown in the figure. Then extend the graph to the first lower class boundary, 99.5, on the *x* axis.



Cumulative frequency graphs are used to visually represent how many values are below a certain upper class boundary. For example, to find out how many record high temperatures are less than 114.5_F, locate 114.5_F on the *x* axis, draw a vertical line up until it intersects the graph, and then draw a horizontal line at that point to the *y* axis. The *y* axis value is 28, as shown in the figure.

5. Relative Frequency Graphs

The histogram, the frequency polygon, and the ogive shown previously were constructed by using frequencies in terms of the raw data. These distributions can be converted to distributions using *proportions* instead of raw data as frequencies. These types of graphs are called **relative frequency graphs.**

Example:

Construct a histogram, frequency polygon, and ogive using relative frequencies for the distribution (shown here) of the kilometers that 20 randomly selected runners ran during a given week.

Class boundaries	Frequency
5.5–10.5	1
10.5–15.5	2
15.5–20.5	3
20.5–25.5	5
25.5–30.5	4
30.5–35.5	3
35.5–40.5	2
	20

Solution:

Step 1: Convert each frequency to a proportion or relative frequency by dividing the frequency for each class by the total number of observations.

For class 5.5–10.5, the relative frequency is $\frac{1}{20} = 0.05$; for class 10.5–15.5, the relative frequency is $\frac{2}{20} = 0.10$; for class 15.5–20.5, the relative frequency is $\frac{3}{20} = 0.15$; and so on.

Place these values in the column labeled Relative frequency.

Class boundaries	Midpoints	Relative frequency
5.5–10.5	8	0.05
10.5–15.5	13	0.10
15.5–20.5	18	0.15
20.5–25.5	23	0.25
25.5–30.5	28	0.20
30.5–35.5	33	0.15
35.5–40.5	38	0.10
		1.00

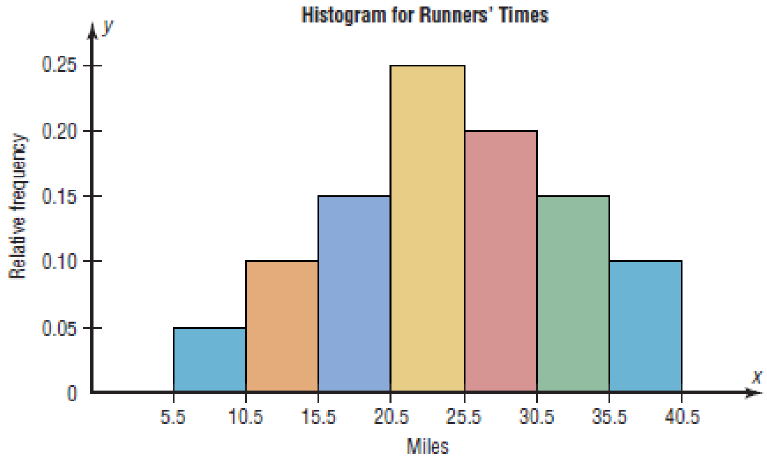
Step 2: Find the cumulative relative frequencies. To do this, add the frequency in each class to the total frequency of the preceding class. In this case, 0 +0.05 = 0.05, 0.05 + 0.10 = 0.15, 0.15 +0.15 =0.30,

0.30+0.25=0.55, etc. Place these values in the column labeled Cumulative relative frequency.

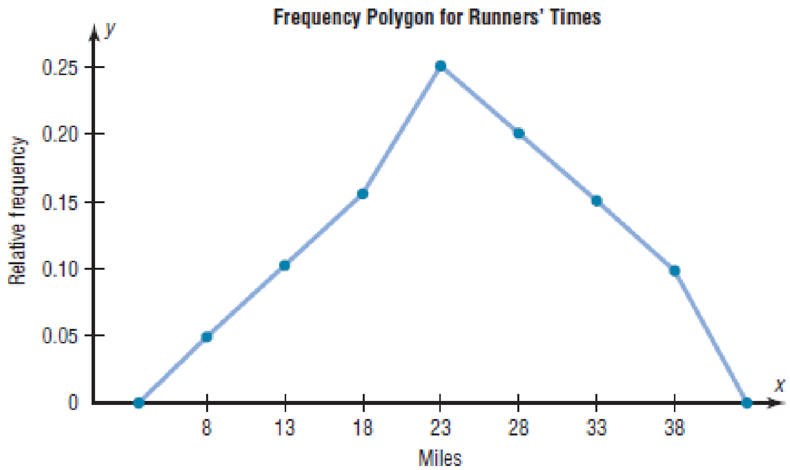
An alternative method would be to find the cumulative frequencies and then convert each one to a relative frequency.

	Cumulative frequency	Cumulative relative frequency
Less than 5.5	0	0.00
Less than 10.5	1	0.05
Less than 15.5	3	0.15
Less than 20.5	6	0.30
Less than 25.5	11	0.55
Less than 30.5	15	0.75
Less than 35.5	18	0.90
Less than 40.5	20	1.00

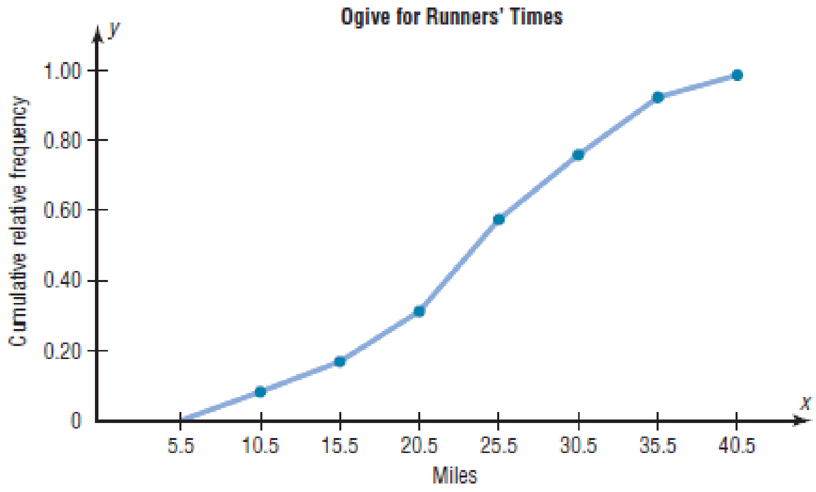
Step 3: Draw each graph as shown in Figure 2–7. For the histogram and ogive, use the class boundaries along the *x* axis. For the frequency polygon, use the midpoints on the *x* axis. The scale on the *y* axis uses proportions.



(a) Histogram



(b) Frequency polygon



(c) Ogive

6. Pie chart

A pie chart is used to display a set of categorical data. It is a circle that is divided into sections or wedges according to the percentage of frequencies in each category of the distribution.

$$Angle\ for\ sector\ S = \frac{Frequency\ of\ S.360^0}{Total\ frequency}$$

Since there are 360^0 in a circle, the frequency for each class must be converted into a proportional part of the circle. This conversion is done by using the formula

$$Degrees = \frac{f}{n}.360^0$$

where f is frequency for each class and n is the sum of the frequencies.

Hence, the following conversions are obtained. The degrees should sum to 360.

Each frequency must also be converted to a percentage by using the formula

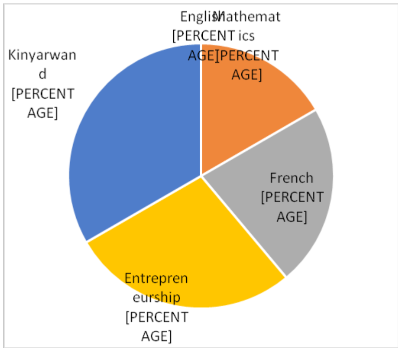
$$\frac{f}{n}.100\%$$

Next, using a protractor and a compass, draw the graph using the appropriate degree measures found in step 1, and label each section with the name and percentages.

Example:

1) One tutor in TTC want to check the level of how his/her student-teachers like different subject taught in Language Education option. The survey done on 60 student-teachers in English, Mathematics, French, Entrepreneurship and Kinyarwanda.

Here is the results he/she obtained respectively after making survey 12, 24, 6, 10, and 8. Present the data on pie chart.



2) Construct a pie graph showing the blood types of the army inductees described above. The frequency distribution is repeated here.

Class	Frequency	Percent
A	5	20
B	7	28
O	9	36
AB	4	16
	25	100

Solution:

Step 1: Find the number of degrees for each class, using the formula.

$$Degrees = \frac{f}{n}.360^0$$

For each class then the following results are obtained:

$A: \frac{5}{25}.360^0 = 72^0$; B: $\frac{7}{25}.360^0 = 100.8^0$;

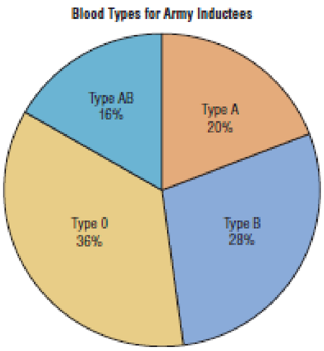
O: $\frac{9}{25}.360^0 = 129.6^0$; $AB: \frac{4}{25}.360^0 = 57.6^0$

Step 2: Find the percentages.

$$\textbf{A:}\ \frac{5}{25}\times 100\% = 20\% \qquad \textbf{B:}\ \frac{7}{25}\times 100\% = 28\%$$

$$\textbf{O:}\ \frac{9}{25}\times 100\% = 36\% \qquad \textbf{AB:}\ \frac{4}{25}\times 100\% = 16\%$$

Step 3: Using a protractor, graph each section and write its name and corresponding percentage, as shown in the Figure



7. Stem and Leaf Plots

The stem and leaf plot is a method of organizing data and is a combination of sorting and graphing. It uses part of the data value as the stem and part of the data value as the leaf to form groups or classes. It has the advantage over a grouped frequency distribution of retaining the actual data while showing them in graphical form.

Examples:

1) At an outpatient testing center, the number of cardiograms performed each day for 20 days is shown. Construct a stem and leaf plot for the data.

25	31	20	32	13
14	43	02	57	23
36	32	33	32	44
32	52	44	51	45

Step 1: Arrange the data in order: 02, 13, 14, 20, 23, 25, 31, 32, 32, 32, 32, 33, 36, 43, 44, 44, 45, 51, 52, 57

***Note**:* Arranging the data in order is not essential and can be cumbersome when the data set is large; however, it is helpful in constructing a stem and leaf plot. The leaves in the final stem and leaf plot should be arranged in order.

Step 2 Separate the data according to the first digit, as shown.

02 13, 14 20, 23, 25 31, 32, 32, 32, 32, 33, 36, 43, 44, 44, 45 51, 52, 57

Step 3: A display can be made by using the leading digit as the *stem* and the trailing digit as the *leaf.*

For **example**, for the value 32, the leading digit, 3, is the stem and the trailing digit, 2, is the leaf. For the value 14, the 1 is the stem and the 4 is the leaf. Now a plot can be constructed as follows:

Leading digit (stem)	Trailing digit (leaf)
0	2
1	3 4
2	0 3 5
3	1 2 2 2 2 3 6
4	3 4 4 5
5	1 2 7

It shows that the distribution peaks in the center and that there are no gaps in the data. For 7 of the 20 days, the number of patients receiving cardiograms was between 31 and 36. The plot also shows that the testing center treated from a minimum of 2 patients to a maximum of 57 patients in any one day.

If there are no data values in a class, you should write the stem number and leave the leaf row blank. Do not put a zero in the leaf row.

2) The mathematical competence scores of 10 student-teachers participating in mathematics competition are as follows: 15, 16, 21, 23, 23, 26, 26, 30, 32, 41. Construct a stem and leaf display for these data by using 2, 3, and 4 as your stems.

Solution

Stem	Leaf
1	5 6
2	1 3 3 6 6
3	0 2
4	1

This means that data are concentrated in twenties.

3) The following are results obtained by student-teachers in French out of 50.

37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20,

20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6

Use stem and leaf to display data

Solution:

Numbers 3, 2, 1, and 0, arranged as a stems to the left of the bars. The other numbers come in the leaf part.

Stem	Leaf
3	2 3 3 7
2	0 0 1 1 1 2 2 2 3 8 8 9
1	2 2 4 4 4 5 6 8 8 8 8 9 9
0	6 9

From the table, we see that data are concentrated in tens and twenties.

Application activity 3.2

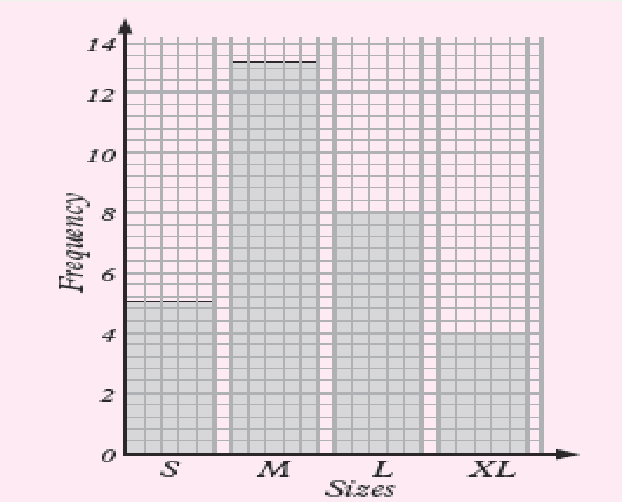
1) Suppose that a tutor conducted a test for student-teachers and the marks out of 10 were as follows: 3 3 3 5 6 4 6 7 8 3 8 8 8 10 9 10 9 10 8 10 6
   a) Draw a frequency table;
   b) Draw a relative frequency table and calculate percentage for each;
   c) Present data in cumulative frequency table, hence show the number of student-teachers who did the test.
2) During the examination of English, student-teachers got the following results out of 80: 54, 42, 61, 47, 24, 43, 55, 62, 30, 27, 28, 43, 54, 46, 25, 32, 49, 73, 50, 45.

   Present the results using stem and leaf.
3) A firm making artificial sand sold its products in four cities: 5% was sold in Huye, 15% in Musanze, 15% in Kayonza and 65% was sold in Rwamagana.
   a) What would be the angles on pie chart?
   b) Draw a pie chart to represent this information.
   c) Use the pie chart to comment on these findings.

3.3 Graph interpretation and Interpretation of statistical data

Activity 3.3

The graph below shows the sizes of sweaters worn by 30 year 1 students in a certain school. Observe it and interpret it by answering the questions below it:



a) How many students are with small size?

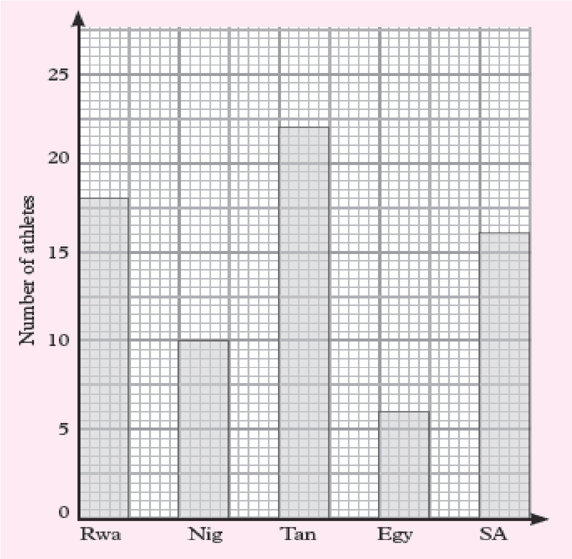
b) How many students with medium size, large size and extra large size are there?

CONTENT SUMMARY

Once data has been collected, they may be presented or displayed in various ways including graphs. Such displays make it easier to interpret and compare the data.

Examples

1) The bar graph shows the number of athletes who represented five African countries in an international championship.



a) What was the total number of athletes representing the five countries?

b) What was the smallest number of athletes representing one country?

c) What was the most number of athletes representing a country?

d) Represent the information on the graph on a frequency table.

Solution:

We read the data on the graph:

a) Total number of athletes are: 18 + 10 + 22 + 6 + 16 = 72 athletes

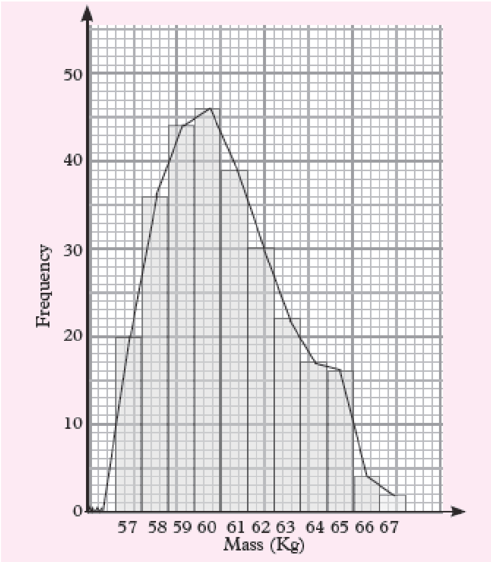
b) 6 athletes

c) 22 athletes

d) Representation of the given information on the graph on a frequency table.

Country	**Number of athletes**
Rwanda	18
Nigeria	10
Tanzania	22
Egypt	6
South Africa	16
Total	**72**

2) Use a scale vertical scale 2cm: 10 students and Horizontal scale 2cm: 10 represented on histogram below to answers the questions that follows



a) Estimate the mode

b) Calculate the range

Solution:

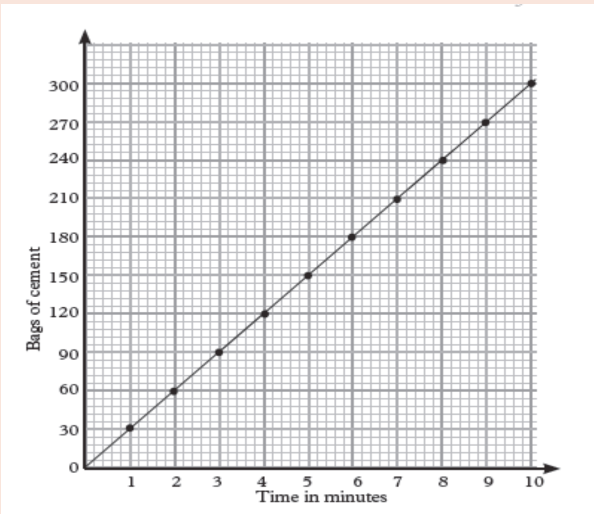
a) To estimate the mode graphically, we identify the bar that represents the highest frequency. The mass with the highest frequency is 60 kg. It represents the mode.

b) The highest mass = 67 kg and the lowest mass = 57 kg

Then, The range=highest mass-lowest mass= $67kg - 57kg = 10kg$

Application activity 3.3

The line graph below shows bags of cement produced by CIMERWA (industry cement factory) in a minute.



a) Find how many bags of cement will be produced in 8 minutes, 3 minutes12 seconds, 5 minutes and 7 minutes.

b) Calculate how long it will take to produce 78 bags of cement.

c) Draw a frequency table to show the number of bags produced and the time taken.

3.4 Measures of central tendencies for ungrouped data

Activity 3.4

Conduct a research in the library or on the internet and explain measures of central tendency, their types and provide examples.

Insist on explaining how to determine the Mean, Mode, Median and their role when interpreting statistic data.

CONTENT SUMMARY

Measures of central tendency were studies in S1 and S2.

1. The mean

The *mean,* also known as the *arithmetic average,* is found by adding the values of the data and dividing by the total number of values.

Suppose that a fruit seller earned the flowing money from Monday to Friday respectively: 300, 200, 600, 500, and 400 Rwandan francs. The mean of this money explains the same daily amount of money that she should earn to totalize the same amount in 5 days.

$$\bar{X} = \frac{X_1 + X_2 + X_3 + ...X_n}{n} = \frac{\sum X}{n} = \frac{300+200+600+500+400}{5} = 400$$

Or $\bar{x} = \frac{1}{n}\sum xfi$

This is called the mean and is equivalent to sharing out all data evenly.

2. The median:

If the data is well arranged in an order from the smallest to the largest, the median is the middle number or the central number of the range. When total observation ($\sum fi = n$) is odd the median is given by $\left(\frac{n+1}{2}\right)^{th}$ number which is located on this position. On the other side when n is even, $\frac{1}{2}\left[\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}\right]$, then the median is a half of the sum of number located on those two positions.

Examples

1) Calculate the median of the following numbers: 4, 5, 7,2, 1

Solution:

Arrange data from lowest to highest number as 1, 2, 4, 5, 7. The median is given by

$$Me = \left(\frac{n+1}{2}\right)^{th} \text{ element/value}$$

$$Me = \left(\frac{5+1}{2}\right)^{th} = 3^{rd}$$ element value, then $Me = 4$

2) Calculate the median of the following numbers: 4, 5, 7, 2, 1 and 8

Solution:

Arrange numbers in ascending order: 1, 2, 4, 5, 7, 8. Total observation $(n) = 6$

since the total observation is even then, $Me = \frac{1}{2}\left(\left[\left(\frac{n}{2}\right) + \left(\frac{n}{2}+1\right)\right]^{th} \text{ element}\right)$

$$Me = \frac{4+5}{2} = 4.5$$

3. The mode:

The mode is the number that appears the most often from the set of data. It represents the value which appears more frequently in the data.

Examples:

Calculate the mean and mode of the following set of numbers: 3, 4, 4, 6, 8, 5, 4, and 8.

Solution: $\bar{x} = \frac{1}{n}\sum xfi$

x	fi	xfi
3	1	3
4	3	12
5	1	5
6	1	6
8	2	16
	$\sum fi = n = 8$	$\sum xfi = 42$

The mean is given by $\bar{x} = \frac{1}{n}\sum xfi$, $\bar{x} = \frac{42}{8} = 5.25$

The median: Arrange data first 3, 4, 4, 4, 5, 6, 8,8. Total observation is 8, Mode is 4

Application activity 3.4

1) A group of student-teachers from language education were asked how many books they had read in previous year, the results are shown in the frequency table below. Calculate the mean, median and mode of the number books read.

Number of books	0	1	2	3	4	5	6	7	8
Frequency (number of student teachers)	5	5	6	9	11	7	4	2	1

2) During oral presentation of internship report for year three student-teachers, the first 10 student-teachers scored the following marks out of 10:

   8, 7, 9, 10, 8, 9, 8, 6, 7 and 10

   Calculate the mean and the median of the group.

3.5 Measures of central tendencies for grouped data: mode, mean, median and midrange

Activity 3.4

1) Conduct a research in the library or on the internet and explain measures of central tendency for grouped data and provide examples.

   Insist on explaining how to determine the Mean, Mode, Median and their role when interpreting statistic data.

2) Using the frequency distribution given below, find the Mean. The data represent the number of kilometers run by a sample of 20 runners during one week:

A Class	B Frequency f	C Midpoint X_m	D $f \cdot X_m$
5.5–10.5	1		
10.5–15.5	2		
15.5–20.5	3		
20.5–25.5	5		
25.5–30.5	4		
30.5–35.5	3		
35.5–40.5	2		
	$n = 20$		

   What does this Mean represent considering the class in which it is located in the data?

1. The mean

The process of finding the mean is the same as the one applied in the ungrouped data with the exeption that the midpoints x_m of each class in grouped data plays the role of x_i used in ungrouped data.

$$\bar{X} = \frac{\sum f.X_m}{n}$$

2. The mode

The mode for grouped data is the modal class. The **modal class** is the class with the largest frequency. The mode can be determined using the following formula:

$$Mode = L + \left(\frac{f_m - f_1}{(2f_m - f_1 - f_2)}\right)w$$

Where:

L: the lower limit of the modal class

f_m: the modal frequency

f_1: the frequency of the immediate class below the modal class

f_2: the frequency of the immediate class above the modal class

w: modal class width.

Example:

Find the modal class for the frequency distribution of kilometers that 20 runners ran in one week.

Class	Frequency
5.5–10.5	1
10.5–15.5	2
15.5–20.5	3
20.5–25.5	5 ← Modal class
25.5–30.5	4
30.5–35.5	3
35.5–40.5	2

The modal class is 20.5–25.5, since it has the largest frequency. Sometimes the midpoint of the class is used rather than the boundaries; hence, the mode could also be given as 23 km per week.

3. Median of grouped data

In the case of continuous frequency of distribution, we first locate the median

by cumulating the frequency until $(\frac{n}{2})^{th}$ point is reached. Finally, the median is determined within this class by using formula. The procedures thus involve the following steps:

1) Compute cumulative frequencies

2) Find the size of $\left(\frac{n}{2}\right)^{th}$ item, see that $\left(\frac{n+1}{2}\right)^{th}$ is not used in this case. N is the total frequency.

3) Locate the median class in cumulative frequency column where the size of $\left(\frac{n}{2}\right)^{th}$ item falls.

4) Obtain the median value by applying the formula

$$Mediam = l_1 + \frac{\frac{n}{2} - cufi}{fi}(l_2 - l_1)$$

where l_1 is lower limit of the median class, l_2 is upper limit of the median class, fi is the frequency and $cufi$ is the cumulative frequency of the class preceding the median class.

Example

The following table shows the weekly consumption of electricity of 56 families

weekly consumption	0-10	10-20	20-30	30-40	40-50
Number of families	16	12	18	6	4

Calculate the median weekly consumption.

Solution

Weekly consumption	**Frequency**	**Cumulative frequency**
0-10	16	16
10-20	12	28
20-30	18	46
30-40	6	52
40-50	4	56
	Summation=56	

Median is the value of $\left(\frac{n}{2}\right)^{th}$ position = $\left(\frac{56}{2}\right)^{th} = 28^{th}$ position, which lies in the class 10-20. Thus 10-20 is the median class

$$Mediam = l_1 + \frac{\frac{n}{2} - cufi}{fi}(l_2 - l_1)$$

$$Median = 10 + \frac{28 - 16}{12}(20 - 10)$$

$$Me = 10 + \frac{12}{12} * 10 = 10 + 10 = 20$$

Hence the median is 20 units.

Question

1) Calculate the median for the following distribution

Class interval	130-140	140-150	150-160	160-170	170-180	180-190	190-200
Frequency	5	9	17	28	24	10	7

4. The Midrange

The *midrange* is a rough estimate of the middle. It is found by adding the lowest and highest values in the data set and dividing by 2. It is a very rough estimate of the average and can be affected by one extremely high or low value.

$$MR = \frac{Lowest\ value + highest\ value}{2}$$

5. Weighted mean

It is a mean of a data set in which not all values are equally represented. Find the weighted mean of a variable X by multiplying each value by its corresponding weight and dividing the sum of the products by the sum of the weights.

$$\bar{X} = \frac{w_1X_1 + w_2X_2 + w_3X_3 + \ldots w_n X_n}{w_1 + w_2 + w_3 + \ldots . + w_n} = \frac{\sum wX}{\sum w}$$

Where $w_1, w_2, \ldots w_n$ are the weights and $X_1, X_2, \ldots X_n$ are the values.

Example

A student received an A in English Composition I (3 credits), a C in Introduction to Psychology (3 credits), a B in Biology I (4 credits), and a D in Physical Education (2 credits). Assuming A=4 grade points, B=3 grade points, C =2 grade points, D =1 grade point, and F=0 grade points, find the student's grade point average.

Solution

Course	Credits (w)	Grade (X)
English Composition I	3	A (4 points)
Introduction to Psychology	3	C (2 points)
Biology I	4	B (3 points)
Physical Education	2	D (1 point)

$$\bar{X} = \frac{\Sigma wX}{\Sigma w} = \frac{3 \cdot 4 + 3 \cdot 2 + 4 \cdot 3 + 2 \cdot 1}{3 + 3 + 4 + 2} = \frac{32}{12} = 2.7$$

$$\bar{X}=\frac{\sum wX}{\sum w}=\frac{3.4+3.2+4.3+2.1}{3+3+4+2}=\frac{32}{12}=2.7$$

Application activity 3.5

The data below shows the marks scored by a group of students in a mathematics out of 100: 72; 63; 51; 25; 31; 49; 51; 27; 46; 42; 25; 39; 38; 39; 55; 38; 35; 64;67;37.

Use a grouped data of 5 intervals and determine:

a) The mean mark
b) The median
c) The modal class
d) The range.

3.6 Measures of dispersion for ungrouped data and for grouped data

Activity 3.6

Before starting the third term, tutor calculated the mean mark of five student-teachers got in second term in Mathematics and he/she obtained that the mean mark is $\bar{x}=16.875$. Use this mean to complete the table below:

x	f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
12	4			
13	2			
15	1			
19	4			
21	5			
	$\sum f=$			$\sum f(x-\bar{x})^2=$

Explain the expression $\sum f\left(x-\bar{x}\right)^2$ in your own words.

CONTENT SUMMARY

The word dispersion has a technical meaning in statistics. The average measures the center of the data. It is one aspect of observations. Another feature of the observations is how the observations are spread about the center. The observation may be close to the center or they may be spread away from the center. If the observation are close to the center (usually the arithmetic mean or median), we say that dispersion, scatter or variation is small. If the observations are spread away from the center, we say that dispersion is large.

The study of dispersion is very important in statistical data. If in a certain factory there is consistence in the wages of workers, the workers will be satisfied. But if some workers have high wages and some have low wages, there will be unrest among the low paid workers and they might go on strikes and arrange demonstrations. If in a certain country some people are very poor and some are very high rich, we say there is economic disparity. It means that dispersion is large.

The extent or degree in which data tend to spread around an average is also called the dispersion or variation. Measures of dispersion help us in studying the extent to which observations are scattered around the average or central value. Such measures are helpful in comparing two or more sets of data with regard to their variability.

Properties of a good measure of dispersion

i) It should be simple to calculate and easy to understand

ii) It should be rigidly defined

iii) Its computation be based on all the observations

iv) It should be amenable to further algebraic treatment

Some measures of dispersion are Quartiles, variance, Range, standard deviation, coefficient of variation.

1. Quartile

A measure which divides any array into four equal parts is known as quartile. Each proportion contains equal number of items. The fist and the second and the third points are termed as first quartile. The first quartile has 25% of the items of the distribution below it and 75% of the items are greater than it. The second quartile which is the median has 50% of the observations above it and 50% of the observations below it. For the arranged data in ascending order and quartiles are calculated as follows:

$Q_1 = \frac{1}{4}(n+1)^{th}$ observation , $Q_2 = \frac{1}{2}(n+1)^{th}$ observation , $Q_3 = \frac{3}{4}(n+1)^{th}$ observation

The **inter-quartile range** is given by the difference between third quartile and the first quartile $Q_3 - Q_1$.

Examples

1) Find the first and the second quartiles of the data set: 1, 3, 4, 5, 5, 6, 9, 14, 21.

Solution:

$Q_1 = \frac{1}{4}(n+1)^{th}$ value $= \frac{1}{4}(9+1)^{th} = (2.5)^{th}$ value, then $Q_1 = 4$

$Q_2 = Me$, $Q_2 = \frac{1}{2}(n+1)^{th}$ value $= \frac{1}{2}(9+1)^{th} = 5^{th}$ value, then $Q_2 = 5$

2) In the series given below, calculate the first quartile, second quartile and inter-quartile range

   2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

Solution

$Q_1 = \frac{1}{4}(n+1)^{th} = \left(\frac{11+1}{4}\right)^{th} = 3^{rd}$ Observation, then the first quartile is $Q_1 = 6$

$Q_3 = \frac{3}{4}(n+1)^{th} = \left(\frac{3}{4}(11+1)\right)^{th} = 9^{th}$ Observation, then the third quartile is $Q_3 = 18$

Inter-quartile range $= Q_3 - Q_1 = 18 - 6 = 12$

2. Variance

Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other.

The variance is denoted and defined by

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}$$

Developing this formula we have

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \left(\bar{x}\right)^2\right)}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n}2\bar{x}\sum_{i=1}^{n} x_i + \frac{1}{n}\left(\bar{x}\right)^2\sum_{i=1}^{n} 1$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\bar{x}\bar{x} + \left(\bar{x}\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Thus, the variance is also defined by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

Recall that the mean of the set of *n* values $x_1, x_2, x_3, ..., x_n$ is denoted and defined by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} \qquad = \sum_{i=1}^{n}\frac{x_i}{n} \qquad = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Examples

1) Calculate the variance of the following distribution: 9, 3, 8, 8, 9, 8, 9, 18

Solution

$$\bar{x} = \frac{9+3+8+8+9+8+9+18}{8} = 9$$

$$\sigma^2 = \frac{(9-9)^2 + (3-9)^2 + (8-9)^2 + (8-9)^2 + (9-9)^2 + (8-9)^2 + (9-9)^2 + (18-9)^2}{8} = 15$$

3. Standard deviation

The standard deviation has the same dimension as the data, and hence is comparable to deviations from the mean. We define the **standard deviation** to be the square root of the variance.

Thus, the standard deviation is denoted and defined by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n}} \quad \text{or} \quad \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2}$$

The following results follow directly from the definitions of mean and standard deviation:

- When all the data values are multiplied by a constant *a*, the new mean and new standard deviation are equal to *a* times the original mean and standard deviation. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $a\left(\bar{x}\right)$ and the standard deviation is $a\sigma$.
- When a constant value, *b*, is added to all data values, then the new mean is increased by *b*. However standard deviation does not change. That is, the mean of $ax_1, ax_2, ax_3, ..., ax_n$ is $\bar{x} + b$ and the standard deviation is σ.

Examples

1) The six runners in a 200 meter race clocked times (in seconds) of 24.2, 23.7, 25.0, 23.7, 24.0,24.6.

   a) Find the mean and standard deviation of these times.

Solution

a) $\bar{x} = \dfrac{24.2 + 23.7 + 25.0 + 23.7 + 24.0 + 24.6}{6} = 24.2 \text{ seconds}$

$$\sigma = \sqrt{\frac{(24.2-24.2)^2 + (23.7-24.2)^2 + (25.0-24.2)^2 + (23.7-24.2)^2 + (24.0-24.2)^2 + (24.6-24.2)^2}{6}}$$

$= 0.473 \text{ seconds}$

The method which uses the formula for the standard deviation is not necessarily the most efficient.

Consider the following: $\frac{\sum\left(x-\bar{x}\right)^2}{n}=\frac{1}{n}\sum x^2-\left(\bar{x}\right)^2$

2) The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the mean height and the standard deviation of the heights.

Solution:

$$\text{Mean} = \frac{1}{6}(1.42+1.35+1.37+1.50+1.38+1.30)=1.39\ m$$

$$\text{Variance} = \frac{1}{6}(1.42^2+1.35^2+1.37^2+1.50^2+1.38^2+1.30^2)-1.39^2=0.00386\ m^2$$

$$\text{Standard deviation} = \sqrt{0.00386\ m^2}=0.0621\ m$$

Variance and Standard Deviation for Grouped Data

The variance for grouped data is given by $\delta^2=\frac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f}$ and the standard deviation for grouped data is given by $\delta=\sqrt{\frac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f}}$

Example:

The frequency distribution result of Mathematics test on 30 marks of 48 students-teachers is as shown in the table below. Calculate the variance and the standard deviation from the mean marks of results

Class	Frequency
20.5-20.9	3
21.0-21.4	10
21.5-21.9	11
22.0-22.4	13
22.5-22.9	9
23.0-23.4	2
Total	48

Solution:

From the data the distribution mean value $\bar{x} = 21.92$,

The "$x - values$" are the class mid-points values, i.e

20.7; 21.2; 21.7; 22.2; 22.7; 23.2

Thus, the $\left(x-\bar{x}\right)^2$ values are:

$$(20.7-21.92)^2;(21.2-21.92)^2;(21.7-21.92)^2;(22.2-21.92)^2;(22.7-21.92)^2;(23.2-21.92)^2$$

and the $f\left(x-\bar{x}\right)^2$ values are:

$$3(20.7-21.92)^2;10(21.2-21.92)^2;11(21.7-21.92)^2;13(22.2-21.92)^2;9(22.7-21.92)^2;2(23.2-21.92)^2$$

The $\sum f\left(x-\bar{x}\right)^2$ are:

$$4.4652+5.1840+0.5324+1.0192+5.4756+3.2768=19.9532$$

$\delta^2 = \dfrac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f} = \dfrac{19.9532}{48} = 0.41569$ and the standard deviation is

$$\delta = \sqrt{\frac{\sum\left\{f\left(x-\bar{x}\right)^2\right\}}{\sum f}} = \sqrt{0.41569} = 0.645$$

Note that the standard deviation for the sample data is given by:

$\delta = \sqrt{\dfrac{\sum X^2 - [(\sum X)^2 / n}{n-1}}$ and the standard deviation for grouped data is given by: $\delta = \sqrt{\dfrac{\sum f.X_m{}^2 - [(\sum f.X_m)^2 / n]}{n-1}}$

4. Coefficient of variation

The coefficient of variation measures variability in relation to the mean (or average) and is used to compare the relative dispersion in one type of data with the relative dispersion in another type of data. It allows us to compare the dispersions of two different distributions if their means are positive. The greater dispersion corresponds to the value of the coefficient of greater variation.

The coefficient of variation is a calculation built as follows:

$$Cv = \frac{\sigma}{\bar{x}} \times 100$$

Where:

- σ is the standard deviation
- $\bar{x}$ is the mean.

Example:

One data series has a mean of 140 and standard deviation 28.28. The second data series has a mean of 150 and standard deviation 24. Which of the two has a greater dispersion?

Solution:

$$CV_1 = \frac{28.28}{140} \times 100 = 20.2\%$$

$$CV_2 = \frac{24}{150} \times 100 = 16\%$$

The first data series has a higher dispersion.

5. Range

The range of a set of observations is the difference in values between the largest and the smallest observations in the set. In the case of grouped data the range is defined as the difference between the upper limit of the highest class and the lower limit of the smallest class.

Example

Calculate the range of the following set of the data set: 1, 3, 4, 5, 5, 6, 9, 14 and 21

Solution:

From the given series the lowest data is 1 and the highest data is 21

The Range $=$ highest value $-$ lowest value

Range $= 21 - 1 = 20$

Application activity 3.6

1) Out of 4 observations done by tutor of English, arranged in descending order, the 5^{th}, 7^{th}, 8^{th} and 10^{th} observations are respectively 89, 64, 60 and 49. Calculate the median of all the 4 observations.
2) In the following statistical series, calculate the standard deviation of the following set of data

   56,54,55,59,58,57,55
3) In the classroom of language education the first ten student-teachers scored the following marks out of 10 in a quiz of French

   5, 6, 5, 2, 4, 7, 8, 9, 7, 5.

   a) Calculate the mean, median and the modal mark
   b) Calculate the quartiles and inter-quartile range
   c) Calculate the variance and the standard deviation
   d) Evaluate the coefficient of variation

3.7 Practical activity in statistics

Activity 3.6

Student teachers were interested in getting information on the number of hours patients spent in the hospital in a certain week. The table below summarizes the data they recorded where the frequency indicates the number of patients.

Class boundaries(of hours)	Frequency (number of patients)
7.5-12.5	3
12.5-17.5	5
17.5-22.5	15
22.5-27.5	5
27.5-32.5	2

Determine and interpret the average, the modal class, the median, the variance and the standard deviation related to the number of hours a sick person spends in that hospital.

What is the advice you can provide to the manager of that hospital if he/she has a few number of beds?

With observation, data are collected through direct observation. Information could also be collected using an existing table that shows types of data to collect in a certain period of time, this method of collecting data is reliable and accurate.

Once data has been collected, they may be presented or displayed in various ways. Such displays make it easier to interpret and compare the data.

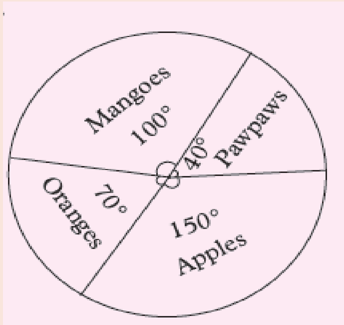
After the collection of data there is need of interpreting them and here there are some tips:

- Collect your data and make it as clean as possible.
- Choose the type of analysis to perform: qualitative or quantitative
- Analyze the data through various statistical methods such as mean, mode, standard deviation or Frequency distribution tables
- Reflect on your own thinking and reasoning and analyze your data and then interpret them referring to the reality.

During interpretation, avoid subjective bias, false information and inaccurate decisions.

Application activity 3.7

1) After selling fruits in a market, Martha had a total of 144 fruits remaining. The pie chart below shows each type of fruit that remained.



   a) Find the total cost of mangoes and pawpaws remaining if a mango sells at 30 FRW and a pawpaw at 160 Frw.

   b) Which type of fruit remained the most?

   c) What was the median number of fruit that remained?

   d) Draw a frequency table to display the information on the pie chart.

2) Use tailor's meters to correct data on the height for 20 people.

   a) Organize the corrected data with a frequency distribution

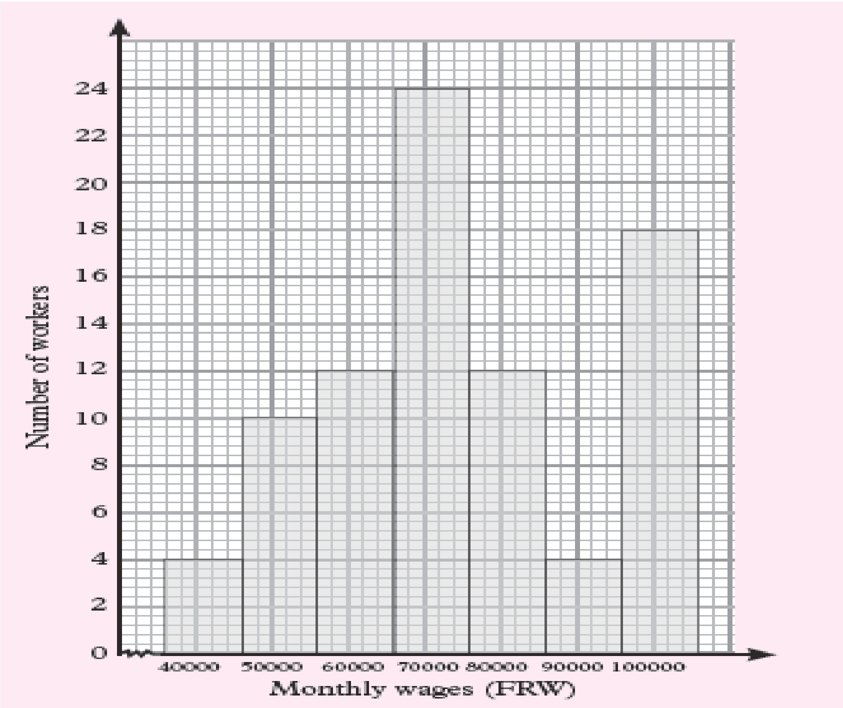
   b) Determine and interpret the mean, the mode and the median for the data.

   c) Determine and interpret the range, the quartiles and the standard deviation.

   d) Organize data into a grouped data distribution of 10 groups and determine: the modal class and the standard deviation. Compare this standard deviation with the one found in (c).

3.8. END UNIT ASSESSMENT

1) Use the graph below to answer the questions that follows



   a) Estimate the mode.

   b) State the range of the distribution.

   c) Draw a frequency distribution table from the graph

   d) Draw the frequency polygon represented by the histogram

2) In test of mathematics, 10 student-teachers got the following marks:

   6, 7, 8, 5, 7, 6, 6, 9, 4, 8

   a) Calculate the mean, mode, quartiles and inter-quartile range

   b) Calculate the variance and standard deviation

   c) Calculate the coefficient of variation.

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