# MATHEMATICS FOR TTCs

TUTOR'S BOOK

## YEAR

## 2

## OPTION:

Early Childhood and Lower Primary Education (ECLPE)

# FOREWORD

Dear Tutor,

Rwanda Education Board is honoured to present the tutor's guide for year two Mathematics in the option of ECLPE. This book serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, TTC curriculum was reviewed to train quality teachers who will confidently and efficiently implement the Competence Based Curriculum in pre-primary and primary education. The rationale of the changes is to ensure that TTC leavers are qualified for job opportunities and further studies in Higher Education in different programs under education career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. After a successful shift from knowledge to a competence based curriculum in general education, TTC textbooks also were elaborated to align them to the new curriculum.

The book provides active teaching and learning techniques that engage student teachers to develop competences. In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable and experienced people.

- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self explanatory so that you can easily use it.

It is divided in 3 parts:

The part I explains the structure of this book and gives you the methodological guidance;

The part II gives the sample lesson;

The part III details the teaching guidance for each concept given in the student book.

Even though this Teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, TTC Tutors, Teachers from general education for their technical support. A word of gratitude goes also to the Head Teachers and TTCs principals who availed their staff for various activities.

**Dr. NDAYAMBAJE Irénée**
Director General, REB

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I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this tutor`s guide was very much valuable.

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**Joan MURUNGI**

Head of CTLR Department

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# PART I. GENERAL INTRODUCTION

## 1.1. The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate student teachers while working on learning activities. More other, all application activities from the textbook have answers in this part.

## 1.2. Methodological guidance

### Developing competences

Since 2015, Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education and recently the TTC curriculum. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences, which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

| **Generic competences** | **Ways of developing generic competences** |
|---|---|
| Critical thinking | All activities that require learners to calculate, convert, interpret, analyse, compare and contrast, etc. have a common factor of developing critical thinking into learners |
| Creativity and innovation | All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners |
| Research and problem solving | All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners. |
| Communication | During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate …have a common character of developing communication skills into learners. |
| Co-operation, interpersonal relations and life skills | All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners. |
| Lifelong learning | All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support. |

| | |
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| Professional skills | Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and non verbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure. |

The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

### *1.2.1 Addressing cross cutting issues*

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture.

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

| Cross-Cutting Issue | Ways of addressing cross-cutting issues |
|---|---|
| **Comprehensive Sexuality Education:** The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour. | Using different charts and their interpretation, Mathematics tutor should lead students to discuss the following situations: "Alcohol abuse and unwanted pregnancies" and advise student teachers on how they can instil learners to fight those abuses.<br><br>Some examples can be given in powers and properties , logarithms and properties , and statistics |
| **Environment and Sustainability:** Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability. | Using Real life models or students' experience, Mathematics Tutor should lead student teachers to illustrate the situation of "population growth" and discuss its effects on the environment and sustainability.<br><br>Some examples in proportional change, logarithms, and polynomial functions |
| **Financial Education:**<br><br>The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life. | Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics Tutor can lead student teachers to discuss how to make appropriate financial decisions.<br><br>Some examples in ratios and proportions, statistics, equations, polynomial functions |

| | |
|---|---|
| **Gender:** At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc. | Mathematics Tutor should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process. |
| **Inclusive Education:** Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity. | Firstly, Mathematics Tutors need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to cater for learners with special educational needs. |
| **Peace and Values Education:** Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society. | Through a given lesson, a tutor should:<br>• Set a learning objective which is addressing positive attitudes and values,<br>• Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants;<br>• Encourage students to respect ideas for others. |
| **Standardization Culture:** Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing. | With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people. |

### *1.2.2 Guidance on how to help students with special education needs in classroom*

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

**Strategy to help learners with developmental impairment:**

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;
- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

**Strategy to help learners with visual impairment:**

- Help learners to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

**Strategy to help learners with hearing disabilities or communication difficulties**

- Always get the learner‘s attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

**Strategies to help learners with physical disabilities or mobility difficulties:**

- Adapt activities so that learners who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

**Adaptation of assessment strategies:**

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

| | |
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| Remedial activities | After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning.<br><br>These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students. |
| Consolidation activities | After introduction of any concept, a range number of activities can be provided to all students to enhance/ reinforce learning. |
| Extended activities | After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity. |

### *1.2.3 Guidance on assessment*

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

## Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

### Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies. There is also end of level/ cycle assessment in form of national examinations.

### When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- Before learning (diagnostic): At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- During learning (formative/continuous): When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- After learning (summative): At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

### Instruments used in assessment.

- **Observation:** This is where the Mathematics tutor gathers information by watching students interacting, conversing, working, playing, etc. A tutor can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the tutor has to continue observing each and every activity.

- **Questioning**

(a) Oral questioning: a process which requires a student to respond verbally to questions

(b) Class activities/ exercise: tasks that are given during the learning/ teaching process

(c) Short and informal questions usually asked during a lesson

(d) Homework and assignments: tasks assigned to students by their tutors to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

### *1.2.4 Teaching methods and techniques that promote active learning*

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Lab method:** Skills lab method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

## What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

| The role of the teacher in active learning | The role of learners in active learning |
|---|---|
| • The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities.<br>• He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods. | A learner engaged in active learning:<br>• Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation);<br>• Actively participates and takes responsibility for his/her own learning;<br>• Develops knowledge and skills in active ways; |

| | |
|---|---|
| • He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.<br>• Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities. | • Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings;<br>• Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking.<br>• Draws conclusions based on the findings from the learning activities. |

## Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

### 1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

### 2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

**Step 1**

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned)

**Step 2**

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of learners' findings/productions**
  - In this episode, the teacher invites representatives of groups to present their productions/findings.
  - After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.
- **Exploitation of learner's findings/ productions**
  - The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
  - Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.
- **Institutionalization or harmonization (summary/conclusion/ and examples)**
  - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.
- **Application activities**
  - Exercises of applying processes and products/objects related to learned unit/sub-unit
  - Exercises in real life contexts

- Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

### 3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

# PART II: SAMPLE LESSON

**School Name:** ……… **Tutor's name:** ……………………

| Term | Date | Subject | Class | Unit No | Lesson No | Duration | Class size |
|---|---|---|---|---|---|---|---|
| III | …./…./202.0 | Mathematics | Year2<br>ECLPE | 2 | 2 of 10 | 40 min | ... |

| | |
|---|---|
| **Type of Special Educational Needs to be catered for in this lesson and number of learners in each category** | 1 slow learner and 1 low vision learners: |
| **Unit title** | Points, lines and geometric shapes in 2D |
| **Key unit competency:** | Determine equations of lines and circles. |
| **Title of the lesson** | Cartesian coordinates of a point |
| **Instructional Objective** | Using a meter ruler and papers , learners will be able to accurately draw XY Plane and locate coordinates of a point in 2D. |
| **Plan for this Class (location: in / outside)** | The lesson is held indoor, the class is organized into groups ,1 slow learner is working with others in group, and 1 low vision learner seats on the front desk near the blackboard in order to see and participate fully in all activities. The low vision learner will be given big print papers on which coordinates a points are located in XY plane |
| **Learning Materials (for ALL learners)** | Textbooks, rulers, working sheets, mathematical sets. |
| **References** | - ECLPE TTC Math syllabus,<br>- Year2 Mathematics textbook and Teacher's guide for ECLPE |

| Timing for each step | Description of teaching and learning activity | | Generic competences and cross cutting issues to be addressed + a short explanation |
|---|---|---|---|
| | - Students work individually the questions in the introduction, and the correction is done on the chalk board by two students, under the guidance of the tutor.<br><br>-Then in different groups, they work out the given activity and discuss on how to locate coordinates of a point in XY plane. Students' presentation by a sampled group will be followed by the interaction of students and harmonization of the results under the facilitation of the tutor. Inn pairs students discuss the solved examples and compare their results with the answer proposed in the book. Finally, the students are assigned individual tasks, and the correction is done on the chalk board, and the tutor winds up the lesson. | | |
| | **Teacher activities** | **Learner activities** | |
| 1. **Introduction:**<br><br>5 minutes | The teacher asks students to work individually and draw a Cartesian plane by indicating X- axis and Y- axis and graduate it properly by using 1 cm.<br><br>- The teacher links the introduction to the lesson of the day | - Students work individually.<br><br>-Two students, one after another, draw a Cartesian plane on the chalkboard: | **Communication** skills developed through the presentation and sharing of ideas. |

| 2. **Development of the lesson** | | | |
|---|---|---|---|
| **2.1 Discovery activity:**<br><br>**10 minutes** | - The teacher organizes the students into groups<br>- Teacher gives students activity 2.1 and gives instructions related to the task<br>- Teacher goes around to monitor the work of each group and provide assistance where needed. | -Learners form groups<br>-Each group analyzes and discuss the activity 2.1 under the direction of the task manager of the group.<br>- Students present to the teacher their eventual problems<br>-students in pairs read through the example in their books on how to represent coordinates of point inn XY plane | • **Cooperation** and **communication** skills through discussions.<br>• **Peace and values education**; Cooperation, mutual respect, tolerance through discussions with people with different views and respect one's views . |
| **2.2 Presentation of student's findings and exploitation:**<br><br>**15 minutes.** | - Teacher invites the one member of a sampled group to present the findings of his/her group<br>- The teacher encourages students to follow attentively<br>- Teacher takes notes on key points from learners' presentation.<br>- The teacher asks students to amend the presentation and to evaluate their work. | -The reporter presents the work on the behalf of the group.<br>**Expected answers**<br>(**Refer** to solution of activity 2.1, in TG)<br>- students follow the presentation<br>- students evaluate the findings of other learners<br>- students evaluate their own findings | • **Cooperation** and **communication**/ attentive listening during presentations and group discussions.<br>• **Critical thinking** through evaluating other's findings. |

| | | | |
|---|---|---|---|
| **2.4.Conclusion/ Summary:**<br><br>**5 minutes**<br><br><br><br><br><br><br>**Assessment**<br>**5 minutes** | -Teacher facilitates the students to elaborate the summary of the presentation<br><br>-Teacher requests students to write down the main points in their books<br><br><br><br><br>- Teacher asks students to work out individually the application activity 2.1 | -The students come to the main point: The coordinates of a point in 2D appear in student-teacher book.<br>- Students take notes in their notebooks.<br>-Individually students work out the application activity 2.1 and finally they make a correction on the chalk board.<br>**Expected answers**<br>**(Refer** to solution of application activity 2.1, in TG) | - **Critical thinking** and **problem-solving** skills are developed through analyzing and solving real life Mathematical problem: e.g. finding the area and the perimeter of the playground;<br><br>-**Financial education** is addressed through good management of the money found when you sell a field given the price of one square meter;<br>-**Standardization culture**: Use correctly the unit of length measurement. |
| **Observation on lesson delivery** | To be completed after receiving the feed-back from the students (what did the students like, what challenged them...). | | |

# PART III: UNIT DEVELOPMENT

# UNIT 1

# SEQUENCES

## 1.1 Key unit competence:

Apply arithmetic and geometric sequences to solve problems in financial mathematics.

## 1.2 Prerequisites

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), Equations and inequalities (unit 2 Year 1), and on limits of functions (unit 4 Year 1).

## 1.3 Cross-cutting issues to be addressed

**Inclusive education** (promote education for all while teaching)

**Peace and value Education** (respect others' view and thoughts during class discussions)

**Gender** (provide equal opportunity to boys and girls in the lesson)

## 1.4 Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 1.0 found in unit 1 of student's book;
- Guide students to read and analyse the questions insisting on the analysis of the given data and to determine the number of insects that will be there in second, third, fourth,…$n^{th}$ generation.
- Invite some group members to present groups' findings, then try to harmonize their answers; try to insist on the list formed by the number of insects at any generation and the generalisation (number of insects at $n^{th}$ generation).

- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be leant in this unit.

**Answer for introductory activity:**

Number of insects in is given by:

$1^{st}$ generation $\rightarrow 126$, $2^{nd}$ generation $\rightarrow 126\times 2 = 252$,

$3^{rd}$ generation $\rightarrow 252\times 2 = 504$, $4^{th}$ generation $\rightarrow 1008$,

*At* $n^{th}$ *generation* $\rightarrow 126\times 2^{(n-1)} = 126.n^{(n-1)}$

$5^{th}$ generation $\rightarrow 2016$, $6^{th}$ generation $\rightarrow 4032$,

$7^{th}$ generation $\rightarrow 8064$, $8^{th}$ generation $\rightarrow 16128$,

$9^{th}$ generation $\rightarrow 32256$, $10^{th}$ generation $\rightarrow 64512$

## 1.5. List of lessons and sub-heading

| No | Lesson title | Learning objectives | Number of periods |
|---|---|---|---|
| 0 | Introductory activity | To arouse the curiosity of student - teachers on the content of unit 2. | 1 |
| 1 | Generalities on sequences and series | To define a sequence and series, infinite sequence and to determine terms of a sequence | 2 |
| 2 | Convergent or divergent sequences | Use basic concepts and formulas of sequences to identify the Convergent and divergent of series | 2 |
| 3 | Monotonic sequences | Use basic concepts of sequences to identify the Monotonic of sequences. | 2 |
| 4 | Arithmetic sequence and its general term | Use basic concepts of sequences to determine terms of an arithmetic sequence. | 3 |
| 5 | Arithmetic means of arithmetic means | Use basic concepts and formulas of arithmetic sequences to calculate arithmetic means. | 2 |

| | | | |
|---|---|---|---|
| 6 | Arithmetic series | Use basic concepts and formulas of sequences to calculate the sum of the first "$n$" terms of an arithmetic sequence | 2 |
| 7 | Harmonic sequences and its general term | Define and determine the terms of a harmonic sequence | 2 |
| 8 | Generalities on geometric sequences and its general term | To define and understand a geometric sequence and series.<br>Use basic concepts and formulas of sequences to find the general term of a geometric sequence. | 2 |
| 9 | Geometric means of a geometric sequence | Calculate the geometric means of a geometric sequence. | 3 |
| 10 | Geometric series | Use basic concepts and formulas of sequences to calculate the sum of the first "$n$" terms of a geometric sequence. | 2 |
| 11 | Infinity geometric series and its convergence | Explain infinity geometric series and determine the sum<br>Explain the convergence of series | 2 |
| 12 | Applications of sequences and series | To apply the concepts of sequences to solve problems involving sequences and series.<br>To appreciate the relationship between the sequences and other subjects to understand occurring situations. | 4 |
| 13 | End unit assessment | | 1 |
| Total number of periods | | | 30 |

## Lesson 1: Generalities on sequences and series

### a) Learning objectives

To define sequences and series, infinite sequence and to determine terms of a sequence

### b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the Arithmetic and on equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.1 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification as they have to give the fraction that represents the part they see when they fold a paper n times;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the fraction that represents the part they see when they fold the paper n times;
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to define sequences, series, infinite sequence, and to determine terms of a given sequence.
- After this step, guide students to do the application activity 1.1 and evaluate whether lesson objectives were achieved.

**Answer for activity 1.1**

When they fold once they see $\frac{1}{2}$; When they fold twice they see $\frac{1}{2^2}$;When they fold 3 times they see $\frac{1}{2^3}$; When they fold n times they see $\frac{1}{2^n}$;... When they fold 10 times they see $\frac{1}{2^{10}}$;...

The list of the fractions obtained is: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ..., \frac{1}{2^n}, ...$

**Answer for application activity 1.1**

Given $\{u_n\}: \begin{cases} u_0 = 1 \\ u_n = \dfrac{2n^2}{n^2+1} \end{cases}$

1) $u_1 = \frac{2\times 1^2}{1^2+1} = 1$; $u_2 = \frac{2\times 2^2}{2^2+1} = \frac{8}{5}$; $u_3 = \frac{2\times 3^2}{3^2+1} = \frac{18}{10}$

2) The five first terms of $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{+\infty}$ are: $\sqrt{2}, \sqrt{3}-\sqrt{2}, 2-\sqrt{3}, \sqrt{5}-2, \sqrt{6}-\sqrt{5}$

3) $\{2n-1\}_{n=1}^{+\infty}$

## Lesson 2: Convergent or divergent sequences

### a) Learning objectives

To explore sequences and differentiate the convergent from a divergent sequence.

### b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the Arithmetic and on equations and inequalities learnt in Year 1 Unit 1& Unit 4.

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2 found in their Mathematics books, they have to determine value of a sequence as *n* approaches to $+\infty$;

- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to use the limit of the sequence at the infinity ;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to differentiate the convergent from a divergent sequence.
- After this step, guide students to do the application activity 1.2 and evaluate whether lesson objectives were achieved.

## Answer for activity 1.2

1) $\lim_{n\to\infty}\left(\frac{3n^2-1}{n^2}\right)=3$

2) $\lim_{n\to\infty} n^2 = +\infty$

## Answer for application activity 1.2

1. $\{2+(0.1)^n\}$ converges to 2
2. $\left\{\frac{1-2n}{1+2n}\right\}$ converge to -1
3. $\left\{\frac{1-5n^4}{n^4+8n^3}\right\}$ converges to -5
4. $\{-1^n\}$ diverge
5. $\left\{\frac{2n}{\sqrt{3n+1}}\right\}$ converges to $\frac{2}{\sqrt{3}}$
6. $\frac{\sqrt{7n^2+2}}{n^3+8}$ converge to 0.

## Lesson 3: Monotonic sequences

### a) Learning objectives

T o use basic concepts and formulas of sequences to identify the Monotonic sequences

### b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the convergent or divergence sequence (lesson 2 of this unit).

### d) Learning activities:

- Invite student teachers to work in groups and do the activity 1.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the increasing sequence, the decreasing sequence, or non increasing sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts and formulas of sequences to identify the monotonic sequences as increasing sequences or the decreasing sequences.
- After this step, guide students to do the application activity 1.3 and evaluate whether lesson objectives were achieved.

### Answer for *activity 1.3*

1. 1, 2, 3, 4, 5, 6, ... is an ascending
2. $1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots$ is a descending
3. 1, -1, 1, -1, 1, ... it is both, not monotonic
4. 2, 2, 2, 2, 2, 2, ... is neither, it is stationary.

### Answer for *application activity 1.3*

1) 1, 2, 3, ..., n, ...is increasing
2) $\left\{\frac{n}{n+1}=1-\frac{1}{n+1}\right\}$ is decreasing.
3) $\left\{\frac{1}{2^n}\right\}$ is decreasing
4) 3, 3, 3, 3, ... non increasing
5) 1, -1, 1, -1, ... not monotonic.

## Lesson 4: Arithmetic sequence and its general term

### a) Learning objectives

To use basic concepts of sequences to determine terms of an arithmetic sequence.

### b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson 1, lesson 2 and the lesson 3 of this unit.

**d) Learning activities:**

- Invite student-teachers to work in groups and do the activity 1.4 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the common difference of an arithmetic sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to determine the general term and different terms of an arithmetic sequence.
- After this step, guide students to do the application activity 1.4 and evaluate whether lesson objectives were achieved.

### Answer for *activity 1.4*

a) $\{u_n\}$ : 5, 5+3, 5+3+3, 5+3x3, ..., 5+3n,.... This constant is d = 3

b) $\{v_n\} = 26; 26+5; 26+(5\times 2), 26+(5\times 3),\ldots$ This constant is d = 5

c) $\{w_n\}$: 20, 20-2, 20-2.2, 20-2.3, ...20-2.n,...0.This constant is d = -2

### Answer for application activity 1.4

1) If $\frac{1}{a+b},\frac{1}{a+c},\frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, then

$$\frac{2}{a+c}=\frac{1}{a+b}+\frac{1}{b+c}$$

$$\Leftrightarrow \frac{2}{a+c}=\frac{2b+c+a}{(a+b)(b+c)}$$

$$\Leftrightarrow 2\left(ab+ac+b^2+bc\right)=\left(a+c\right)\left(2b+a+c\right)$$
$$\Leftrightarrow 2ab+2ac+2b^2+2bc=2ab+a^2+ac+2bc+ac+c^2$$
$$\Leftrightarrow 2b^2=a^2+c^2$$

Also $a^2, b^2, c^2$ are 3 consecutive terms of an arithmetic progression if

$2b^2=a^2+c^2$.

Thus, if $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, it will be the same for $a^2, b^2, c^2$.

Solution

2) Let the second term be x. The first term is $x-d$ and the third term is $x+d$ where d is the common difference.

Now, $x-d+x+x+d=30 \Rightarrow 3x=30$ or $x=10$

Also, $\left(x-d\right)^2+x^2+\left(x+d\right)^2=332$

Or $\left(10-d\right)^2+100+\left(10+d\right)^2=332$

Or $2d^2=32 \Rightarrow d=\pm 4$

Therefore, the progression is 6, 10, 14 or 14, 10, 6

3) We need to find x such that $\left(1+x\right)^2, \left(q+x\right)^2, and \left(q^2+x\right)^2$ form an arithmetic progression.

$$2\left(q+x\right)^2=\left(1+x\right)^2+\left(q^2+x\right)^2$$

$$\Leftrightarrow 2\left(q^2+2qx+x^2\right)=1+2x+x^2+q^4+2xq^2+x^2$$
$$\Leftrightarrow 2q^2+4qx+2x^2=1+2x+x^2+q^4+2xq^2+x^2$$
$$\Leftrightarrow 2q^2+4qx=1+2x+q^4+2xq^2$$
$$\Leftrightarrow 4qx-2x-2xq^2=1-2q^2+q^4$$
$$\Leftrightarrow x\left(4q-2-2q^2\right)=\left(1-q^2\right)^2$$
$$\Leftrightarrow x=\frac{\left(1-q^2\right)^2}{-2\left(1-2q+q^2\right)}$$

$$\Leftrightarrow x=\frac{(1-q)^2(1+q)^2}{-2(1-q)^2}$$

$$\Leftrightarrow x=\frac{(1+q)^2}{-2}$$

Thus, $x=\frac{-(1+q)^2}{2}$

## Lesson 5: Arithmetic Means of an arithmetic sequences

### a) Learning objectives

To use basic concepts of sequences to find an arithmetic mean of two numbers.

### b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson 1, lesson 2, the lesson 3 and the lesson 4 of this unit.

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.5 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to explain the arithmetic means of such sequences;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to find arithmetic means of two terms of an arithmetic sequences.

- After this step, guide students to do the application activity 1.5 and evaluate whether lesson objectives were achieved.

## Answer for activity 1.5

$u_1 = 2, u_7 = 20$

$u_n = u_1 + (n-1)d \Rightarrow u_7 = u_1 + 6d$

$\Rightarrow 20 = 2 + 6d$

$\Rightarrow d = 3$

$u_1$, $A = u_2 = 2+3 = 5$, $B = u_3 = 2+6 = 7$, $C = u_4 = 2+9 = 11$, $D = u_5 = 2+12 = 14$

$E = u_6 = 2+15 = 17$, $u_7 = 2+18 = 20$.

## Answer for application activity 1.5

1) $-3,-1,1,3,5,7$

2) $2,5,8,11,14,17,20,23,26,29,32$

3) The arithmetic sequence has $n+2$ terms $t_1, t_2, t_3, ..., t_{n+2}$,

$3; a_1, a_2, ..., a_8, ... a_{n-2}, a_{n-1}, a_n; 54$, where

$t_1 = 3; t_2 = a_1; t_3 = a_2; ...; t_9 = a_8; ...; t_{n-1} = a_{n-2}; ...$

Let $d$ be the common difference. Then, from $t_{n+2} = t_1 + (n+1)d$, we have $54 = 3 + (n+1)d$ Solving for $nd$, we have: $nd = 51 - d$ (1)

But also, $\frac{a_8}{a_{n-2}} = \frac{3}{5} \Leftrightarrow 5a_8 = 3a_{n-2}$

$\Leftrightarrow 5(3+8d) = 3[3+(n-2)d]$

$\Leftrightarrow 3nd = 6 + 46d$ (2)

Solving simultaneously (1) and (2): $\begin{cases} nd = 51 - d \\ 3nd = 6 + 46d \end{cases}$; $d = 3$ and $n = 16$.

4) Using the same method as in question (3) we get $n = 14$

5) If $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the the arithmetic mean between $a$ and b, then

$$\frac{a+b}{2} = \frac{a^{n-1} + b^{n-1}}{a^n + b^n}$$

Solving the equation, we get $n = 0$

## Lesson 6: Arithmetic Series

### a) Learning objectives

To use basic concepts of sequences to calculate the sum of the first "$n$" terms of an arithmetic sequence.

### b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 5 of this unit).

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.6 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to establish how to determine the sum $s_n$ for the first n terms of the arithmetic sequence $\{u_n\}$;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover

how to use basic concepts of sequences to calculate the sum of the first "$n$" terms of different arithmetic sequences.

- After this step, guide students to do the application activity 1.6 and evaluate whether lesson objectives were achieved.

## Answer for activity 1. 6

The sequence $2,5,8,11,14,\ldots.$

a) The first term is 2, $d=3$, the general term $u_n = 2+3(n-1)$

b) $S_6 = 2+(2+3)+(2+2.3)+..+(2+5.3) = 6.2+3(1+2+3+4+5) = 6.2+3\frac{6(5)}{2}$

as we apply the sum of all first positive integers not greater than n.

c) $s_n = 2n+3(1+2+3+\ldots.+n) = 2n+3\frac{n(n-1)}{2}$ .

## Answer for application activity 1. 6

1) $2n(n+3)$

2) Knowing that $u_n = u_1 + (n-d)$ and that $s_n = \frac{n}{2}(u_1+u_n) \Rightarrow s_n = 860$

3) $S_n = \frac{n}{2}(2u_1+(n-1)d) = \frac{396}{2}(2+(n-1)7) \Leftrightarrow n =$ 11

4) The bottom row requires 100 tiles and the top row, 50 tiles. Since each successive row requires two less tiles, the total number of tiles required is $S = 100+98+96+\ldots+(100-2.(n-1))+\ldots+50$.

As $100-2.(n-1)=50$, we have:

$100-2n+2=50$ Which gives $n=21$.

This is the sum of an arithmetic sequence; the common difference is $-2$. The number of terms to be added is $n=21$ with the first term $u_1=100$ and the last term $u_n=50$

The sum $S$ is $S = \frac{n}{2}(u_1 + u_n) = \frac{21}{2}(100 + 50) = 1575$.

In all, 1575 tiles will be required.

## Lesson 7: Harmonic sequence and its general terms

### a) Learning objectives

To use basic concepts and formulas of sequences to find the terms of an harmonic sequence.

### b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this unit (from the lesson 1 to lesson 6 of this unit).

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.7 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to discover that an harmonic sequence is made by the reciprocals of terms of the arithmetic sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find the terms of a harmonic sequence.
- After this step, guide students to do the application activity 1.7 and evaluate whether lesson objectives were achieved.

## Answer for activity 1.7

a) $\frac{1}{2},\frac{1}{4},\frac{1}{6},..,\frac{1}{2n},...$

b) Its first term is $\frac{1}{2}$, the third is $\frac{1}{6}$, the general term is $\frac{1}{2n}$.

There is no common method of determining the relationship between two consecutive terms of a harmonic sequence, but it is easy to show that in our example, $u_n = \left(1-\frac{1}{n}\right)u_{n+1}$.

Indeed, $u_{n+1} - u_n = \frac{1}{2n+2} - \frac{1}{2n} = \frac{-1}{(2n+2)(n)}$

$$n(u_{n+1} - u_n) = u_{n+1} \Leftrightarrow u_n = \left(1-\frac{1}{n}\right)u_{n+1}$$

## Answer for application activity 1.7

1. The sequence is $6,4,3,\frac{12}{5},2,\frac{12}{7},\frac{3}{2},\frac{4}{3}$. The $4^{th}$ *term is* $\frac{12}{5}$, $8^{th}$ *term is* $\frac{4}{3}$

2. $3,\frac{90}{23},\frac{90}{16},10$

4. $\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{4},\frac{1}{\sqrt{5}},\dots,\frac{\sqrt{5}}{13}$

5. $6\, and\; 2$

6. $\frac{60}{16-n}$

## Lesson 8: Generalities on Geometric sequences and their general terms

### a) Learning objectives

To use basic concepts and formulas of sequences to determine terms of a geometric sequence.

### b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 7 of this unit) and to the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.8 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group to present their work;
- As a tutor, harmonize the findings from presentation and guide students to make a geometric sequence of numbers, its the general term and the common ratio;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine terms of a geometric sequence.
- After this step, guide students to do the application activity 1.8 and evaluate whether lesson objectives were achieved.

### Answer for activity 1.8

Learners will take a piece of paper and cut it into two equal parts. Take one part and cut it again into two equal parts. When they continue in this manner the fraction corresponding to the obtained parts according to the original piece of paper are as follows:

$\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},...$

**Answer for application activity 1.8**

1. 98304
2. $\frac{\sqrt[5]{16}}{4}$
3. $-21.87$
4. $\frac{1}{16}$
5. $(u_n): u_n = \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$ , $u_8 = \frac{2187}{256}$
6. $p = 5$

## Lesson 9: Geometric means of a geometric sequence

### a) Learning objectives

To use basic concepts and formulas of sequences to find geometric means of two numbers.

### b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.9 found in their Mathematics books;

- Move around in the class for facilitating students where necessary and give more clarifications on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to explain geometric means of two terms in a geometric sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find geometric mean of two numbers.
- After this step, guide students to do the application activity 1.9 and evaluate whether lesson objectives were achieved.

### Answer for activity 1.9

$u_1 = 1, u_6 = 243$

$u_n = u_1 \cdot r^{n-1} \Rightarrow u_6 = u_1 \cdot r^5$

$\Rightarrow 243 = r^5$

$\Rightarrow 3^5 = r^5$

$\Rightarrow r = 3$

The sequence is $1, 3, 9, 27, 81, 243$

### Answer for application activity 1.9

1) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$

2) $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \frac{2}{243}, \frac{2}{729}$

3) a) $14$ b) $\frac{9}{2}$

4) $12$ and $108$

5) $64$ and $4$

## Lesson 10: Geometric Series

### a) Learning objectives

To use basic concepts and formulas of sequences to calculate the sum of the first "$n$" terms of a Geometric sequence

### b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.10 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to determine the sum of n terms of the sequence they found;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to Use basic concepts and formulas of sequences to calculate the sum of the first "$n$" term of a Geometric sequence.
- After this step, guide students to do the application activity 1.10 and evaluate whether lesson objectives were achieved.

## Answer for activity 1.10

a)

| Place | Money gotten |
|---|---|
| $1^{st}$ | $u_1 = 100,000 Frw$ |
| $2^{nd}$ | $u_2 = \frac{1}{2}(100,000 Frw) = 50,000 Frw$ |
| $3^{rd}$ | $u_3 = \frac{1}{2}(50,000 Frw) = 25,000 Frw$ |
| $4^{th}$ | $u_4 = \frac{1}{2}(25,000 Frw) = 12,500 Frw$ |
| $5^{th}$ | $u_5 = \frac{1}{2}(12,500 Frw) = 6,250 Frw$ |

b) The total of their money is

$$u_1 + u_2 + u_3 + u_4 + u_5$$

$$= 100,000 + \frac{1}{2}(100000) + \frac{1}{2^2}100000 + \frac{1}{2^3}100000 + \frac{1}{2^4}100000$$

c) The money for the first is $100,000 Frw$, this is greater than the money for the fifth student which is $6,250 Frw$. When you win at the first place, the

d) The money for the student who passed at the $n^{th}$ place is $u_n = \frac{1}{2^{(n-1)}}100,000 Frw$

To determine the total amount of money for n students, students teachers will make the sum from $u_1 = 100,000 Frw$ to $u_n = \frac{1}{2^{(n-1)}}100,000 Frw$ and they will find

$$S_n = 100000 \times \frac{\left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)}$$

### Answer for application activity 1.10

1) 21.25

2) 39.1

3) $1, \frac{5}{4}$

4) –32

5) The interest is compounded each quarter. So n=1 and the interest rate per period is $\frac{6\%}{4} or\ 1.5\%$. The common ratio r for the geometric series is then $\left(1+0.015\right) or\ 1.015$

The first term $u_1$ in this series is the account balance at the end of the first quarter. Thus, $u_1 = 500(1.015)\ or\ 507.05$.

Apply the formula for the sum of a geometric series $S_n = \frac{u_1 - u_1 r^n}{1-r}$, we have:

$$S_n = \frac{507.5 - 507.5(1.015)^4}{1-1.015}; n = 4; r = 1.015$$

$$S_4 = 2076.13$$

Aloys's account balance at the end of one year is \$2076.13.

## Lesson 11: Infinity geometric series and its convergence

### a) Learning objectives

To explain infinity geometric series and determine their sum.

### b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

## d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.11 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to determine the limit of a given series;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to Explain infinity Geometric series and determine the their sum.
- After this step, guide students to do the application activity 1.11 and evaluate whether lesson objectives were achieved.

### Answer for activity 1.11

a) $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[5\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\left(\frac{1}{2}\right)\right)}\right]=10$

b) If $-1<r<1, -1<r<1$, thus $\lim_{n\to\infty}\frac{u_1\left(1-r^n\right)}{1-r}=\frac{u_1}{1-r}$

## Answer for application activity 1.11

1. a) $\sum_{n=1}^{\infty} 10\left(1-\frac{3x}{2}\right)^{n}$ this exists if $\left|1-\frac{3x}{2}\right|<1$. Solving this inequality we find

$0<x<\frac{4}{3}$

b) If x = 1.3, the sum can be $\frac{u_1}{1-r}=\frac{10}{1-\left(1-\frac{3(1.3)}{2}\right)}=\frac{200}{39}$

2) The decimal $0.999\ldots=0.9+0.09+0.009+\ldots=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\ldots$ is an infinite geometric series. We will write it in the form $\sum_{k=1}^{\infty} a_1 r^{k-1}$, then

$$0.999\ldots=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\ldots=\sum_{k=1}^{\infty}\frac{9}{10^k}=\sum_{k=1}^{\infty}\frac{9}{10.10^{k-1}}=\sum_{k=1}^{\infty}\frac{9}{10}\left(\frac{1}{10}\right)^{k-1}$$

Now we can compare this series to $\sum_{k=1}^{\infty} a_1 r^{k-1}$ and conclude that $a_1=\frac{9}{10}$ and $r=\frac{1}{10}$. Since $|r|<1$, the series converges and its sum is $0.999\ldots=\frac{\frac{9}{10}}{1-\frac{1}{10}}=\frac{\frac{9}{10}}{\frac{9}{10}}=1$

The repeating decimal $0.999\ldots$ equal 1.

3) $\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}=2+\frac{4}{3}+\frac{8}{9}+\ldots$ This geometric series converges as $\frac{2}{3}<1$. Its sum is its limit $\frac{2}{1-\frac{2}{3}}=6$

**Lesson 12: Application of sequences in real life**

### a) Learning objectives

To Apply the concepts of sequences and series to solve problems related to finance or Economics.

### b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

### c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

### d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.12 found in their Mathematics books;
- Visit each group for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to solve problems related to finance or Economics.
- After this step, guide students to do the application activity 1.12 and evaluate whether lesson objectives were achieved.

**Answers for activity 1.12**

Refer to the student's book to highlight some applications of sequences and series. They are also used to determine t*he monthly payments made to pay off an automobile or home loan with interest portion, the list of maximum daily temperatures in one area for a month, etc. Sequences are used in calculating interest, population growth, half-life and decay in radioactivity.*

## Answer for application activity 1.1 2

1) $P = 1300,\ r = 7\% = 0.07,\ k = 1$

$$A = 1300\left(1+\frac{0.07}{1}\right)^{1\times17} = 4106.46$$

The account will contain $4,106.46.

2) We apply the formula $P = P_0e^{rt}$ With initial population $P_0 = 153,800$ rate of growth $r = 0.05$, and time $t = 2000-1970 = 30\,years$. Thus, a prediction for the population of the city in the year 2000 is
$153,800e^{(0.05)(30)} = 153,800e^{1.5} \approx 689,284$

3) This is an ordinary annuity with $\mathrm{n} = 30$ annual deposits of $P = \$2000$. The rate of interest per payment period is $i = \frac{0.04}{1} = 0.04.$ The amount A of the annuity after 30 deposits is

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right] = 2000\left[\frac{\left((1+0.04)^{30} - 1\right)}{0.04}\right] = 112,169.88.$$

4) This is an example of a sinking fund. The payment P required twice a year to accumulate 4,000,000Frw in 12 years (24 payments at a rate of interest of $i = \frac{0.04}{2} = 0.02$ per payment period) obeys:

$$A = P.\left[\frac{(1+i)^n - 1}{i}\right]$$

$$4,000,000 = P\left[\frac{\left((1+0.02)^{24} - 1\right)}{0.02}\right]$$

$$4,000,000 = P(30.4218)$$

$$P = 131,484.39\,Frw$$

The school leader will need to make a payment of $131,484.39 every 6 months to redeem the bonds in 12 years.

## 1.6 Unit summary

1. Numbers in sequence are denoted $u_1, u_2, u_3, \ldots, u_{n-1}, u_n, \ldots$ and shortly $\{u_n\}$.

The natural number n is called **term number** and value $u_n$ is called a **general term** of a sequence and the term $u_1$ is the **initial term**.

2. As a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{+\infty}$.

3. A sequence $\{u_n\}$ is said to be

- increasing if $u_1 < u_2 < u_3 < \ldots < u_n < \ldots$
- non decreasing if $u_1 \le u_2 \le u_3 \le \ldots \le u_n \le \ldots$
- decreasing if $u_1 > u_2 > u_3 > \ldots > u_n > \ldots$
- non increasing $u_1 \ge u_2 \ge u_3 \ge \ldots \ge u_n \ge \ldots$

4. A numerical sequence is said to be **convergent** if the limit exist whereas if the limit does not exist (or is infinity) the sequence is said to be **divergent.** A number $L$ *is called* **a limit** of a numerical sequence $\{u_n\}$ if $\lim\limits_{n\to\infty} u_n = L$

5. One of the most famous and important of all diverging series is the **harmonic series**, $\sum_{k=1}^{+\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

6. Sequences of numbers that follow a pattern of adding a fixed number from one term to the next are called **arithmetic sequences** or **arithmetic progressions**.

7. For an arithmetic sequence $u_{n-1}, u_n, u_{n+1}$, we have $2u_n = u_{n-1} + u_{n+1}$.

8. If $u_p$ is any $p^{th}$ term of a sequence then the $n^{th}$ term is given by $u_n = u_p + (n-p)d$

9. The sum of first $n$ terms of a finite arithmetic sequence with initial term $u_1$ is given by $s_n = \frac{n}{2}[u_1 + u_n]$

10. Sequences of numbers that follow a pattern of multiplying a fixed number from one term to the next are called geometric sequences.

11. For a geometric sequence $u_{n-1}, u_n, u_{n+1}$, we have $u_n^2 = u_{n-1} \cdot u_{n+1}$

12. The n[th] term, $u_n$, of a geometric sequence $\{u_n\}$ with common ratio r and initial term $u_1$ is given by $u_n = u_1 r^{n-1}$

13. The sum of first n terms of a geometric sequence with initial term $u_1$ and common ratio r is given by: $s_n = \frac{u_1(1-r^n)}{1-r}$ with $r \neq 1$
14. Also, the product of first n terms of a geometric sequence with initial term $u_1$ and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$
15. For the formula $s_n = \frac{u_1(1-r^n)}{1-r}$

    If $-1 < r < 1$, $S_\infty = \frac{u_1}{1-r}$

Sequences are used in calculating interest, population growth, half-life and decay in radioactivity.

## 1.7. Additional information for the tutor

The tutor has to emphasize the application of sequences and series in solving problems related to finance or Economics in real situation life.

## 1.8 End unit assessment

**1)** $0, -\frac{1}{4}, -\frac{2}{9}, -\frac{3}{16}$ b) $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}$ c) $1, 3, 1, 3$

**2)** a) $(-1)^n$, $n = 0, 1, 2, ..$ b) $n^2 - 1$, $n = 1, 2, 3, ...$ c) $4n - 3$, $n = 1, 2, 3, ...$

**3)** a) converges to $\sqrt{2}$ b) Converges to 0 c) Converges to 1

**4)** £11 million

**5)** 2048000

**6)** $99.8^0 F$

**7)** 1800

## 1.9 Additional activities

### *1.9.1 Remedial activities*

1. Find the 20th term of the following arithmetic progressions and calculate the sum of first 20 terms

a. 2, 6, 10, 14, ...

b. -5, -3.5, -2, -0.5, ...

**Solution:**

a) $u_{20}=78, S_{20}=800$ b. $u_{20}=23.5, S_{20}=185$

2. In an arithmetic progression, the sum of the 8th and 14th terms is 50. The 5th term is equal to 13. Find that progression.

**Solution:**

$5,7,9,11,13,15,17,19,21,23,25,27,29,31,\ldots$

### *1.9.2 Consolidation activities*

1) Find *x* consecutive integers numbers known that the first number is 8 and their sum is $x^3$.

**Solution:** $8,9,10$

2) In a geometric progression, we have

a. $u_1=3, r=4, n=5$; find $u_n$ and sum of terms.

b. $u_n=\frac{3}{64}, u_1=12, n=9$; find r and sum of terms.

**Solution:**

a) $u_5=768, S_5=341$ b) $r=\frac{1}{2}, S_n=\frac{1533}{64}$

### *1.9.3 Extended activities*

1. In an arithmetic progression, we have

a. $u_1=4, d=2, n=8$; find $u_n$ and sum of terms

b. $d=4, u_n=39, n=10$; find $u_1$ and sum of terms

c. $u_1=3, u_n=21, S_n=120$; find n and d

d. $u_n=199, n=100, S_n=10000$; find $u_1$ and d.

**Solution:**

a) $u_8 = 18, S_8 = 88$  b) $u_1 = 3, S_{10} = 210$  c) $n = 10, d = 2$  d. $u_1 = 1, d = 2$

2. A mine worker discovers an ore sample containing 500 mg of radioactive material. It is discovered that the radioactive material has a half life of 1 day. Find the amount of radioactive material in the sample at the beginning of the 7th day.

**Solution:**

Half life of one day means that the half of the amount remains after 1 day.

| | | | |
|---|---|---|---|
| Begin of day 1: | Begin of day 2: | Begin of day 3: | ... |
| 500 mg | 250 mg | 125 mg | |
| End of day 1: | End of day 2: | End of day 3: | ... |
| 250 mg | 125 mg | 62.5 mg | |

Decide to either work with the "beginning" of each day, or the "end" of each day, as each can yield the answer.  Only the starting value and number of terms will differ. We will use "beginning": The beginning of the 7th day corresponds to

$$u_7 = 500\left(\frac{1}{2}\right)^{7-1} = 7.8125mg.$$

# UNIT 2

# POINTS, STRAIGHT LINES AND CIRCLES IN 2 DIMENSIONS

## 2.1 Key Unit competence

Determine equations of lines and circles.

## 2.2. Prerequisite

Student - teachers will perform well in this unit if they have a good background on:

- Linear equations learnt in S1;
- Vectors and their properties learnt in S2;
- Translations learnt in S3;
- Solving algebraically and graphically simultaneous linear equations in two unknowns learnt in year one.

## 2.3. Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all student-teachers while teaching;
- **Peace and value Education**: During group activities, the teacher will encourage student teachers to help each other and to respect opinions of colleagues;
- **Financial education**: Guide students to discuss how to establish an equation illustrating the variation of money;
- **Gender:** Give equal opportunities to all students (girls and boys) to participate actively in all learning activities from the beginning to the end of the lesson.

## 2.4. Guidance on introductory activity 2

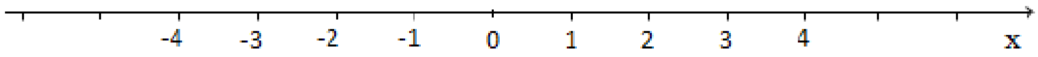
- Invite student teachers to form groups and guide them to work on the introductory activity.

- Give time to students to analyse the activity;
- Invite group representatives to present findings in a whole class discussion;
- Harmonize students' answers and guide them to enhance how to make effective graduation of lines and axes for a Cartesian plane;
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to arouse their curiosity on what is going to be leant in this unit.

## Answer for introductory activity 2

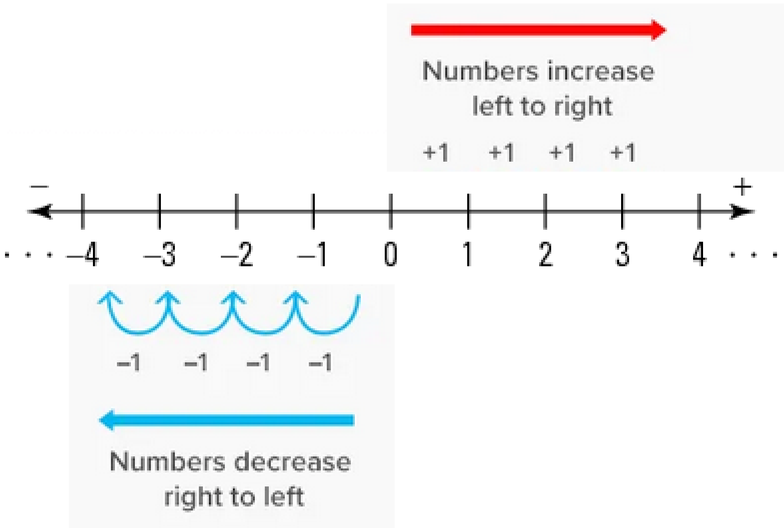
a) We all know that, numbers on the number line increase as one moves from left to right and decrease on moving from right to left.

Basing on this, Rukundo made a mistake. The correct number line he should have done is the following:




b) A number line is a straight with numbers placed at equal intervals or segments along its length. A number line can be extended infinitely in any direction.

c) The numbers on the number line increase as one moves from left to right and decrease on moving from right to left.



d) The Cartesian plane is a plane made by two fixed and perpendicular number lines (i.e. two directions) in which every point P is characterized by P (x, y) where x is measured as horizontal displacement on the axis OX and y is measured as vertical displacement on the y-axis.

## 2.5 List of lessons and sub-heading

| No | Lesson title | Learning objectives | Number of periods |
|---|---|---|---|
| 0 | Introductory activity | To arouse the curiosity of student teachers on the content of unit 2. | 1 |
| 1 | Cartesian coordinates of a point | Define and plot the coordinates of a point in 2D. | 1 |
| 2 | Distance between two points | Calculate the distance between two points | 1 |
| 3 | Midpoint of a line segment | Determine the coordinate for the mid-Point of a segment in 2D. | 1 |
| 4 | Vector in 2D and dot product | Describe a vector in 2 D and determine the dot product of 2 vectors. | 2 |
| 5 | Equation of a line in 2D:<br><br>(Vector, Parametric and Cartesian ) | Determine equations of a straight line (vector equation, parametric equation, Cartesian equation). | 6 |
| 6 | Problems on points and straight lines in 2D | Explain and calculate the angles between two lines in 2D.<br><br>Determine the intersection, perpendicularity or parallelism of two lines | 2 |
| 7 | Circle and its equation in 2D | Explain and determine the center, radius, and diameter of a circle in a Cartesian plan. | 2 |
| 8 | | Establish the equation of a circle. | 2 |
| 9 | | Determine the intersection of a line and a circle | 2 |
| 10 | End unity assessment | | **1** |
| Total number of periods | | | **21** |

## Lesson 1: Cartesian coordinates of a point

### a) Learning objectives:

Define and plot the coordinates of a point in 2D

### b) Teaching resources

Learner's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, Math Type and GeoGebra software, smart class, …

### c) Prerequisites / Revision / Introduction

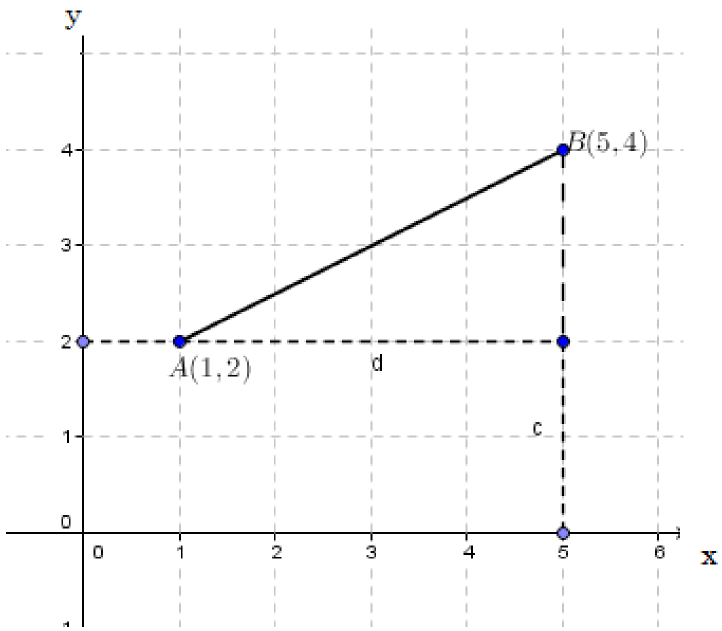
Student - teacher will easily learn this unit, if they refer to:

- Define the coordinate of a point in 2D as it is taught in S1 (UNIT3),
- Calculate the distance between two points in 2D S2 (UNIT7)
- Determine equations of a straight line S3 (UNIT6),
- Appreciate the importance of a point and a line in a plane (S2 and S3).

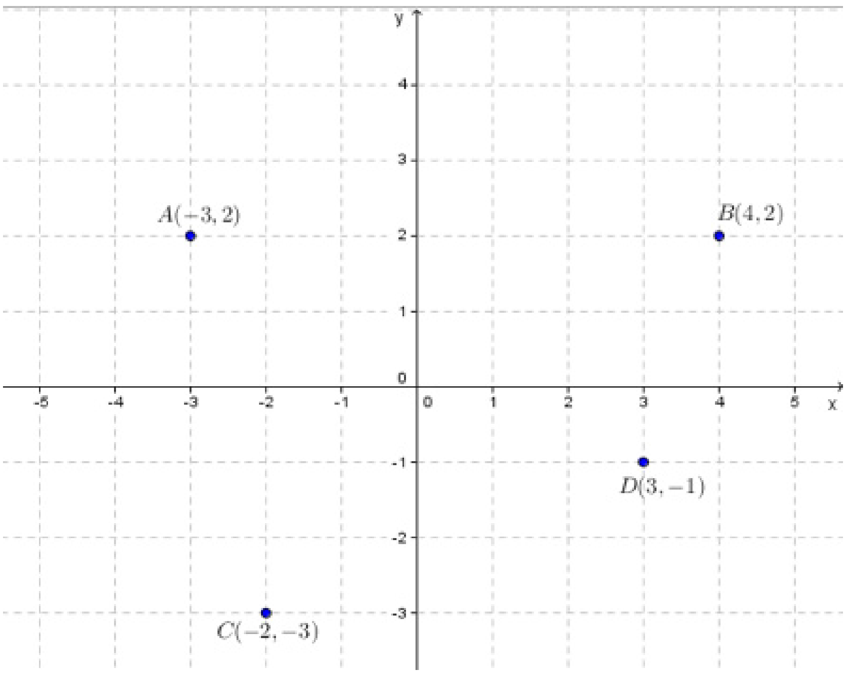
### d) Learning activities

- Organize the student-teachers into small groups;
- Ask them to do the activity 3.1 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, help student-teachers plot a point and draw a line joining two points in the Cartesian plan;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance the coordinate of a point and identify the point when coordinates were given;
- After this step, guide students to do the application activity 3.1, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for Activity 2.1




## Answers for application activity 2.1



## Lesson 2: Distance between two points in 2D

### a) Learning objectives:

Calculate the distance between two points.

### b) Teaching resources

Learner's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils.

### c) Prerequisites / Revision / Introduction:

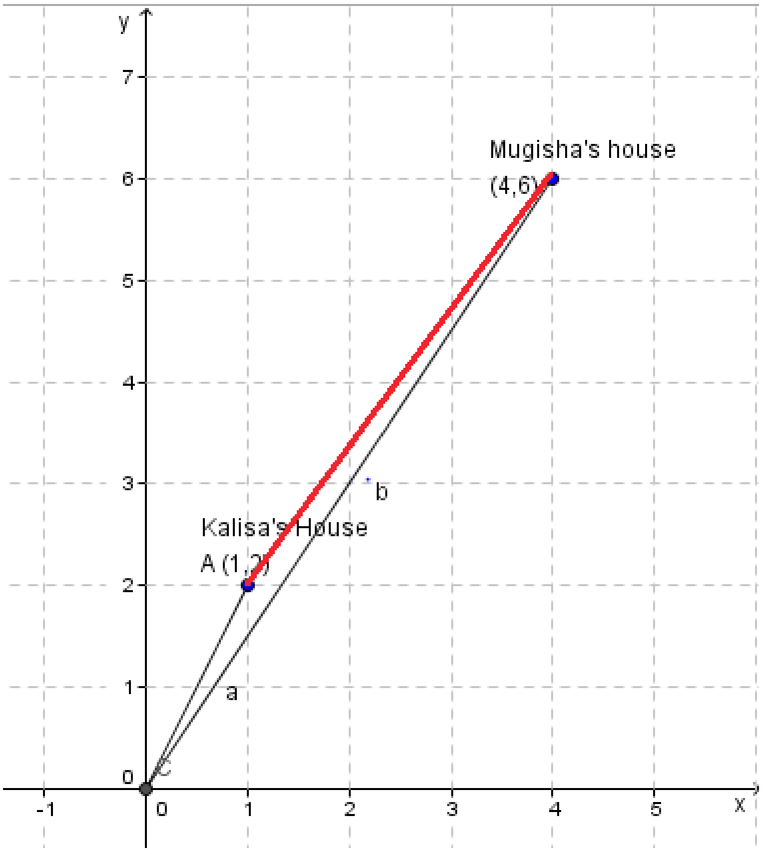
Student - teacher will easily learn this unit, if they are able to:

- Define the coordinate of a point in 2D as it is taught in S1 (UNIT3),
- Calculate the distance between two points in 2D learnt in S2 (UNIT7)
- Determine equations of a straight line. S3 (UNIT6),
- Appreciate the importance of a point and a line in a plane.

### d) Learning activities

- Organize the student-teachers into small groups;
- Ask them to do the activity 3.2 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, help student-teachers to measure and to calculate the distance between two points in the Cartesian plane;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance how to find the distance "$d$" between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the Cartesian plane;
- After this step, guide students to do the application activity 3.2, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for Activity 3.1




a) Here the distance is

$$d_{AB}=\sqrt{(4-1)^2+(6-2)^2} \qquad d_{AB}=\sqrt{25}=|5|=5$$

## Answers for Application activity 2.2

1. a). $d_{SQ}=\sqrt{(-2-7)^2+(-5-(-2))^2}=\sqrt{90}=9.5$

   b). $d_{SQ}=\sqrt{(-2-7)^2+(-5-(-2))^2}=\sqrt{90}=9.5$

   c). $d_{AB}=\sqrt{(2-(-3))^2+(7-5)^2}=\sqrt{29}$

   d). $d_{AB}=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{x-(x+4)^2+(y-(y-1))^2}=\sqrt{17}$

2. The missing coordinates given CD, are the following

a).. $d_{CD} = \sqrt{(6-x)^2 + (-2-2)^2}$

$$\begin{array}{l} 5^2 = 36 - 12x + x^2 + 16 \\ 0 = (x-3)\ (x-9) \end{array} \text{, therefore } \left| \begin{array}{l} x = 3 \\ x = 9 \end{array} \right.$$

b).. $5 = \sqrt{(4-1)^2 + (y+1)^2}$

$$\begin{array}{l} 5^2 = 9 + y^2 + 2y + 1 \\ 0 = (y-3)\ (y+5) \end{array}, \quad \text{therefore } \left| \begin{array}{l} y = 3 \\ y = -5 \end{array} \right.$$

## Lesson 3: Mid-point and Distance between two points in 2D

### a) Learning objectives:

Determine the coordinate for the mid-Point of a segment in 2D.

### b) Teaching resources

Student's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils.

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

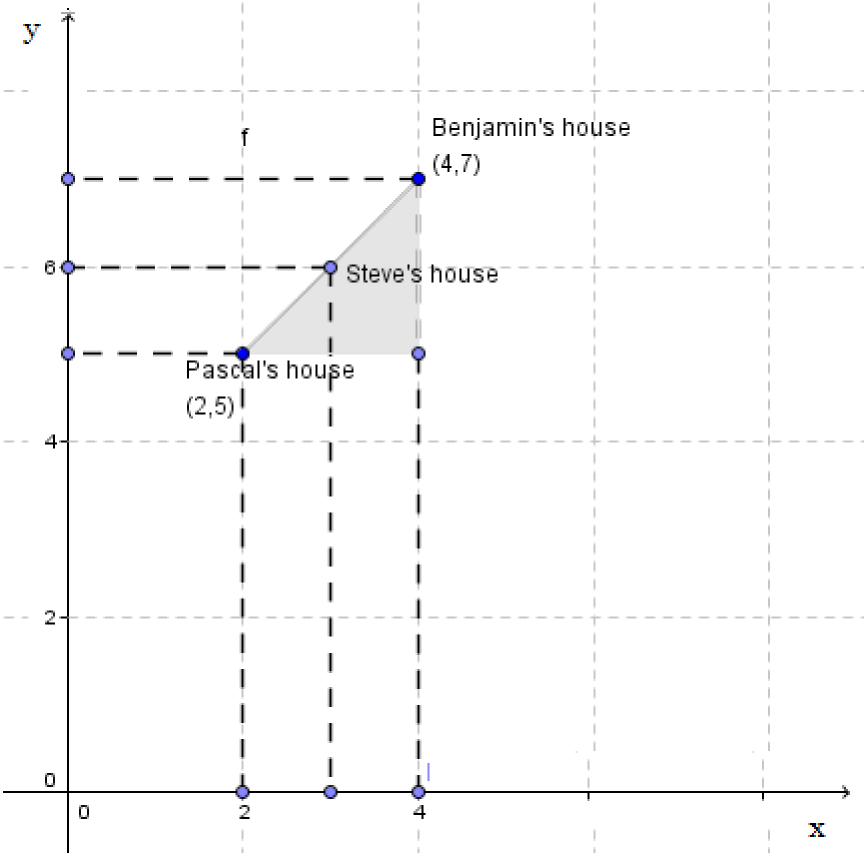
- Define the coordinate of a point in 2D as it is taught in S1 (UNIT3),
- Calculate the distance between two points in 2D S2 (UNIT7)
- Appreciate the importance of a point and a line in a plane.

### d) Learning activities

- Organize the student-teachers into groups;
- Ask them to do the activity 3.3 from student-teacher's book;
- After a given time, ask some groups to present their findings to the whole class;
- During the harmonization, help student-teachers identify the middle point of a line segment and its coordinates;

- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance how to determine the coordinate of midpoint of a line segment;
- After this step, guide students to do the application activity 3.3, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for Activity 3.3




Steve's location is given by: $\left(\frac{2+4}{2}, \frac{5+7}{2}\right) \Rightarrow (3,6)$

## Answers for Application activity 2.3

Michael and Sarah live in different cities and one day they decided to meet up for lunch. Because they both wanted to travel as little as possible, they will have to meet at:

$$\left(\frac{3100+5120}{2},\frac{500+125}{2}\right) \Rightarrow (4110\,,312.5)$$

a) $(4110,312.5)$

b) $(4110,375)$

c) $(2020,375)$

d) $(8220,625)$

The right answer is sub question $a)$ $(4110\,,312.5)$, the rest answers are false.

## Lesson 4: Vector in 2D and dot product

### a) Learning objectives:

Describe a vector in 2 D and determine the dot product of 2 vectors.

### b) Teaching resources

Student's book and other reference textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

- Define different concepts of a vector and their properties, as it is taught in S2 (UNIT 7),
- Plot vectors in a Cartesian plane
- Effectively do operations on vectors
- Calculate the magnitude of vectors as its length.
- Calculate the distance between two points in 2D S2 (UNIT7)

## d) Learning activities

- Organize the student-teachers into groups;
- Ask them to do the activity 2.4.1and activity 2.4.2 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, help student-teachers to have the ability of plotting and describing a vector in a Cartesian plane;
- Guide students to do the activity 2.4.2 related to dot product;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance the concepts: vector position, components of a vector, characteristics of a vector in 2D, operation of vectors, dot product of two vectors related properties and the magnitude of a vector;
- After this step, guide students to do the application activity 2.4.1 and 2.4.2, assess their competences and evaluate whether lesson objectives were achieved.

### Answers for Activity 2.4.1

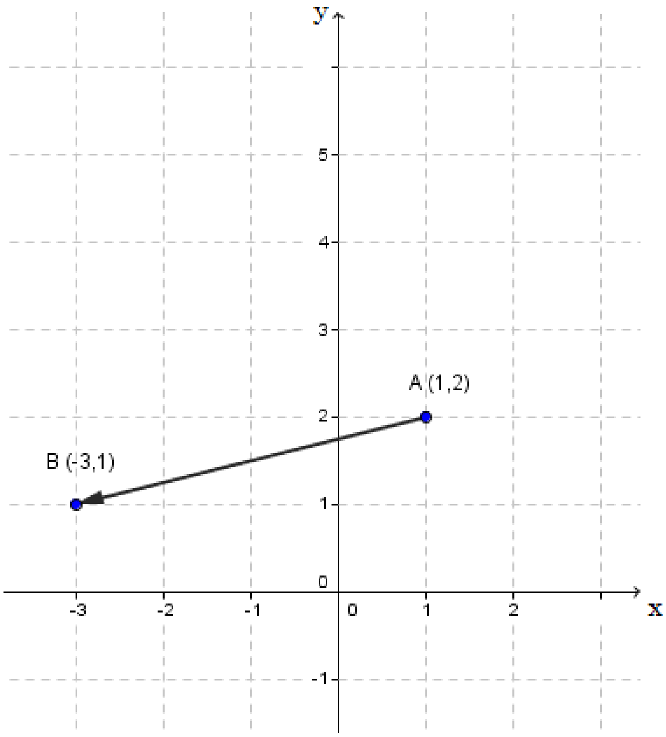
A vector is any quantity that has both magnitude and direction. Two examples of vectors are force and velocity. Both force and velocity are in a particular direction. The magnitude of the force indicates the strength of the force. For velocity, the speed is the magnitude. Other examples include displacement, acceleration.

Note that magnitude and direction are the two properties of a vector.

Quantities with magnitude only are only called scalars.

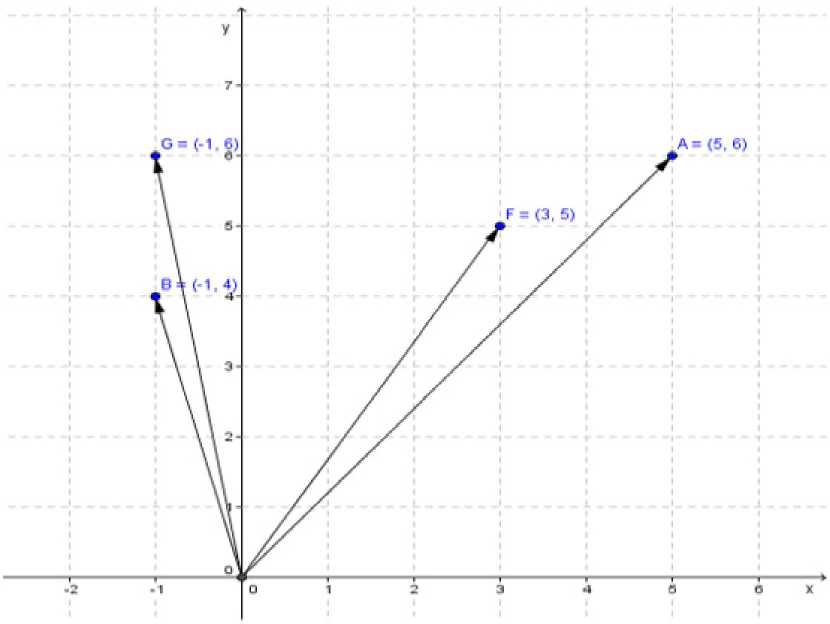
Examples of scalar quantities are: distance, mass, time.

Geometrically, we represent a vector as a directed line segment, whose length is proportional to the magnitude of the vector and with an arrow indicating the direction.

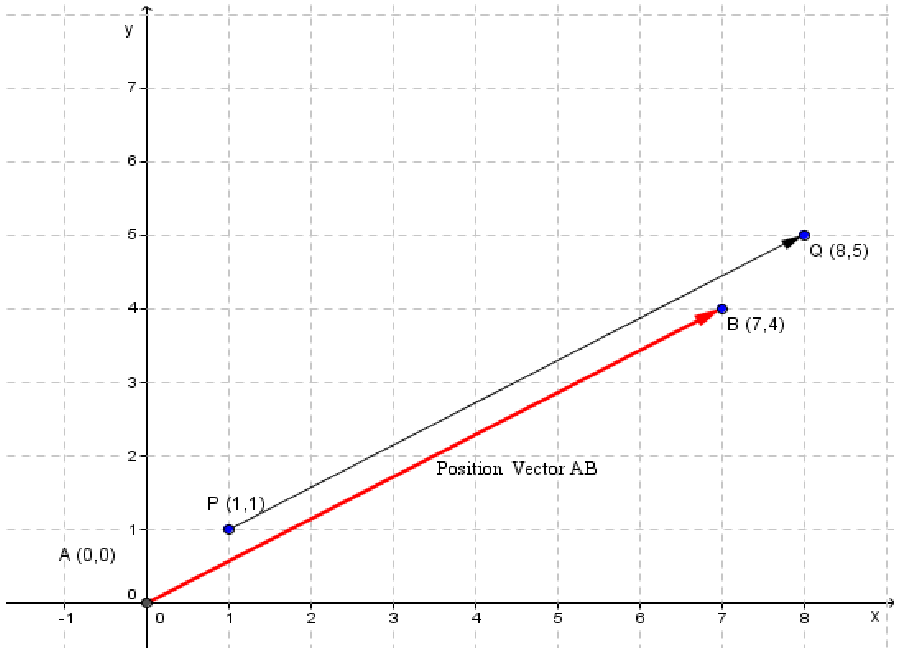



## Answers for application activity 2.4.1

a).

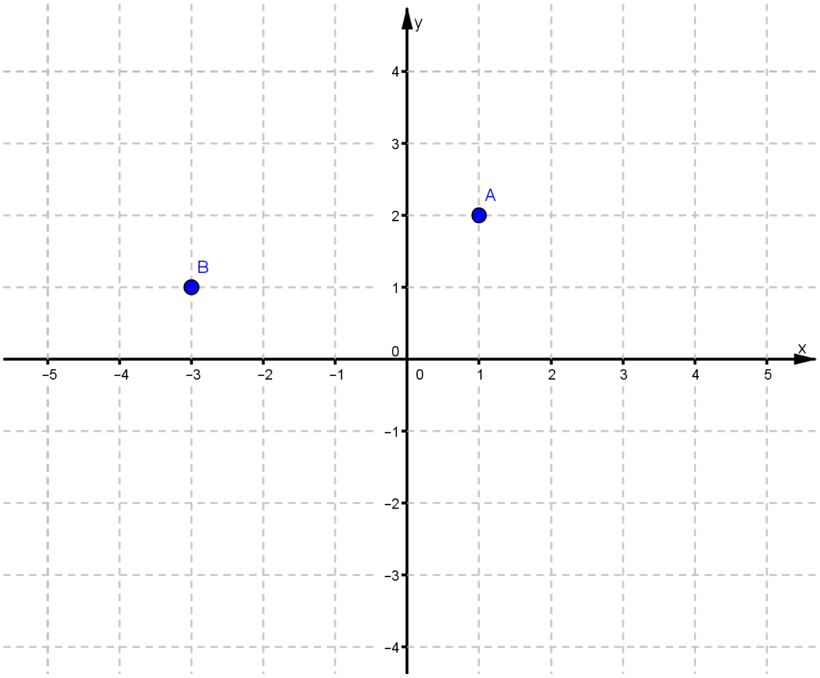


b).




## Answers for activity 2.4.2

1. a).



b) $\sqrt{17}$

2. a) $a)\ -1 \qquad b)\ 0 \qquad c)\ \vec{u}.\vec{v} = (a_1b_1 + a_2b_2)$ since $\vec{u}(a_1, a_2)$ and $\vec{v}(b_1, b_2)$

## Answers for application activity 2.4.2

1. a) Given the vector $\vec{v} = (3,4)$.

The norm is $\|\vec{v}\| = \sqrt{9+16} = 5$

2. b) Given the vector $\vec{v} = (3,4)$.

The norm is $\|\vec{v}\| = \sqrt{9+1} = \sqrt{10}$

3. $\overrightarrow{AB} = (-5,-1)$ b) $(-25,-16)$ c) $\|\overrightarrow{AB}\| = \sqrt{26}$, $\|\vec{w}\| = \sqrt{881}$ d) 7 e) 59

# Lesson 5: Straight line passing through a point and parallel to a direction vector

### a) Learning objectives

Determination of equation of a straight line given 2 points and a direction vector.

### b) Teaching resources:

Student's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

- Define different concepts of vectors and its properties seen in unit 7 and parallel projection of a point to a line in unit 8 leant in S2
- Effectively Plot lines and vectors in a Cartesian plane
- Appreciate the importance of a point and a line in a plane.

### d) Learning activities

- Organize the student-teachers into small groups;
- Ask them to do the activity 2.5.1 from student-teacher's book;

- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, help student-teachers establish a vector parallel to a given vector and translated to a given point;
- Use different probing questions and guide students to explore examples and the content given in the student's book to establish equations of a line given a direction vector and a through it passes that line;
- After this step, guide students to do the application activity 2.5.1, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for activity 2.5.1

a) Given director vector $\vec{u}(2.-3)$ and $\vec{w} \parallel \vec{v}$ , it means there exist a parameter r such that $\vec{w} = r\vec{v}$ .

As $\vec{w}$ is translated (passes) to the point $(1,6)$, we have $\vec{w} = r\vec{v} + (1,6)$.

Thus, the form of a vector $\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix}$ is $(x, y) = r(2,-3) + (1,6)$

b) Examples of such a vector $\vec{w}$

If $r = 1, \vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

If $r = 2, \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

Here the answer may vary depending on the choice of the value of the parameter r.

c) If D is a line with the direction vector $\vec{w}$, the equation D should be given by

$$\begin{cases} x = 2r + 1 \\ y = -3r + 6 \end{cases} \text{ or } \frac{x-1}{2} = r \text{ and } \frac{y-6}{-3} = r$$

Thus as $r = r$ , The Cartesian equation of D becomes $\frac{x-1}{2} = \frac{y-6}{-3}$. This equation can be written in the form of $y = ax + b$ as it was learnt in S1.

$$-3(x-1)=2(y-6)$$
$$\Rightarrow -3x+3=2y-12$$
$$\Rightarrow 2y+3x-15=0$$
$$2y+3x=15$$
$$y=\frac{-3}{2}x+\frac{15}{2}$$

## Answers for application activity 2.5.1

1) Vector equation of a given line is $(x,y)=(3,5)+r(1,6)$

A line L parallel to this line has the same direction vector $\vec{v}=\begin{pmatrix}1\\6\end{pmatrix}$.

As L passes to the point (2,3), every point P (x, y) of L is such that

$$(x,y)=(2,-3)+r(1,6)$$

Therefore, the vector equation of L is $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\-3\end{pmatrix}+r\begin{pmatrix}1\\6\end{pmatrix}$ or

$$x\vec{i}+y\vec{j}=(2+r)\vec{i}+(6r-3)\vec{j}$$

Parametric equations:

$$\begin{cases}x=2+r\\y=-3+6r\end{cases}$$

Cartesian equation:

$$x-2=\frac{y+3}{6} \text{ or } y=6x-15$$

2) a. $(x,y)=(-1,2)+r(-2,3)$

$$\begin{cases}x=-1+3r\\y=2+3r\end{cases}$$

$$r=\frac{x+1}{3} \text{ and } r=\frac{y-2}{3}$$

Thus, $\frac{x+1}{3}=\frac{y-2}{3}$

$\Rightarrow y = x + 3$

b) $\begin{cases} x = 3 + 3r \\ y = 2 - r \end{cases}$

$r = \frac{x-3}{3}$ and $r = 2 - y$

Thus, $\frac{x-3}{3} = 2 - y$

$\Rightarrow y = 2 - \frac{x-3}{3}$

$\Rightarrow y = \frac{9-x}{3}$

$\Rightarrow y = -\frac{1}{3}x + 3$.

## Lesson 6: Equation of a straight line given 2 points and a direction vector

### a) Learning objectives:

Determination of equation of a straight line given 2 points and a direction vector.

### b) Teaching resources:

Student's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

**Note:** The Tutor can help student-teachers to use GeoGebra software to draw lines.

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

- Define different concepts of vectors and its properties seen in unit 7 and parallel projection of a point to a line in unit 8 leant in S2
- Effectively plot lines and vectors in a Cartesian plane.

### d) Learning activities

- Organize the student-teachers into small groups;
- Ask them to do the activity 2.5.2 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization guide students to determine the vector AB passing joining two given points (A and B) and another vector V parallel to the vector AB;
- Use different probing questions and guide students to explore examples and the content given in the student's book to determine the equation of a straight line given 2 points and a direction vector;
- After this step, guide students to do the application activity 2.5.2, assess their competences and evaluate whether lesson objectives were achieved.

**Note:** Where it is possible, the tutor can help student-teachers to use GeoGebra software to plot lines.

### Answers for activity 2.5.2

1. a) The vector $\overline{AB} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$.

b). The vector $\overline{w} = \begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} 2 \\ -6 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

c) Here the answer may vary depending on the choice of each and every one

For example, one may choose $r=1$ or another value $r=2$ and the answer differ.

d) $\begin{cases} x = 2r+3 \\ y = -6r-2 \end{cases}$

$r = \frac{x-3}{2}$ *and* $r = \frac{y+2}{-6}$,

**therefore** $\frac{x-3}{2} = \frac{y+2}{-6}$

$\Rightarrow -6(x-3) = 2(y+2)$
$\Rightarrow -6x+18 = 2y+4$
$\Rightarrow 2y+6x = 14$

## Answers for application activity 2.5.2

1) The direction vector is $\overrightarrow{PB} = (1,-6)$,

If r is a parameter, the vector equation is $(x, y) = \overrightarrow{OP} + r\overrightarrow{PB} = (2,4) + r(1,-6)$

Parametric equations: $\begin{cases} x = 2+r \\ y = 4-6r \end{cases}$

The Cartesian equation:

$$\frac{x-2}{1} = \frac{y-4}{-6}$$

or $-6(x-2) = (y-4) \Rightarrow y+6x = 16$

2) We need the equation whose vector equation is $a\vec{i}+b\vec{j}$ such that

$\left(a\vec{i}+b\vec{j}\right)\cdot\left(\vec{i}-2\vec{j}\right) = 0$ $\quad \left(a\vec{i}+b\vec{j}\right)\cdot\left(\vec{i}-2\vec{j}\right) = 0$
$a-2b = 0 \Rightarrow a = 2b$

The direction vector of the needed line is $2\vec{i}+\vec{j}$. The required equation is $x\vec{i}+y\vec{j} = 2\vec{i}+3\vec{j}+r\left(2\vec{i}+\vec{j}\right)$.

## Lesson 7: Equation of a straight line given its gradient

### a) Learning objectives:

Determination of equation of a straight line given its gradient.

### b) Teaching resources:

Student's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

**Note:** Where it is possible, the tutor can help student-teachers to use GeoGebra software to plot lines.

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

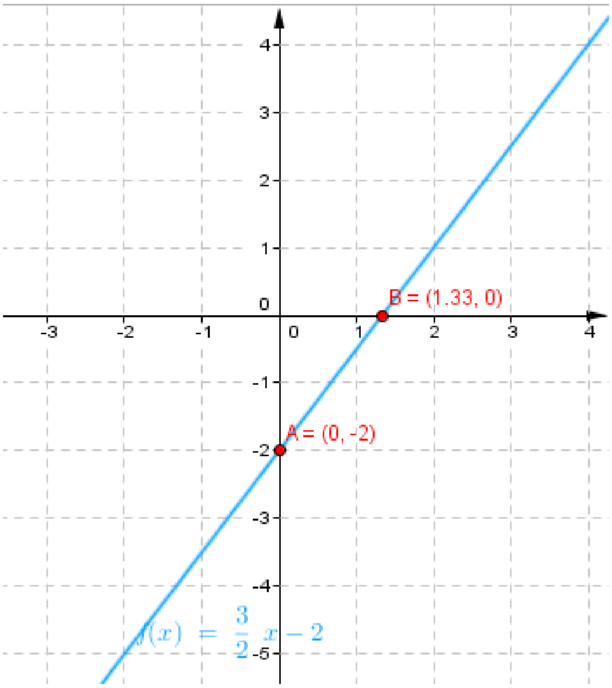
- Previous lessons of this unit;
- Effectively Plot lines and vectors in a Cartesian plane;
- Appreciate the importance of a point and a line in a plane.

### d) Learning activities

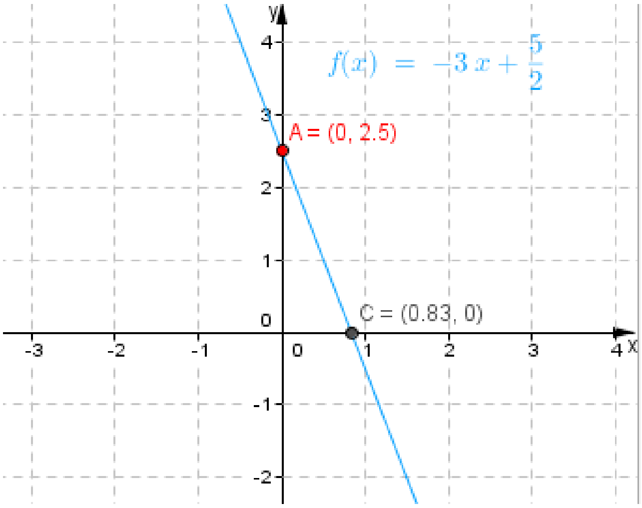
- Organize the student-teachers into groups;
- Ask them to do the activity 2.5.3 from student-teacher's book;
- After a given time, ask groups with different working steps to present their findings to the whole class;
- During the harmonization, help student-teachers plot graph of a line passing to a point given its gradient;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance how to plot a line parallel to a given line (as its gradient is known);
- After this step, guide students to do the application activity 2.5.3, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for activities 3.5.3

a) The slope is $m = \frac{3}{2}$ for $y = \frac{3}{2}x - 2$ and y-intercept is $y = -2$ and $x = 0$




b)The slope $m = -3$ for $y = -3x + \frac{5}{2}$ and y-intercept is $y = \frac{5}{2}$ and $x = 0$



## Answer for application activities 2.5.3

1. a) $y+4=6(x-3)$ $y=6x-22$

b) $y+7=\frac{-3}{2}(x+2)$

$\Rightarrow y=\frac{-3}{2}x-\frac{6}{2}-7$

$\Rightarrow y=\frac{-3}{2}x-10$

c) $y-2=0$

$\Rightarrow y=2$

d) $y-2=\frac{-5}{3}(x-4)$

$y=-\frac{5}{3}x-\frac{20}{3}+2$

$y=-\frac{5}{3}x-\frac{14}{3}$

e) $y=4x$

f) $y+8=-\frac{1}{5}(x-1)$

$y=-\frac{1}{5}x+\frac{1}{5}-8$

$y=-\frac{1}{5}x-\frac{39}{5}$

## Lesson 8: intersection, perpendicularity or parallelism of lines in 2D

### a) Learning objectives:

Perform operations to determine the intersection, perpendicularity or parallelism of lines

### b) Teaching resources

Student's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

**Note:** Where it is possible, the tutor can help student-teachers to draw graphs using GeoGebra software.

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

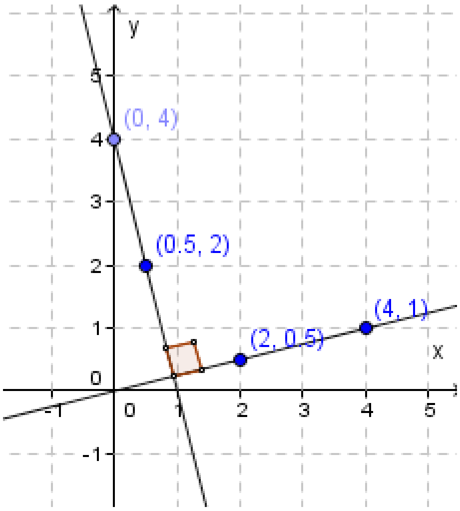
- Define different concepts of a parallel and / or perpendicular projection in 2D as it is taught in S2 (UNIT 8).
- Effectively Plot lines and vectors in a Cartesian plane
- Appreciate the importance of a point and a line in a plane.

### d) Learning activities

- Organize the student-teachers into groups;
- Ask them to do the activity 2.6.1 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, guide student-teachers to determine the equation of a line parallel or perpendicular to a given line and to determine the intersection of lines where it is possible;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance the development of their competences.
- After this step, guide students to do the application activity 2.6.1, assess their competences and evaluate whether lesson objectives were achieved.

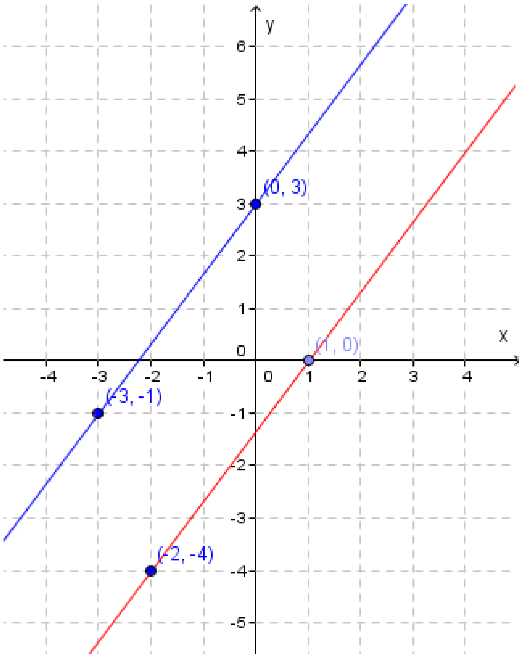
## Answer for activity 2.6.1

1)




The two lines are perpendicular

2)




The two lines are parallel

## Answers for application activity 2.6.1

1. The equation of a line perpendicular to $2x+4y+7=0$ is $4x-2y+k=0$

(i) Where k is an arbitrary constant.

According to the problem equation of the perpendicular line $4x - 2y + k = 0$ passes through the point (-2, 3)

Then,

$4 \cdot (-2) - 2 \cdot (3) + k = 0$

$\Rightarrow -8 - 6 + k = 0$

$\Rightarrow - 14 + k = 0 \quad \Rightarrow \quad k = 14$

Now putting the value of k = 14in (i) we get, $4x - 2y + 14 = 0$

Therefore, the required equation is $4x - 2y + 14 = 0$.

2. The given two equations are $x + y + 9 = 0$ (i)

and $3x - 2y + 2 = 0$ (ii)

Multiplying equation (i) by 2 and equation (ii) by 1 we get

$2x + 2y + 18 = 0$

$3x - 2y + 2 = 0$

Adding the above two equations we get, $5x = - 20 \quad \Rightarrow \quad x = - 4$

Putting $x = -4$ in (i) we get, $y = -5$

Therefore, the co-ordinates of the point of intersection of the lines (i) and (ii) are (- 4, - 5).

Since the required straight line is perpendicular to the line $4x + 5y + 1 = 0$, hence we assume the equation of the required line as $5x - 4y + \lambda = 0$ (iii)

Where $\lambda$ is an arbitrary constant.

By problem, the line (iii) passes through the point (- 4, - 5); hence we must have,

$\Rightarrow 5 \cdot (- 4) - 4 \cdot (- 5) + \lambda = 0$

$\Rightarrow -20 + 20 + \lambda = 0$

$\Rightarrow \lambda = 0.$

Therefore, the equation of the required straight line is 5x - 4y = 0.

3. The equation of any straight line parallel to the line

3x - 2y + 10 = 0 is 3x - 2y + k = 0 (i)

(k is an arbitrary constant).

According to the problem, the line (i) passes through the point (5, - 6) then we shall have,

$3 \cdot 5 - 2 \cdot (-6) + k = 0$

$\Rightarrow 15 + 21 + k = 0$

$\Rightarrow 36 + k = 0$

$\Rightarrow k = -36$

Therefore, the equation of the required straight line is 3x - 2y - 36 = 0.

4. $\vec{u} \cdot \vec{v} = 3 \cdot (-8) + 4 \cdot 6 = 0$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{0}{\sqrt{3^2+4^2}\sqrt{(-8)^2+6^2}}\right) = \cos^{-1}(0)$$

$$\Rightarrow \quad \alpha = 90^\circ$$

## Lesson 9: Distance between point and lines, lines and lines in 2D

**a) Learning objectives:** Determine the distance between point and lines, lines and lines in 2D

### b) Teaching resources

Student's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

- Apply previous lessons in this unit.
- Define different concepts of a parallel and / or perpendicular projection

in 2D as it is taught in S2 (UNIT 8),

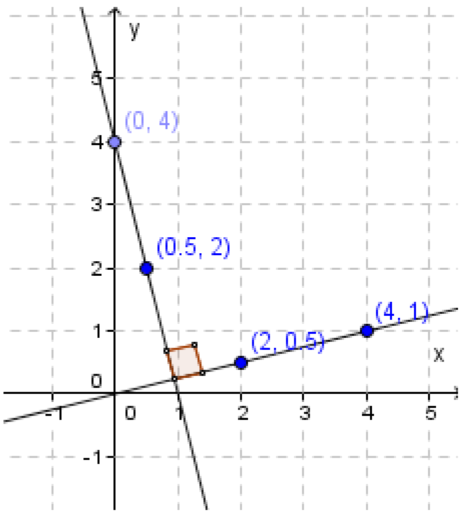
- Effectively Plot lines and vectors in a Cartesian plane
- Appreciate the importance of a point and a line in a plane.

### d) Learning activities

- Organize the student-teachers into groups;
- Ask them to do the activity 2.6.2 from student-teacher's book;
- After a given time, ask groups with different working steps to present their findings to the whole class;
- During the harmonization, help student-teachers establish how to determine the distance between two lines;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance the development of their competences.
- After this step, guide students to do the application activity 2.6.2, assess their competences and evaluate whether lesson objectives were achieved.

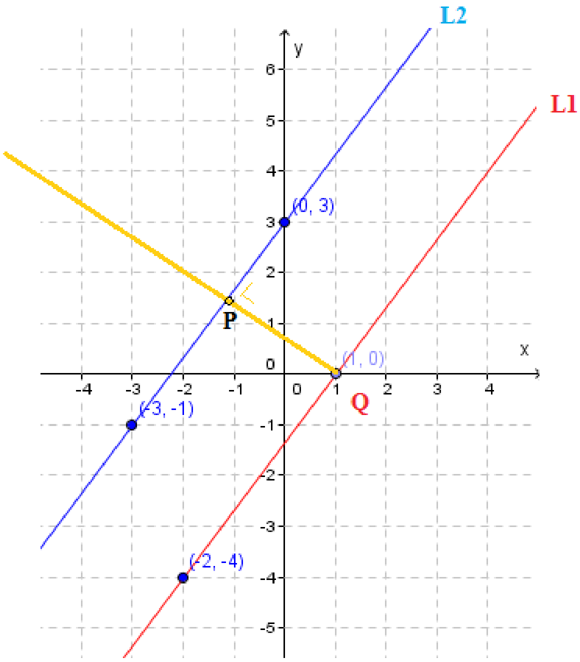
### Answers activity 2.6.2

1.




the distance between the two lines is zero because they intersect.

2) When the two lines are parallel: For example, the line L1 and L2, the distance between them is the length of a segment $PQ$ where the line PQ is perpendicular to L1 or L2.




2) Considering one line L1 from the two we can find the normal line QP between the two parallel lines for which the gradient m is $-\frac{1}{m}$ given by

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0+4}{1+2} = \frac{4}{3}$ (red line) is the gradient of L1. Thus, the equation of PQ can be found as it passes through the point Q (1,0): This is $PQ \equiv y = -\frac{1}{m}x + b$ .

As it passes for example at a point (1, 0), we can find the value of b

$$0 = -\frac{3}{4} + b$$

$$b = \frac{3}{4}$$

Thus, the equation of the Normal line PQ is $y = -\frac{3}{4}x + \frac{3}{4}$ .

The equation of line L2 passing through the points (0,3) and (-3,-1) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y + 1 = \frac{-1-3}{-3}(x - 0)$$

$$y = \frac{4}{3}x - 1$$

Now we can determine the point P intersection of L2 and PQ.

$$\begin{cases} y = \frac{4}{3}x - 1 \\ y = -\frac{3}{4}x + \frac{3}{4} \end{cases}$$

$$-\frac{3}{4}x + \frac{3}{4} = \frac{4}{3}x - 1,$$

$$x = \frac{21}{25},$$

$$y = \frac{4}{3}\left(\frac{21}{25}\right) - 1, \quad y = \frac{3}{25}$$

Now the intersection point is $P\left(\frac{21}{25}, \frac{3}{25}\right)$

The distance between $P\left(\frac{21}{25}, \frac{3}{25}\right)$ and $Q(1,0)$ is $\sqrt{(1-\frac{21}{25})^2 + (0-\frac{3}{25})^2} = \frac{1}{5}$ unit length.

Note: This distance can be shown as the distance between Q(1,0) and the line L1with equation $y = \frac{4}{3}x - 1$ that can be written as $-4x + 3y + 3 = 0$

And $PQ = \frac{|-4(1) + 3(0) + 3|}{\sqrt{(-4)^2 + (3)^2}} = \frac{1}{5}$.

This result can be generalized as the distance between a line L of equation $L \equiv ax + by + c = 0$ and the point $Q(x_0, y_0)$. This distance $d(Q, L) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$

3) Applying the same process as the previous question, on can show that

Distance between a point and line is given by $d=\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$, here point $(4,2)$ to the line $-2x+3y=7$

So, $d=\frac{|-2(4)+3(2)-7|}{\sqrt{(-2)^2+(3)^2}}=\frac{|-8+6-7|}{\sqrt{4+9}}=\frac{9\sqrt{13}}{13}$ unit of length

The shortest distance is $\frac{9\sqrt{13}}{13}\approx 2.5$ unit of length

## Answers for applications 2.6.2

1. First of all, we find the slopes of the two lines. We convert the equations into slope intercept form

$$3x+4y=9$$
$$\Rightarrow 4y=-3x+9$$
$$\Rightarrow y=\frac{-3}{4}x+\frac{9}{4}$$

and

$$6x+8y=15$$
$$\Rightarrow 8y=-6x+15$$
$$\Rightarrow y=\frac{-6}{8}x+\frac{15}{8}$$
$$\Rightarrow y=\frac{-3}{4}x+\frac{15}{8}$$

Slope $m$ *of the lines* $=\frac{-3}{4}$

Hence both lines are parallel

Y intercept of the first line $=\frac{9}{4}$

Y intercept of the second line $=\frac{15}{4}$

Difference between the y- intercepts

$$=\left|\frac{9}{4}-\frac{15}{8}\right|=\left|\frac{18}{8}-\frac{15}{8}\right|=\frac{3}{8}$$

$$\sqrt{1+m^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$

Distance between the lines $=\frac{3}{8}\div\frac{5}{4}=\frac{3}{8}\times\frac{4}{5}=\frac{3}{10}$

2. The slope of the lines $3x+4y=5$ and $6x-8y=45$ is $\frac{-3}{4}$ and $\frac{3}{4}$ respectively.

So, these lines are not parallel but are perpendicular therefore the distance between the two lines is 0.

3. Look at closely, the second equation is actually first multiplied by 2 on both LHS and RHS.

Then the lines represented by these two equations are also same so the distance between them is 0

4. If $P(x_p, y_p)$, its perpendicular distance from $ax+by+c=0$ is $d=\left|\frac{ax_p+by_p+c}{\sqrt{a^2+b^2}}\right|$

So, if 0 $\Leftrightarrow$ $(0,0)$, its perpendicular distance from $3x-4y+15=0$ is,

$$d=\left|\frac{3\times 0-4\times 0+15}{\sqrt{(3)^2+(-4)^2}}\right|=\frac{15}{5}=3$$

5. The distance between the point (-2,3) and the line $x-y=5$?

line $x-y-5=0$ , point (-2,3)

distance $d=\left|\frac{ax_p+by_p+c}{\sqrt{a^2+b^2}}\right|$

$$=\left|\frac{(1\times -2)+(-1\times 3)+(-5)}{\sqrt{(1)^2+(-1)^2}}\right|$$

$(\because\ a=1, b=-1, c=-5, x=-2, y=3)$

$$d=\left|\frac{-2-3-5}{\sqrt{2}}\right|=\left|\frac{-10}{\sqrt{2}}\right|=\frac{10}{\sqrt{2}}=5\sqrt{2}\text{ unit lenght}$$

6. As mentioned, points (-3, -4) to the line $3x-4y-1=0$

Let $x=-3$ and $y=-4$ , So comparing the line with equation

$Ax+By+C=0 \qquad A=3,\ \ B=-4$ and $C=-1$

Equation for distance between line and coordinate axes points. Is

$$d = \left| \frac{Ax + By + C}{\sqrt{A^2 + B^2}} \right|$$

$$d = \left| \frac{3(-3) + 4(-4) + (-1)}{\sqrt{(-3)^2 + (-4)^2}} \right| \quad d = \frac{6}{5} \text{ unit lenght}$$

7. Distance between the line $3x + 4y - 6 = 0$ and point $(2, -1)$ is given by

$$d = \left| \frac{3(2) + 4(-1) - 6}{3^2 + 4^2} \right| = \left| \frac{6 - 4 - 6}{\sqrt{25}} \right| = \frac{4}{5}$$

## Lesson 7: Circle and its equation in 2D

### a) Learning objectives:

- Explain and determine the center, radius, and diameter of a circle in a Cartesian plan
- Determine the intersection of a line and a circle
- Establish the equation of a circle.
- Determine the intersection of a line and a circle

### b) Teaching resources

Learner's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

**Note:** Where it is possible, the tutor can guide students to draw circles using GeoGebra software and present their findings using PowerPoint;

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

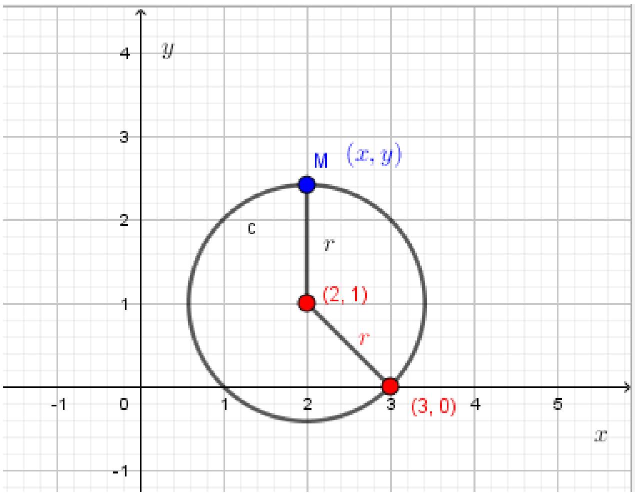
- Recognize different circle theorems and perform operations on circle as it is taught in S3 (UNIT 9) ,
- Effectively Plot lines and vectors in a Cartesian plane
- Appreciate the importance of a point and a line in a plane.
- Be accurate in plotting/graphing and calculations.

## d) Learning activities

- Organize the student-teachers into small groups; …
- Ask them to do the activity 2.7.1 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, guide student-teachers discover the characteristics of a circle in a Cartesian plane and to establish equation of circle of radius r and centred at $C(a,b)$;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance how to establish Cartesian equation of a circle,
- After this step, guide students to do the application activity 2.7.1, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for activity 2.7.1

In $XY$ plane,



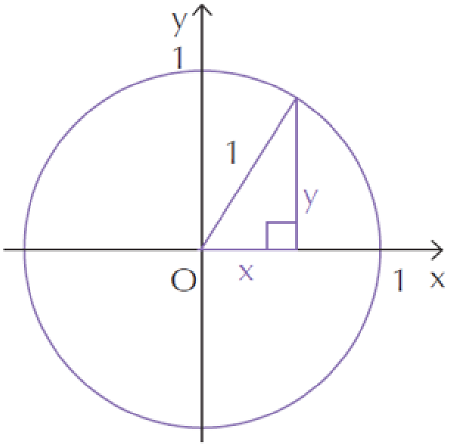

The distance between C and P is: $d(C,P)=\sqrt{(3-2)^2+(0-1)^2}=\sqrt{2}$

The distance between $d(C,M)=r=\sqrt{(x-2)^2+(y-1)^2}$

Or $r^2=(x-2)^2+(y-1)^2$

This is the form for the equation of the circle of radius is $r$ and the centre $C(2,1)$ which can be generalized to $(x-a)^2+(y-b)^2=r^2$ as the standard form for the equation of the circle, the radius is $r$ and the centre $C(a,b)$.

If we place the circle centre at $(0,0)$ and set the radius to 1 we get:




We know that equation of the circle is $(x-a)^2+(y-b)^2=r^2$

Putting $a=0$ and $b=0$, we get:

$$(x-0)^2+(y-0)^2=r^2$$
$$x^2+y^2=r^2$$

Generally, the circle is defined as the locus of all points, $P(x,y)$, which are equidistant from some given point $C(a,b)$. .

## Answers for application activity 2.7.1

1. $x^2+y^2-6x-4y-3=0$

2. $x^2+y^2+4x-4y+4=0$

Or $x^2+y^2-6x+4y-51=0$

3. Centre: $(2,-4)$, radius: $4\sqrt{2}$

4. Centre: $(3,2)$, radius: $\sqrt{22}$

## Lesson 8: Parametric equation of a Circle

### a) Learning objectives:

- Establish the parametric equations of a circle of radius r and center C(a,b);

### b) Teaching resources

Learner's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

**Note:** Where it is possible, the tutor can guide students to draw circles using GeoGebra software and present their findings using PowerPoint;

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

- Recognize different circle theorems and perform operations on circle as it is taught in S3 (UNIT 9) and in the previous lesson of this unit,
- Effectively Plot lines and vectors in a Cartesian plane
- Appreciate the importance of a point and a line in a plane.
- Be accurate in plotting/graphing and calculations.

### d) Learning activities

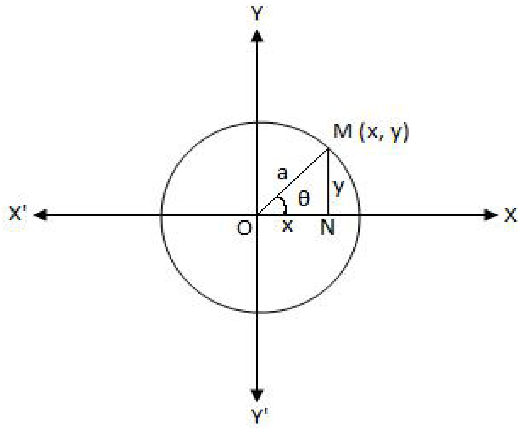
- Organize the student-teachers into small groups; …
- Ask them to do the activity 2.7.2 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, guide student-teachers discover the characteristics of a circle in a Cartesian plane and to establish parametric equation of circle of radius r and centred at $C(a,b)$;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance how to establish parametric equation of different types of a circle,
- After this step, guide students to do the application activity 2.7.2, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for activity 2.7.2

a) $(x-0)^2+(y-0)^2=\left(\sqrt{19}\right)^2 \rightarrow r=\sqrt{19}$ and $C(0,0)$

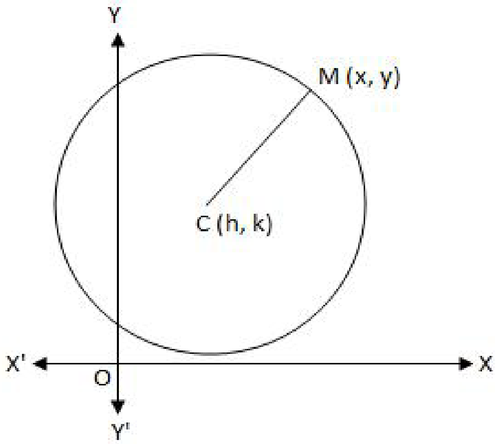
b) $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \rightarrow \begin{cases} x=\sqrt{19}\cos\theta \\ y=\sqrt{19}\sin\theta \end{cases}$ for

c) Let learners discover that in 2D, a circle whith centre at the origin and radius , the equation of a circle is given or ; ()




Expressing the equation of a circle using a parameter , we write parametric equations for

For a circle with centre at the point and radius , the equation of a circle



Hence, we write parametric equations

for

- Lead them to do examples to emphasize the skills for finding equation of a circle.
- Call learners to do Application activity 2.7.2 to master the content.

## Answers for application activity 2.7.2

1. a) $x^2+y^2-25=0$

$$\begin{cases} x=5\cos\theta \\ y=5\sin\theta \end{cases} \quad \text{for}$$

b) $x^2+y^2-8x+4y-44=0$

$$\begin{cases} x=4+8\cos\theta \\ y=-2+8\sin\theta \end{cases} \quad \text{for}$$

c) . $r=\sqrt{(1+4)^2+(3+2)^2}=5\sqrt{2}$ then

$x^2+y^2+8x+2y-30=0$

$$\begin{cases} x=4+5\sqrt{2}\cos\theta \\ y=2+5\sqrt{2}\sin\theta \end{cases} \quad \text{for}$$

2. Let substitute the coordinates of each of the three points into the general equation of circle $x^2+y^2+ax+by=c$

We get: $\begin{cases} 1+b=c \\ 16+9+4x+3y=c \\ 1+1+x-y=c \end{cases}$ and we solve the system,

$$\Leftrightarrow \begin{cases} b=c-1 \\ 4a+3b=c-25 \\ a-b=c-2 \end{cases} \qquad \Leftrightarrow \begin{cases} 4a+3c-3=c-25 \\ a-c+1=c-2 \end{cases}$$

$\Leftrightarrow \begin{cases} 4a+2c=-22 \\ a-2c=-3 \end{cases}$ $\Leftrightarrow \begin{cases} 4a+2c=-22 \\ a-2c=-3 \end{cases}$

$5a=-25 \Rightarrow a=-5$

$-5-2c=-3 \Rightarrow c=-1$ $\Rightarrow$ $b=c-1=-2$

Thus the equation is $x^2+y^2-5x-2y=-1$

3. (1) a) $(x-5)^2+(y-4)^2=6^2$

b) $C(5,4)$ and the radius $\text{r}=6$

c) $\begin{cases} x=5+6\cos\theta \\ y=4+6\sin\theta \end{cases}$ for

(2). a) $(x-3)^2+(y-4)^2=0^2$

b) $C(3,4)$ and the radius $\text{r}=0$

c) no parametric equations

## Lesson 9: Intersection of a line and a circle

### a) Learning objectives:

Establish the intersection of a circle and a line using their positions or their equations;

### b) Teaching resources

Student's book and other Reference Textbooks to facilitate research, Mathematical set, Calculator, Manila paper, Markers, Pens and Pencils.

**Note:** Where it is possible, the tutor can guide students to draw circles using GeoGebra software and present their findings using PowerPoint;

### c) Prerequisites / Revision / Introduction:

Student-teacher will easily learn this unit, if they are able to:

- Recognize different circle theorems and perform operations on circle as it is taught in S3 (UNIT 9) and in the previous lesson of this unit,
- Effectively Plot lines and vectors in a Cartesian plane
- Appreciate the importance of a point and a line in a plane.

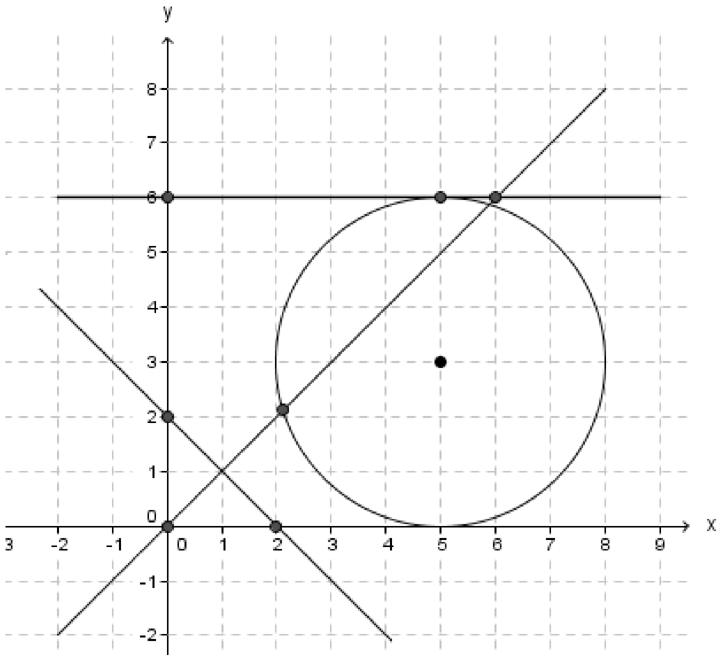
- Be accurate in plotting/graphing and calculations.

## d) Learning activities

- Organize the student-teachers into small groups; …
- Ask them to do the  activity 2.7.3 from student-teacher's book;
- After a given time, ask randomly some groups to present their findings to the whole class;
- During the harmonization, guide student-teachers discover the intersection of a circle and a line after drawing their graphs;
- Use different probing questions and guide students to explore examples and the content given in the student's book to enhance how to establish the intersection of a line and a circle whose equations are given;
- After this step, guide students to do the application activity 2.7.3, assess their competences and evaluate whether lesson objectives were achieved.

## Answers for activity 2.7.3

1) (a)



(b).Intersection points with the circle and the lines:

i) with the line $y = x$ , the points are $\left\{\left(\frac{-\sqrt{31}+9}{2},\frac{-\sqrt{31}+9}{2}\right),\left(\frac{\sqrt{31}+9}{2},\frac{\sqrt{31}+9}{2}\right)\right\}$

ii) with the line $y + x = 2$ , there is no intersection point

iii) with the line $y = 6$, there is one intersection point $\left\{\left(2\sqrt{3}+5,6\right),\left(-2\sqrt{3}+5,6\right)\right\}$

**Note:** Looking carefully at the figure above, it observable that:

- There are two points of intersection between the circle and the line $y = x$
- There are no points of intersection between the circle and the line $x + y = 2$
- There is one point of intersection between the circle and the $y = 6$

## Answers for application activity 2.7.3

(1) Solve simultaneously the system

$$\begin{cases} x^2+y^2-14x+12y+69=0 & (1) \\ y=-2 & (2) \end{cases}$$

$(2)$ in $(1)$:

$$x^2+(-2)^2-14x+12(-2)+69=0$$

$$\Leftrightarrow x^2+4-14x-24+69=0$$

$$\Leftrightarrow x^2-14x+49=0$$

$$\Delta = 196-196=0$$

Since $\Delta = 0$, there is unique point of intersection.

$$x=\frac{14}{2}=7 \text{ and } y=-2$$

The point of intersection is $(7,-2)$

(2) We solve the following simultaneous equations:

$$\begin{cases} x^2+y^2-14x+12y+69=0 & (1) \\ y=x & (2) \end{cases}$$

$(2)$ in $(1)$:

$$x^2+x^2-14x+12x+69=0$$

$$\Leftrightarrow 2x^2-2x+69=0$$

$$\Delta=4-552=-548<0$$

Since $\Delta<0$, there is no point of intersection.

(3) Solve simultaneously the system

$$\begin{cases} x^2+y^2-14x+12y+69=0 & (1) \\ x+y=1 & (2) \end{cases}$$

From $(2)$: $y=1-x \quad (3)$

$(3)$ in $(1)$:

$$x^2+(1-x)^2-14x+12(1-x)+69=0$$

$$\Leftrightarrow x^2+1-2x+x^2-14x+12-12x+69=0$$
$$\Leftrightarrow 2x^2-28x+82=0$$
$$\Leftrightarrow x^2-14x+41=0$$

$$\Delta=196-164=32$$

Since $\Delta>0$, there are two points of intersection.

$$x_1=\frac{14+4\sqrt{2}}{2}=7+2\sqrt{2}\Rightarrow y_1=1-x_1=-6-2\sqrt{2}$$
$$x_2=\frac{14-4\sqrt{2}}{2}=7-2\sqrt{2}\Rightarrow y_2=1-x_2=-6+2\sqrt{2}$$

The points of intersection are $\left(7+2\sqrt{2},-6-2\sqrt{2}\right)$ and $\left(7-2\sqrt{2},-6+2\sqrt{2}\right)$

## 2.6. Summary of the Unit

1. The Cartesian coordinates of a point in the plane are written as $(x,y)$.

2. If A and B are two points the distance between these two points denoted is

$$d(A,B)=\left\|\overrightarrow{AB}\right\|=\sqrt{(b_1-a_1)^2+(b_2-a_2)^2}$$

3. The midpoint M of the line segment from point A to point B given by

$$M=\frac{1}{2}(A+B).$$

4. Line passing through point $P(x_0,y_0)$ and parallel to the direction vector, $\vec{v}=(a,b)$:

Vector equation

$$(x,y)=(x_0,y_0)+r(a,b)$$

Parametric equation

$$\begin{cases} x=x_0+ra \\ y=y_0+rb \end{cases}$$

Cartesian equatios

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}$$

5. Consider the point $P(x_0,y_0)$ and the line with equation $ax+by=c$. The shortest distance from P point to the line $ax+by=c$ is given by the relation

$$d=\frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$$

6. Let $\alpha_1$ and $\alpha_2$ be the angles that the two lines make with the positive x-axis respectively, then if $\alpha$ is the acute angle between the two lines

$$\alpha=\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$$

**Note:**

If the direction vectors of two lines are known the angle between the two lines is the angle formed by their direction vectors.

7. The equation of the circle with centre $C(a,b)$ and radius $r$ is given by

$$(x-a)^2+(y-b)^2=r^2$$

8. General equation of the circle is given by $x^2+y^2+kx+ly+m=0$

In this case,the centre is given by $C=\left(-\frac{k}{2},-\frac{l}{2}\right)$ and the radius is given by

$r=\frac{1}{2}\sqrt{k^2+l^2-4m}$.

9. Given the equations of a line and a circle, we can solve the two equations simultaneously. If

- there are two **distinct roots** then the line cuts the circle in two distinct points and the intersection is the set of these two points. In this case part of the line is a **chord** of the circle and the distance from the centre of the circle to the line is less than the radius of the circle.
- there is a **repeated root** then the line touches the circle and the intersection is this root. In this case the line is **tangent** to the circle and the distance from the centre of the circle to the line is equal to the radius of the circle.
- there is **no real root** then the line neither cuts nor touches the circle. In this case there is no intersection and the distance from the centre of the circle to the line is greater than the radius of the circle.

## 2.7. Additional information for the tutor

- Suppose that $\bar{v}$ is a vector position with initial point $P_1(x_1,y_1)$ not necessary the origin, and terminal point $P_2(x_2,y_2)$. If $\vec{v}=\overrightarrow{P_1P_2}$, the $\vec{v}$ is equal to position vector $\vec{v}=\langle x_2-x_1, y_2-y_1\rangle$
- Emphasize on the new notation of vector $\{\vec{i}=(1,0),\vec{j}=(0,1)\}$ leads the equation of the line to be rewritten as $x\vec{i}+y\vec{j}=x_0\vec{i}+y_0\vec{j}+r(a\vec{i}+b\vec{j})$
- These are orthonormal basis of $IR^2$ is $\{\vec{i}=(1,0),\vec{j}=(0,1)\}$ .Where $\vec{i}$ is on $x-axis$ while $\vec{j}$ is on $y-axis$.

- The parameter   in parametric equations of a circle is given by

  with  and .

  with  and .

- A circle that has radius $r = 0$, is called **point circle.**
- A circle that has radius $r = 1$, is called **unit circle.**

## 2.8 End unit assessment

Answer for End Unit assessment

1. a) $6\sqrt{2}$  b) $6$  c) $\sqrt{41}$  d) $\sqrt{x^2 + y^2}$

2. a) $C(0,0); r = \sqrt{10}$  b) $(3,0); r = 5$  c) $(2,-1); r = 3\sqrt{2}$  d) $(-1,1); r = 0$

3. 8 units

4. a. $(5,7)$  b. $(6,6)$  c. $(3,4)$  d. $\left(\frac{5}{2},4\right)$

5. The solution is  3,5

6. a). $y = 4x - 10$  b). $y = -\frac{3}{2}x + \frac{1}{2}$

7. i). $(x, y) = (2,1) + \lambda(3,4)$

   ii). $(x, y) = (5,-3) + \lambda(1,-4)$

8. The following are those represent the circles:  b. and c.

## 2.9 Additional activities

### *2.9.1 Remedial activities*

1. Find the length of the straight lines joining each of the following pairs of points:

a. $(1,3)$ and $(4,7)$

b. $(1,8)$ and $(7,0)$

c. $(4,0)$ and $(-3,0)$

d. $(1,5)$ and $(-2,1)$

**Solution:**

a). 5 b). 10 c). 7 d). 5

2. Find the Cartesian equations of the lines whose vector equations are given below. Give your answers in the form $y = mx + c$

a. $x\vec{i} + y\vec{j} = 3\vec{i} + 2\vec{j} + r(3\vec{i} - \vec{j})$

b. $x\vec{i} + y\vec{j} = 4\vec{i} - 5\vec{j} + r(2\vec{i} + 3\vec{j})$

**Solution:**

a). $y = -\frac{1}{3}x + 3$ b). $y = \frac{3}{2}x - 11$

3. Find the centre and radius of each of the following circles

a) $x^2 + y^2 = 10$

b) $(x-3)^2 + y^2 = 25$

c) $(x-2)^2 + (y+1)^2 = 18$

d) $x^2 + y^2 + 2x - 2y = 2$

**Solution:**

a. $(0,0)$, $\sqrt{10}$ b. $(3,0)$, $5$ c. $(2,-1)$, $3\sqrt{2}$ d. $(-1,1)$, $2$

### 2.9.2 Consolidation activities

1) State the equation of the circle that can satisfy the given conditions. Hence, write parametric equations for each.

a. $C(2,0); r = 5$

b. $C(-1,-2); r = 7$

c. $C(4,2)$ and passes through $P(1,3)$

**Solution:**

a) $y - 2 = \frac{4}{5}(x - 0)$ b) $y + 8 = -3(x - 5)$ c) $y - 1 = \frac{2}{3}(x + 6)$

2) Find the possible values of k given that the point $(4,k)$ is the same distance from $9x+8y+1=0$ as $(2,5)$ is from $y=12x+2$

**Solution:**

The possible values of "k: are : 2, and $-\frac{29}{4}$

3) Find the lengths of the sides of the triangle with vertices at $(1,1),(4,5)$ and $(9,-5)$. Hence prove that the triangle is right-angled, and state which angle, $\widehat{A},\widehat{B}$ or $\widehat{C}$ is the right angle.

**Solution:**

The lengths are 5, 10, and $5\sqrt{5}$ . The triangle is right angled at , $\widehat{A}$

### *2.9.3 Extended activities*

1. Find the equation of the circle passing through the given three points

a. $(1,0),(0,1),(3,4)$

b. $(2,-1),(1,3),(1,-4)$

c. $(2,2),(-2,1),(2,3)$

**Solution:**

a. $x^2+y^2-4x-4y+3=0$

b. $x^2+y^2+9x+y=22$

c. $2x^2+2y^2+x-10y+2=0$

2. Find the equation of the circle having *AB* as diameter where *A* is the point $(1,8)$ and *B* is the point $(3,14)$

**Solution:**

$x^2+y^2-4x-22y+115=0$

3. Points $A(0,2)$ and $B(4,-2)$ lie on the circumference of a given circle. Points

$C(-3,-3)$ and $D(7,2)$ lie outside the circle but the centre of the circle lies on $CD$. Find the equation of the circle.

**Solution:**

$x^2 + y^2 - 2x + 2y = 8$

4. The points A, B and C have coordinates $(2,1),(7,3),$ *and* $(5,k)$ respectively. If AB and BC are of equal length, find the possible values of k.

**Solution**

$k = 8$ or $k = -2$

5. Find the equations of the two straight lines that make an angle of $45^\circ$ with the line $y = 3x - 2$ and that pass through the point $(6,4)$.

**Solution:**

a) $2y = x + 2$ b). $y + 2x = 16$

# UNIT 3

# APPLICATION OF TRIGONOMETRIC CONCEPTS IN SOLVING RELATED PROBLEMS

## 3.1. Key unit competence

Apply trigonometric concepts in solving problems on triangles and real-life situation.

## 3.2. Prerequisite

Student - teachers will perform well in this unit if they have a good background on

- Use correctly simple language structure, vocabulary and suitable symbols of algebraic expressions, learnt in learnt in ECLPE year one, Student-teacher book Unit 5 and Unit 7.
- Carry out numerical calculations correctly;
- Vector representation learnt from S2 in Unit 7;
- Right-angled triangle learnt from S3 ;
- Set of numbers learnt in unit 1;
- Algebraic fractions learnt from Senior 3, in unit 3;
- Isometries learnt from S2, unit 9.

## 3.3. Cross-cutting issues to be addressed

- **Inclusive education**: promote the participation of all student-teachers while teaching
- **Peace and value Education**: During group activities, the teacher will encourage student teachers to help each other and to respect opinions of colleagues.
- **Financial education**: Guide students to discuss how to invest money into a common project. This should be addressed via problems, it consists on several ways of using money that encourage learners to deal with money financially

- **Gender:** Give equal opportunities to all learners (girls and boys) to participate actively in all learning activities from the beginning to the end of the lesson.

## 3.4. Guidance on introductory activity 1

- Student-teachers work on the introductory activity to understand the use of trigonometry.
- Put learners in groups.
- Let them read and do the introductory activity 3 in the Student teacher's book.
- Make sure that all student-teachers are participating and performing well.
- Through class discussions, let student-teachers think of different ways of application of trigonometry.
- Through different examples, help student-teachers to understand the importance of trigonometry by showing their application in real life for example in construction, satellite systems and astronomy, naval and aviation industries, land surveying and in cartography (creation of maps) and so on.
- Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to arouse their curiosity on what is going to be leant in this unit.

### Answer for introductory activity 3

Pythagoras theorem is not enough for finding the height of the given cathedral. By using sine rule, the required height can be determined as

$$\frac{\sin 60^0}{h} = \frac{\sin 30^0}{280} \Leftrightarrow h\sin 30^0 = 280\sin 60^0 \Leftrightarrow \frac{h}{2} = \frac{280\sqrt{3}}{2} \Leftrightarrow h = 280\sqrt{3} \simeq 484.97m$$

## 3.5. List of lessons and sub-heading

| No | Lesson title | Learning objectives | Number of periods |
|---|---|---|---|
| 0 | Introductory activity | To arouse the curiosity of student teachers on the content of unit 3 | 1 |
| 1 | Angle and its measurements | Identify angles as acute, obtuse, right, or straight;<br><br>Measure and estimate accurately angles using a protractor. | 2 |
| 2 | Unit conversion | Convert angles measured in degrees, minutes and seconds into radian and grades. | 2 |
| 3 | Unit circle | Understand and Explain unit circle, reference angle, terminal side, standard position of any angle whose reference angle measures 30°, 45°, or 60°.<br><br>Determine the quadrants where sine, cosine, and tangent are positive and negative. | 1 |
| 4 | Trigonometric ratio of special angles $\left(30^0, 45^0, 60^0\right)$ | Evaluate trigonometric ratios of special angles $\left(30^0, 45^0, 60^0\right)$. | 2 |
| 5 | Trigonometric identities | Define and use trigonometric identity to solve related problem and to simplify mathematics expressions | 2 |
| 6 | Reduction to positive of acute angles | Solve problems involving positive acute angles | 2 |

| | | | |
|---|---|---|---|
| 7 | Transformation formulae: | Calculate the value of any angle using the **addition and subtraction formulae**. | 2 |
| | | Use trigonometric identities such as angle addition /subtraction and **double angle formulas**, to express values of trigonometric functions | 2 |
| | | Use trigonometric identities such as angle addition /subtraction and **half angle formulas**, to express values of trigonometric functions | 2 |
| | | Convert a **sum into product** and apply the formulae to solve related problems | 2 |
| | | Convert a **product into a sum the** formulae to solve related problems | 2 |
| 8 | Trigonometric equations | Extend the concepts of trigonometric ratios and their properties to trigonometric equations to solve related problems | 3 |
| 9 | Triangles and applications | Use trigonometry to solve problems involving triangles<br><br>Use trigonometry to solve problems involving bearings<br><br>Use trigonometry to solve problems involving air navigation, inclined plane, etc. | 4 |
| 10 | End unit assessment | | 1 |
| Total number of periods | | | 30 |

## Lesson 1: Angle and its measurements

### a) Learning objectives:

- Identify angles as acute, obtuse, right, or straight;
- Measure and estimate accurately angles using a protractor.

### b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares,…

c) **Prerequisites / Revision / Introduction:**

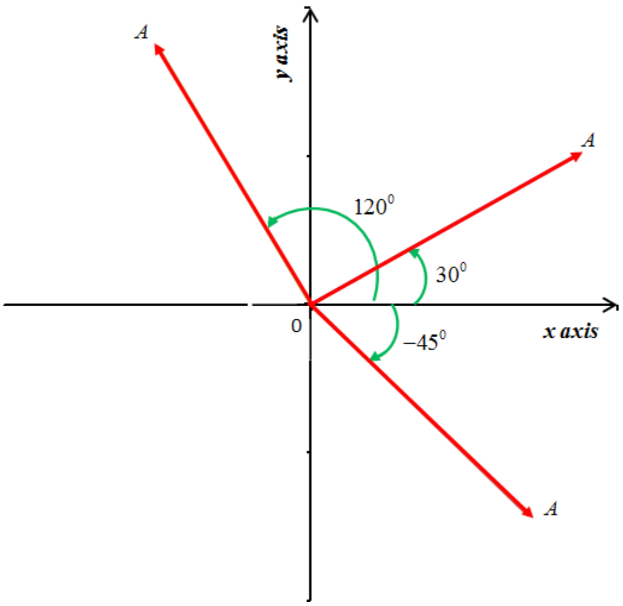
Student-teachers will learn better this lesson if they have a good understanding concept of

- A vector learnt from S2 in Unit 7 ;
- Right-angled triangle learnt from S3 in Unit 8.

d) **Learning activities**

- Organize the student-teachers into groups and ask them to attempt the **Activity 3.1** from student-teacher's book and introduce the concept of angles
- Introduce the topic by giving some examples of angles and their measurements
- Discuss with student-teachers on how a given angle is measured.
- Guide student-teachers in drawing the given angles.
- Invite groups for presentation of their work.
- Facilitate them to do the provided examples given in **Student-teacher's book** and work individually application activity 3.1 to check the skills they have acquired.

## Answers for Activity 3.1



## Answers for Application activity 3.1

**1.** a) $10\pi\ cm$ b) $9.2\pi\ cm$

**2.** a) $\frac{7}{\pi}cm$ b) $\frac{8}{\pi}cm$

**3.** The angle in degrees that describes the compass bearing

a) SSW (south-southwest) is 202.5°

b) WNW (west-northwest) is 292.5°

c) NNW (north-northwest) is 337.5° .

d) ESE is closest at 112.5°.

e) SW is closest at 225°.

f) In 8h, the hour hand of a clock rotates $\frac{4\pi}{3}$ .

In 1 week, the hour hand of a clock rotates $28\pi$ .

**4.** At 4:00, the hour hand of a clock has rotated $\frac{2\pi}{3}$. At 2:30, the hour hand of a clock has rotated $\frac{5\pi}{12}$. At 10:12, the hour hand of a clock has rotated $\frac{17\pi}{10}$.

## Lesson 2: Unit Conversion

### a) Learning objectives:

Convert angles measured in degrees, minutes and seconds into radians and grades.

### b) Teaching resources

Learner's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding concept of

- Set of numbers learnt in unit 1year 1
- Algebraic fractions learnt from Senior 3, in unit 3.

### d) Learning activities

- Organize the student-teachers into groups / or give them task individually and ask them to attempt the **Activity 3.2** from student-teacher's book and introduce the concept of angles;
- Make sure that everybody is engaged/ involved;
- Facilitate student-teachers on cutting a piece of thread or string with the same size as the radius;
- Invite groups for presentation of the work to whole class ;
- Harmonize their work and guide student-teachers to define a radian; degree, grade and how to establish a relationship between them for converting each unit to another;
- As they are discussing, concentrate on slow student-teachers for further explanation and provide assistance to groups in need;
- Facilitate them to do the provided examples given in **Student-teacher's book** and work individually application activity 3.2 to check the skills they have acquired.

#### Answers for Activity 3.2

Show how a piece of string and graph paper may be used to demonstrate radian measure. Measure a radius-length of string and then show, by wrapping the string around the circle, that it takes a little more than 6 radius-lengths to go all the way around the circle

1) $2\pi r$

2) A little more than 6 radians

3) The change in radius affects change of circumference but does not for radians (Answer may vary)

4) $\pi$ radians

## Answers for application activity 3.2

1. Without using a calculator,

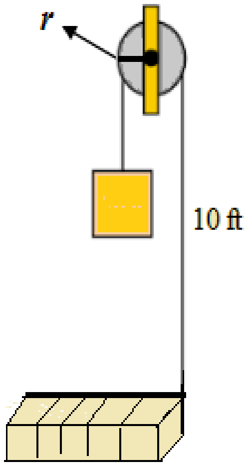
a) $\frac{\pi}{4}=45^0$ b) $\frac{2\pi}{3}=120^0$ c) $\frac{\pi}{6}=30^0$ d) $\frac{\pi}{8}=22.5^0$ e) $\frac{5\pi}{12}=75^o$

f) $\frac{3\pi}{8}=67.5$ g) $\frac{2\pi}{5}=72^0$ h) $\frac{5\pi}{4}=225$

2. Leaving the result in terms of $\pi$ ( without using a calculator),

a) $150^0=\frac{5\pi}{6}$ b) $225^0=\frac{5\pi}{4}$ c) $45^0=\frac{\pi}{4}$ d) $90^0=\frac{\pi}{2}$ e) $30^0=\frac{\pi}{6}$

3. $\frac{17\pi}{30}$




a) $h = arc\ length = 2\pi r.\frac{\theta}{360} = \theta.r(if\ \theta\ is\ given\ in\ radians)$

$h = 16\pi cm$

$h = 16\pi\ cm \simeq 50.265\,cm$

b) Applying the same formula we get:

$$h = \pi r = 2\pi m \simeq 6.283\,m$$

## Lesson 3: Unit Circle

### a) Learning objectives:

- Understand and explain unit circle, reference angle, and terminal side, standard position of any angle whose reference angle measures 30°, 45°, or 60°.
- Determine the quadrants where sine, cosine, and tangent are positive and negative.

### b) Teaching resources

Learner's book and other reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding concept of

Elements of a circle learnt from S3 in Unit 9.

### d) Learning activities

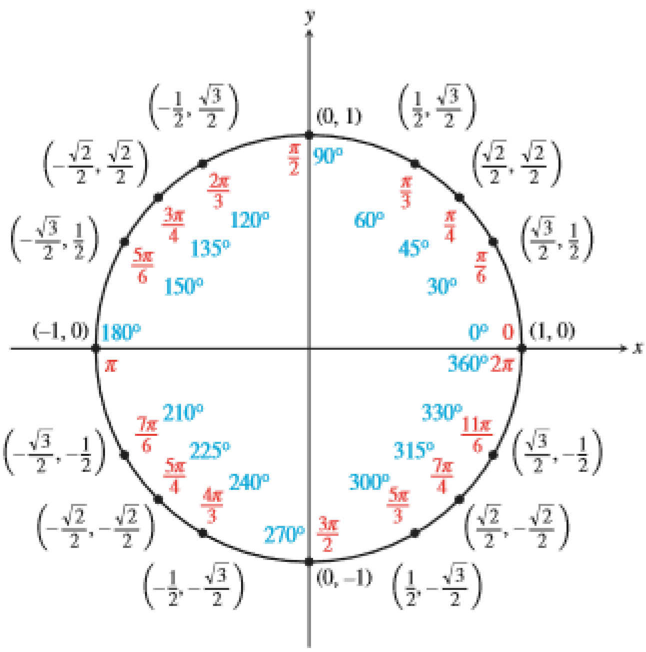
- Organize student-teachers into groups / or give them task individually and ask them to attempt the Activity 3.3 from student-teacher's book and introduce the concept of angles and guide them
- Make sure that everybody is engaged/ involved.
- Discuss with student-teachers on how to construct a unit circle.
- Guide student-teachers in drawing the given angles in a unit circle.
- Invite group representatives for presentation.
- Harmonize their answers and guide students to identify quadrants in a unit circle;
- Facilitate them to discuss about examples given in Student-teacher's book to emphasize the skills, he/she has got.

- Invite them to work out the application activity 3.3 to check their understanding

## Answers for activity 3.3

While, the wheel turns, the graph for groups may vary but the height of the point from the centre remains constant.

## Answers for application activity 3.3



## Lesson 4: Trigonometric ratio of special angles ($30^0, 45^0, 60^0$)

### a) Learning objectives:

Evaluate trigonometric ratios of special angles $\left(30^0, 45^0, 60^0\right)$.

### b) Teaching resources

Learner's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding concept of

- Right-angled triangle learnt from S3, in unit 8.

### d) Learning activities

Note: this lesson can be taught in two steps or two different sessions:

- Organize the student-teachers into groups / or give them task individually and ask them to attempt the **Activity 3.4 1** from student-teacher's book and introduce the concept of angles;
- Make sure that everybody is engaged/ involved.
- Discuss with student-teachers on how to construct a unit circle.
- Guide student-teachers in drawing the given angles.
- Invite groups for presentation of their work;
- Facilitate them to work out exercises given in **application activity 3.4.1** in **Student teacher's book** to emphasize the skills, he/she has got.
- Retake the same process for the activity 3.4.2, the related content and examples.
- Invite them to work on the **application activity 3.4** 2.to check their understanding.

#### Answers for activity 3.4.1

The figures may vary. For any point $P(x, y)$, $\frac{x}{1} = \cos\theta$ and $\frac{y}{1} = \sin\theta$.

$\left(\frac{x}{1}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$, we deduce that $(\cos\theta)^2 + (\sin\theta)^2 = 1$ for any value of $\theta$.

## Answers for application activity 3.4.1

1. a) Pairs: sine and cosecant, cosine and secant, tangent and cotangent

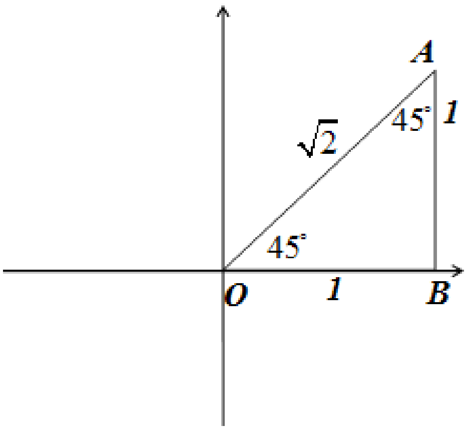
   b) $\tan\theta$ c) $\sec\theta$ d) 1 e) $\sin\theta$ and $\cos\theta$

2. $\csc\theta = 1{,}000$ and $\theta$ is in $II^{nd}$ quadrant

3. $\frac{5\sqrt{61}}{61}, \frac{6\sqrt{61}}{61}, \frac{5}{6}, \frac{6}{5}, \frac{\sqrt{61}}{6}, \frac{\sqrt{61}}{5}$ and $\theta$ is in $I^{st}$ quadrant

## Answers for activity 3.4.2

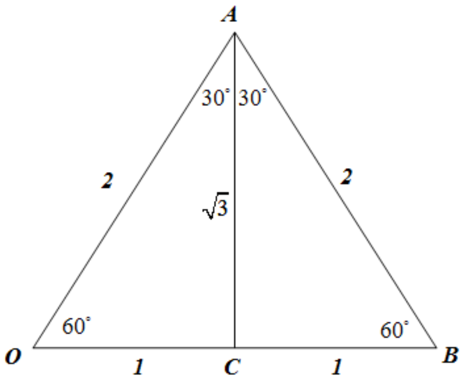
1) $\sin 45^\circ, \cos 45^\circ$ *and* $\tan 45^\circ$




From the diagram

$\sin 45^\circ = \frac{1}{\sqrt{2}}$ $\cos 45^\circ = \frac{1}{\sqrt{2}}$ $\tan 45^\circ = \frac{1}{1} = 1$

2) The six trigonometric values of $60^\circ$ and $30^\circ$




From $\Delta OAC$,

$$\sin 60^0 = \frac{\sqrt{3}}{2}, \cos 60^0 = \frac{1}{2},\ \tan 60^0 = \sqrt{3},\ \csc 60^0 = \frac{2\sqrt{3}}{3},\ \sec 60^0 = 2,\ \cot 60^0 = \frac{\sqrt{3}}{3}.$$

From $\Delta OAC$,

$$\sin 30^0 = \frac{1}{2}, \cos 30^0 = \frac{\sqrt{3}}{2},\ \tan 30^0 = \frac{\sqrt{3}}{3},\ \csc 30^0 = 2,\ \sec 30^0 = \frac{2\sqrt{3}}{3},\ \cot 30^0 = \sqrt{3}.$$

## Answers for application activity 3.4.2

1.

| $\theta$ | $0^0$ | $30^0$ | $45^0$ | $60^0$ | $90^0$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | $\nexists$ |

2. a) $\frac{\sqrt{3}-1}{2}$ b) 0

3. a)1 b) $\sqrt{3}$ c) $\frac{\sqrt{3}}{3}$ d) 0 e) does not exist f) 0

g) does not exist h) does not exist i)0 j) does not exist k) 0

## Lesson 5: Trigonometric identities

### a) Learning objectives:

Define and use trigonometric identities to solve related problem and to simplify mathematics expressions

### b) Teaching resources

Students's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding concept of

- Definition of trigonometric ratios learnt in previous lesson
- Pythagoras theorem learnt from S2, in unit 6.

### d) Learning activities

- Organize the student-teachers into groups / or give them task individually and ask them to attempt the Activity 3.5 from student-teacher's book and introduce the concept of angles;
- Make sure that everybody is engaged/involved.
- Invite groups for presentation of their work to the whole class;
- Facilitate them to work out examples given in Student-teacher's book to emphasize the skills;
- Invite them to work the application activity 3.5 to check their understanding

## Answers for activity 3.5

In this triangle,

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \sin\alpha = \frac{x}{r},\ \cos\alpha = \frac{y}{r}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = (\sin\theta)^2 + (\cos\theta)^2 = \sin^2\theta + \cos^2\theta$$ and then $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\alpha + \cos^2\alpha = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

## Answers for application activity 3.5

1)

$$\frac{\cot\theta}{1+\sin\theta} = \frac{\cot\theta}{1+\sin\theta}.\frac{1-\sin\theta}{1-\sin\theta} = \frac{\cot\theta(1-\sin\theta)}{1-\sin^2\theta} = \frac{\cot\theta(1-\sin\theta)}{\cos^2\theta} = \frac{1-\sin\theta}{\sin\theta\cos\theta}$$

2)

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\left(\sin^2\theta + \cos^2\theta\right) + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}}$$

$$= \frac{2}{\sin\theta}$$ or we know that $\csc\theta = \frac{1}{\sin\theta}$

$$= 2\csc\theta$$

## Lesson 6: Reduction to positive of acute angles

### a) Learning objectives:

- Solve problems involving positive acute angles

### b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding concept of

- Unit circle learnt in lesson 3 of this unit.
- Isometries learnt from S2, unit 9.

### d) Learning activities

- Organize the student-teachers into groups / or give them task individually and ask them to attempt the **Activity 3.6.1** from student-teacher's book;
- Make sure that everybody is engaged/ involved
- Invite groups with different working steps for presentation of their works to the whole class for discussion.
- Harmonize their answers and guide students to establish formula they can use when determining the trigonometric value for equivalent angles.
- Facilitate them to do the provided examples given in **Student-teacher's book** to emphasize the skills;
- Invite them to work on the **application activity 3.6.1** to assess their competences.

**Note**: The process used for the lesson on determining the trigonometric value for equivalent angles can also be applied on the lessons related to Negative angle or Opposite angles, Complementary angles and Supplementary angles where the activities and the applications are 3.6.2, 3.6.3 and 3.6.4.1 respectively.

## Answers for activity 3.6.1

1. $-240^0$ and $840^0$

2. There are many answers. Some of them

a) $-180^0$ and $540^0$ $\left(Generally\ \ 180^0+360^0k, k\in\mathbb{Z}\right)$

b) $219^0$ and $-501^0$ $\left(Generally\ \ -141^0+360^0k, k\in\mathbb{Z}\right)$

c) $-231^0$ and $489^0$ $\left(Generally\ \ 129^0+360^0k, k\in\mathbb{Z}\right)$

d) $-360^0$ and $360^0$ $\left(Generally\ \ 0^0+360^0k, k\in\mathbb{Z}\right)$

3. Trigonometric values of two co-terminal angles are equal.

## Answers for application activity 3.6.1

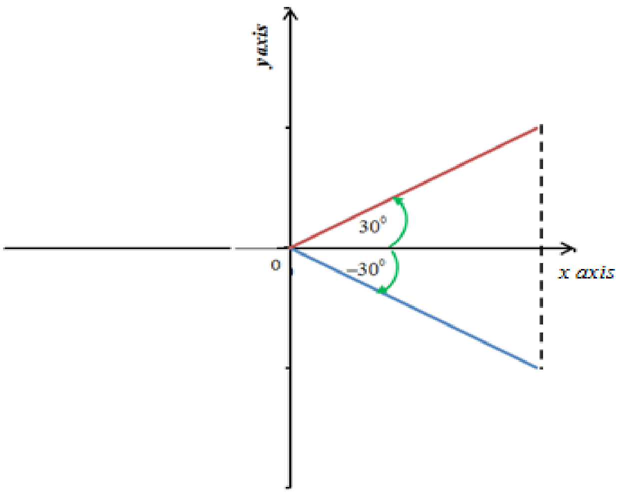
1) A positive angle coterminal with a 55° angle is $415^0+360^0k, k\in\mathbb{Z}$.

A negative angle coterminal with a 55° angle is $-205^0+360^0k, k\in\mathbb{Z}$.

2) A positive angle coterminal with a $\frac{\pi}{3}$ angle is $\frac{7\pi}{3}+2k\pi, k\in\mathbb{Z}$.

A negative angle coterminal with $\frac{\pi}{3}$ angle is $-\frac{5\pi}{3}+2k\pi, k\in\mathbb{Z}$.

## Answers for activity 3.6.2



| | The given angle(in degree) | | New angle(in degree) after reflection | | Observations from trigonometric values of both angles |
|---|---|---|---|---|---|
| | **Figure 1** | **Figure 2** | **Figure 1** | **Figure 2** | |
| sine | $\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | Opposite |
| cosine | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | Equal |
| tangent | 1 | $-\frac{\sqrt{3}}{3}$ | -1 | $\frac{\sqrt{3}}{3}$ | Opposite |

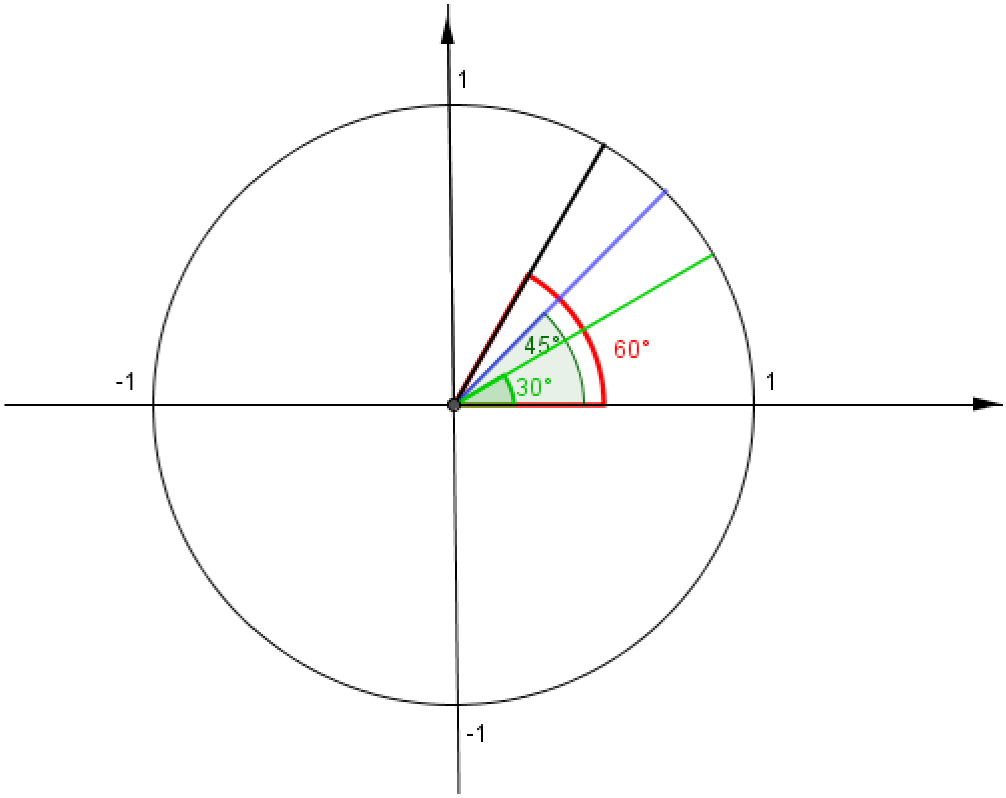
## Answers for application activity 3.6.2

1. If $\sin\theta = -0.1903$, $\sin(-\theta) = 0.1903$
2. If $\cos(\theta) = 0.0133$, $\cos(-\theta) = 0.0133$
3. If $\sec(\theta) = -1.753$, $\cos(-\theta) = 0.570451$
4. If $\csc(-\theta) = \sqrt{3}$, $\sin\theta = \frac{\sqrt{3}}{3}$
5. If $\cos(-\theta) = \frac{1}{7}$, $\sec\theta = 7$
6. If $\cot(\theta) = -5.4219$, $\tan(-\theta) = 0.184437$

## Answers for activity 3.6.3

Examples of pairs angles whose sum is right angle:

$\left(30^0, 60^0\right), \left(45^0, 45^0\right)$ *and* $\left(0^0, 90^0\right)$




After presenting them on circle, we find that
$$\begin{cases} \sin 30^0 = \cos 60^0 = \dfrac{1}{2} \\ \sin 45^0 = \cos 45^0 = \dfrac{\sqrt{2}}{2} \\ \sin 0^0 = \cos 90^0 = 0 \end{cases}$$

Conclusion : If $\theta + \alpha = 90^0 = \dfrac{\pi}{2}$, *then* $\sin\theta = \cos\alpha$ *and* vice versa

## Answers for application activity 3.6.3

1. If $\sin\theta = 0.00213$, $\cos\left(\frac{\pi}{2}-\theta\right) = 0.00213$

2. If $\tan\left(\frac{\pi}{2}-\theta\right) = -0.11221$, $\cot\theta = -0.11221$

3. If $\cot(-\theta) = 1.1482$, $\tan\left(\theta-\frac{\pi}{2}\right) = -1.1482$

4. If $\cos(\theta) = 0.5$, $\csc\left(\theta-\frac{\pi}{2}\right) = -2$

5. If $\sin\left(\frac{\pi}{2}-\theta\right) = \frac{1}{2}$, $\sec\theta = 2$

6. If $\sec\left(\theta-\frac{\pi}{2}\right) = 7$, $\csc\theta = 7$

## Answers for activity 3.6.4

**Figure 1**

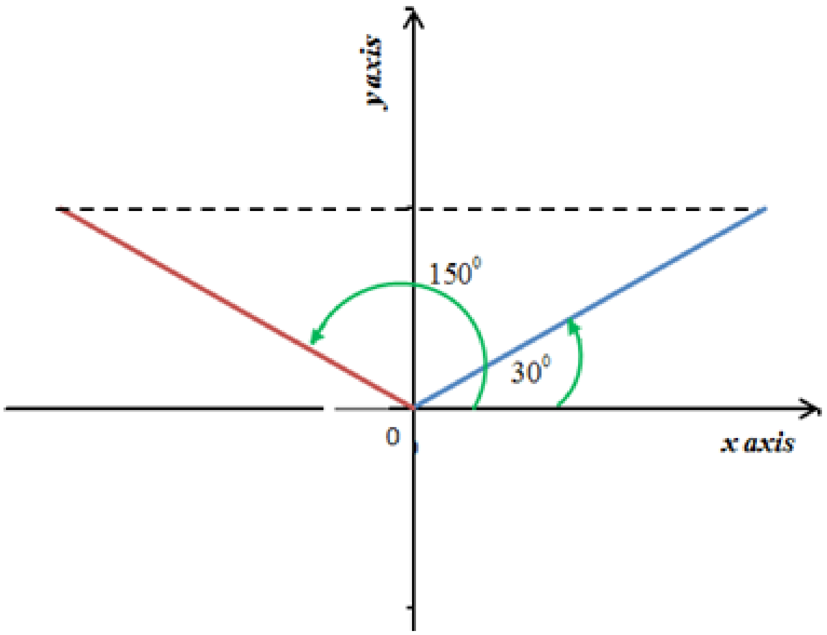
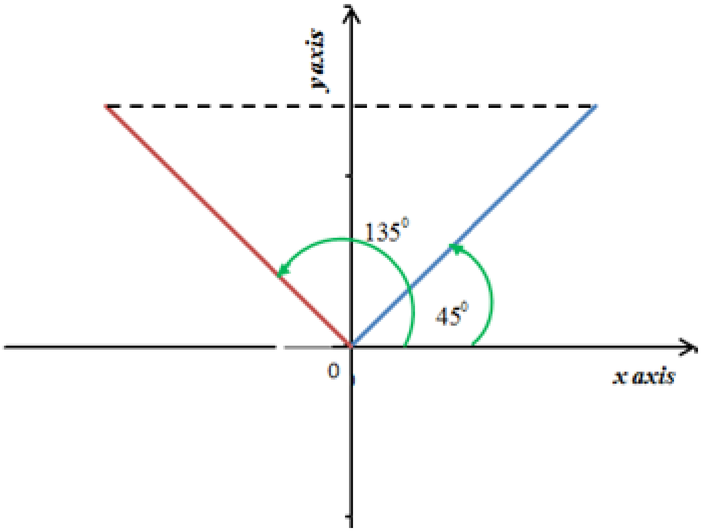


**Figure 2**




| Angle | The given angle(in degree) | | New angle(in degree) after reflection | | Observations from trigonometric values of both angles |
|---|---|---|---|---|---|
| | Figure 1 | Figure 2 | Figure 1 | Figure 2 | |
| sine | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | Equal |
| cosine | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | Opposite |
| tangent | 1 | $\frac{\sqrt{3}}{3}$ | -1 | $-\frac{\sqrt{3}}{3}$ | Opposite |

## Answers for application activity 3.6.4

1. If $\sin\theta = 0.0312$, $\cos(\pi-\theta) = \sin\theta = 0.0312$
2. If $\tan(\pi-\theta) = -0.11221$, $\tan\theta = -0.11221$
3. If $\cot(\theta) = -2.5148$, $\cot(\theta-\pi) = -2.5148$
4. If $\sec(\theta) = 0.5$, $\cos(\theta-\pi) = 2$

5. If $\csc(\pi-\theta)=\frac{1}{2}$, $\csc\theta=\frac{1}{2}$

6. If $\sec(\theta-\pi)=3$, $\sec\theta=3$

## Lesson 7: Transformation formulae

**Note:** Depending on the time available, this lesson can be conducted in 5 different lessons considering how it is organized in the student's book.

### a) Learning objectives:

Calculate the trigonometric value of any angle using the addition and subtraction formulae.

### b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding concept of

- Unit circle learnt in lesson 3 of this unit.
- Isometrics learnt from S2, unit 9.

### d) Learning activities

- Organize the student-teachers into groups / or give them task individually and ask them to attempt the **Activity 3.7.1** from student-teacher's book;
- Turn around different groups, make sure that everybody is engaged/ involved and identify groups with different working steps.
- Invite groups with different working steps for presentation of their works to the whole class for discussion.
- Harmonize their answers and guide students to establish formula they can use when determining the trigonometric value for different types of angles involving transformation formulae: Sum and difference formula.

- Facilitate them to work out examples given in **Student-teacher's book** to emphasize the skills;
- Invite them to work the **application activity 3.7.1**to assess the competences.

**Note**: The process used for the lesson on Sum and difference formula can also be applied on the lessons related to Double angle formulae, Half angle formulae, Product into Sum formulae and Sum into product formulae where the activities and the applications are 3.7.2, 3.7.3, 3.7.4.1 and 3.7.4.2 respectively.

## Answers for activity 3.7.1

1)

a) $\sin\left(\frac{\pi}{2}+\pi\right)=\sin\frac{3\pi}{2}=-1 \qquad \textit{and} \qquad \sin(\pi)+\sin(\frac{\pi}{2})=0+1=1$

b) Both sine are not equal

2) $\sin\left(\pi-\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1 \qquad \textit{and} \qquad \sin(\pi)-\sin(\frac{\pi}{2})=0-1=-1$

3) There are various answers

## Answers for application activity 3.7.1

1. $2\sin\theta\sin 4\theta+2\cos\theta\cos 4\theta=2\left(\sin\theta\sin 4\theta+\cos\theta\cos 4\theta\right)$

$$=2\cos\left(4\theta-\theta\right)=2\cos 3\theta$$

2. a) $\sin 75^0=\sin\left(45^0+30^0\right)=\sin 45^0\cos 30^0+\cos 45^0\sin 30^0$

$$=\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

b) $\cos\frac{13\pi}{6}=\cos\left(2\pi+\frac{\pi}{6}\right)$

$$=\cos 2\pi\cos\frac{\pi}{6}-\sin 2\pi\sin\frac{\pi}{6}$$

$$=\frac{\sqrt{3}}{2}$$

c) $\tan 330^0 = \tan\left(360^0 - 30^0\right)$

$$= \frac{\tan 360^0 - \tan 30^0}{1 + \tan 360^0 \tan 30^0} = \frac{0 - \frac{\sqrt{3}}{3}}{1} = -\frac{\sqrt{3}}{3}$$

3. a) $2+\sqrt{3}$ b) $\frac{\sqrt{6}+\sqrt{2}}{4}$ c) $\frac{\sqrt{3}}{2}$ d) $\frac{1}{2}$

e) $\tan(69^0 + 66^0) = \tan 135^0 = -1$

## Answers for activity 3.7.2

1) $\cos\left(x+x\right) = \cos x \cos x - \sin x \sin x$

$\cos\left(2x\right) = \cos^2 x - \sin^2 x$

2) $\sin\left(x+x\right) = \sin x \cos x + \cos x \sin x$

$\sin\left(x+x\right) = \cos x \cos x - \sin x \sin x$

$\sin\left(2x\right) = 2\sin x \cos x$

3) $\tan\left(x+x\right) = \frac{\tan x + \tan x}{1 - \tan x \tan x}$

$\tan\left(2x\right) = \frac{2\tan x}{1 - \tan^2 x}$

4) $\cot\left(x+x\right) = \frac{\cot x \cot x - 1}{\cot x + \cot x}$

$\cot\left(2x\right) = \frac{\cot^2 x - 1}{2\cot x}$

## Answers for application activity 3.7.2

1. $4\sin x \cos^3 x - 4\cos x \sin^3 x$
2. $\cos^8 x + \sin^8 x - 28\cos^2 x \sin^6 x + 70\cos^4 x \sin^4 x - 28\cos^6 x \sin^2 x$
3. $2\sin 15^0 \cos 15^0 = \sin\left(2\times 15^0\right) = \sin 30^0 = \frac{1}{2}$

4. $\cos 2A = \frac{1}{\sqrt{2}}$

5. a) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$ b) $-\frac{4}{5}, \frac{3}{5}, -\frac{4}{3}$

## Answers for activity 3.7.3

From the double angle formulae, you have

1. $\cos 2x = \cos^2 x - \sin^2 x$

$= \left(1 - \sin^2 x\right) - \sin^2 x$ from $\cos^2 x + \sin^2 x = 1$

$= 1 - 2\sin^2 x$

So $\cos 2x = 1 - 2\sin^2 x$

Letting $\theta = 2x$, $\cos 2x = 1 - 2\sin^2 x$ gives $\cos\theta = 1 - 2\sin^2 \frac{\theta}{2}$

Or $2\sin^2 \frac{\theta}{2} = 1 - \cos\theta \Rightarrow \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ So $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

2. $\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2 x - \left(1 - \cos^2 x\right)$ from $\cos^2 x + \sin^2 x = 1$

$= 2\cos^2 x - 1$

So $\cos 2x = 2\cos^2 x - 1$

Letting $\theta = 2x$, $\cos 2x = 2\cos^2 x - 1$ gives

$\cos\theta = 2\cos^2 \frac{\theta}{2} - 1 \Leftrightarrow 2\cos^2 \frac{\theta}{2} = 1 + \cos\theta$

$\cos^2 \frac{\theta}{2} = \frac{1}{2}\left(1 + \cos\theta\right) \Rightarrow \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

Thus $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

3. $\tan\frac{\theta}{2}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}}$ $\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$

By rationalizing denominator, you get

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}$$

$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{1-\cos^2\theta}}$ $\Leftrightarrow \tan\frac{\theta}{2}=\frac{|1-\cos\theta|}{\sqrt{1-\cos^2\theta}}$ $\Leftrightarrow \tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{\sqrt{1-\cos^2\theta}}$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\pm(1-\cos\theta)}{|\sin\theta|}$$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

So , $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$

From $\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}$, conjugating numerator, you get

$$\tan\frac{\theta}{2}=\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}$$

$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{1-\cos^2\theta}}{\sqrt{(1+\cos\theta)^2}}$ $\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sqrt{\sin^2\theta}}{\sqrt{(1+\cos\theta)^2}}$ $\Leftrightarrow \tan\frac{\theta}{2}=\frac{|\sin\theta|}{|1+\cos\theta|}$

$$\Leftrightarrow \tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$$

So $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$.

Therefore, $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$ or $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$

## Answers for application activity 3.7.3

1. If $\cos A=-\frac{7}{25}$,

$$\sin\frac{1}{2}A=\pm\sqrt{\frac{1-\cos A}{2}}=\pm\sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}}=\pm\sqrt{\frac{\frac{32}{25}}{2}}=\pm\sqrt{\frac{16}{25}}=\pm\frac{4}{5};$$

$$\cos\frac{1}{2}A=\pm\sqrt{\frac{1+\cos A}{2}}=\pm\sqrt{\frac{1-\frac{7}{25}}{2}}=\pm\sqrt{\frac{\frac{18}{25}}{2}}=\pm\sqrt{\frac{9}{25}}=\pm\frac{3}{5};$$

$$\tan A=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}=\pm\sqrt{\frac{1+\frac{7}{25}}{1-\frac{7}{25}}}=\pm\sqrt{\frac{32}{18}}=\pm\frac{4}{3}$$

2. If $\tan 2A=\frac{7}{24}$, $0<A<\frac{\pi}{4}$, to find $\tan A$

$$\tan 2A=\frac{2\tan A}{1-tan^2 A}$$

$$\Rightarrow\frac{7}{24}=\frac{2\tan A}{1-tan^2 A}$$

$$\Rightarrow 7-7tan^2 A=48\tan A$$

$$\Rightarrow 7tan^2 A-48\tan A-7=0$$

$$\Rightarrow(7\tan A-1)(\tan A+7)=0$$

$\Rightarrow \tan A=\frac{1}{7}$ since $\tan A=7$ is impossible for $0<A<\frac{\pi}{4}$

3. $\frac{\sqrt{2-\sqrt{2}}}{2},\frac{\sqrt{2+\sqrt{2}}}{2},1-\frac{\sqrt{2}}{2}$

## Answers for activity 3.7.4.1

1. $\sin(x+y)+\sin(x-y)=\sin x\cos y+\cancel{\cos x\sin y}+\sin x\cos y-\cancel{\cos x\sin y}$

$$=2\sin x\cos y$$

2. $\sin(x+y)-\sin(x-y)=\sin x\cos y+\cos x\sin y-(\sin x\cos y-\cos x\sin y)$

$$=\cancel{\sin x\cos y}+\cos x\sin y-\cancel{\sin x\cos y}+\cos x\sin y$$

$$=2\cos x\sin y$$

3. $\cos(x+y)+\cos(x-y)=\cos x\cos y-\cancel{\sin x\sin y}+\cos x\cos y+\cancel{\sin x\sin y}$

$$=2\cos x\cos y$$

4. $\cos(x+y)-\cos(x-y)=\cos x\cos y-\sin x\sin y-(\cos x\cos y+\sin x\sin y)$

$$=\cancel{\cos x\cos y}-\sin x\sin y-\cancel{\cos x\cos y}-\sin x\sin y$$

$$=-2\sin x\sin y$$

## Answer for application activity 3.7.4.1

1. a) $\sin x\cos 3x=\frac{1}{2}(\sin 4x-\sin 2x)$

b) $\cos 12x\sin 9x=\frac{1}{2}(\sin 21x-\sin 3x)$

c) $-\frac{1}{2}(\cos 20x-\cos 2x)$

d) $\sin 8x-\sin 2x$

e) $\frac{1}{2}(\cos 4x+\cos x)$

## Answers for activity 3.7.4.2

The formulae for transforming product in sum are

$\cos x \cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$ **(Equation 1)**

$\sin x \sin y = -\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$ **(Equation 2)**

$\sin x \cos y = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$ **(Equation 3)**

$\cos x \sin y = \frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$ **(Equation 4)**

$$\begin{cases} x+y=p \\ x-y=q \end{cases} \Rightarrow \begin{cases} x=\frac{p+q}{2} \\ y=\frac{p-q}{2} \end{cases} \qquad \text{(i)}$$

From (i)

Equation (1) becomes $\cos\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\cos p+\cos q)$

Equation (2) becomes $\sin\frac{p+q}{2}\sin\frac{p-q}{2}=-\frac{1}{2}(\cos p-\cos q)$

Equation (3) becomes $\sin\frac{p+q}{2}\cos\frac{p-q}{2}=\frac{1}{2}(\sin p+\sin q)$

Equation (4) becomes $\cos\frac{p+q}{2}\sin\frac{p-q}{2}=\frac{1}{2}(\sin p-\sin q)$

## Answers for application activity 3.7.4.2

1. a) $\cos x+\cos 7x = 2\cos 4x\cos 3x$

   b) $\sin 4x-\sin 9x = -2\cos\frac{13x}{2}\sin\frac{5x}{2}$

   c) $\sin 3x+\sin x = 2\sin 2x\cos x$

   d) $\cos 2x-\cos 4x = 2\sin 3x\sin x$

## Lesson 8: Trigonometric equations

### a) Learning objectives:

Extend the concepts of trigonometric ratios and their properties to trigonometric equations to solve related problems.

### b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding on concept of

- Unit circle learnt in lesson 3 of this unit.
- Solving equations in general
- Trigonometric formulas and identities

### d) Learning activities

- Organize the student-teachers into groups / or give them task individually and ask them to attempt the **Activity 3.8,** from student-teacher's book;
- Make sure that everybody is engaged/ involved.
- Invite groups for presentation of their work to the whole class for discussion.
- Harmonize their answers and guide students on decision to be made when solving such a trigonometric equation;
- Facilitate them to do the examples given in **Student-teacher's book** to emphasize the skills, he/she has got.
- Invite them to work out the **application activity 3.8** to verify their competences.

## Answers for activity 3.8

1. $\begin{cases} \frac{\pi}{6}+2k\pi \\ \frac{5\pi}{6}+2k\pi \end{cases}, \quad k\in\mathbb{Z}$

2. $\pm\frac{\pi}{4}+2k\pi, \quad k\in\mathbb{Z}$

3. $\frac{\pi}{6}+k\pi, \ k\in\mathbb{Z}$

## Answers for application activity 3.8

1. $\pm\frac{\pi}{12}+\frac{k\pi}{4}, \quad k\in\mathbb{Z}$

2. $\left\{0,\frac{\pi}{14},\frac{\pi}{3}\right\}$

3. $\left\{30^0,150^0,199.5^0,340.5^0\right\}$

4. $\left\{170.7^0,350.7^0\right\}$

5. $\frac{k\pi}{2} \ or \frac{\pi}{2}(2k\pi+1), \ k\in\mathbb{Z}$

6. $(2k+1)\frac{\pi}{2}, (2k+1)\frac{\pi}{8}$

7. $\frac{k\pi}{3}, \pm\frac{\pi}{12}+\frac{k\pi}{2}, \ k\in\mathbb{Z}$

## Lesson 9: Application of trigonometry in solving triangle related problems

### a) Learning objectives:

- Use trigonometry to solve problems involving triangles
- Use trigonometry to solve problems involving bearings
- Use trigonometry to solve problems involving air navigation, inclined plane, etc.

### b) Teaching resources

Student's book and other Reference textbooks to facilitate research, Mathematical set, calculator, T-square, ruler, manila paper, markers, pens, pencils, MathType and GeoGebra softwares, strings, scissors or laser blades, pins, a pair of compass, internet.

### c) Prerequisites / Revision / Introduction:

Student-teachers will learn better this lesson if they have a good understanding concept of

- Solving equation in general
- Trigonometric formulas and identities

### d) Learning activities

- Organize the student-teachers into groups / or give them task individually and ask them to attempt the **Activity 3.9.1.1, 3.9.1.2, 3.9.2** from student-teacher's book;
- Make sure that everybody is engaged/ involved and identify groups with different working steps.
- Invite groups with different working steps for presentation of their works to the whole class for discussion.
- Harmonize their answers and guide students to decide on how they can solve triangle related problems using trigonometry: sine law, cosine law and their application in rela life situations;
- Facilitate them to do the provide examples given in **Student-teacher's book** to emphasize the skills;
- Invite them to work the **application activity 3.9.1.1, 3.9.1.2, 3.9.2** to verify their competences.

## Answers for activity 3.9.1.1

1. $\sin B = \frac{h}{a}$, $\sin A = \frac{h}{b}$ .

$h = a\sin B$ and $b\sin A = h$, then $a\sin B = b\sin A$ or $\frac{a}{\sin A} = \frac{b}{\sin B}$

2. $\sin A = \frac{k}{c}$, $\sin C = \frac{k}{a}$ .

$k = c\sin A$ and $k = a\sin C$, then $c\sin A = a\sin C$ or $\frac{c}{\sin C} = \frac{a}{\sin A}$

3. Now, $\frac{a}{\sin A} = \frac{b}{\sin B}$ and $\frac{c}{\sin C} = \frac{a}{\sin A}$. This gives $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

## Answers for application activity 3.9.1.1

The best approach to use while solving this kind of equation you first draw / sketch the figure representing the situation and apply the tangent functions and or sine laws.

1. The height of the hill is 763.94 m.
2. The length of the field is 2460.36 m.
3. Height of the tree is 34.64 m and the breadth of the river is 20 m.
4. 308.11 km

## Answers for activity 3.9.1.2

1. $\cos A = \frac{AX}{b}$

2. $b^2 = h^2 + (AX)^2 \Rightarrow h^2 = b^2 - (AX)^2$

3. $a^2 = h^2 + (XB)^2 \Rightarrow h^2 = a^2 - (XB)^2$

4. $h^2 = b^2 - (AX)^2$ and $h^2 = a^2 - (XB)^2$ gives $b^2 - (AX)^2 = a^2 - (XB)^2$

But $XB = c - AX$, then

$$b^2 - (AX)^2 = a^2 - (c - AX)^2 \Leftrightarrow b^2 - (AX)^2 = a^2 - \left(c^2 - 2cAX + (AX)^2\right)$$
$$\Leftrightarrow b^2 - (AX)^2 = a^2 - c^2 + 2cAX - (AX)^2$$
$$\Leftrightarrow b^2 + c^2 - 2cAX = a^2$$

But $\cos A = \frac{AX}{b} \Rightarrow AX = b\cos A$.

Then, $b^2 + c^2 - 2cb\cos A = a^2 \quad \Leftrightarrow a^2 = b^2 + c^2 - 2bc\cos A$

## Answers for application activity 3.9.1.1

1. $9.43\,cm$
2. $c = 21.7\,cm$
3. 130.42 m
4. 0.6km

## Answers for activity 3.9.2

Strip B width $\simeq 3.244\,m$, Strip C width $\simeq 2.168\,m$

## Answers for activity 3.9.2

1. The tree was *8.91 m* tall.
2. The distance is *369.15 m.*
3. The distance is *45.7 m*
4. The bearing of B from A is S54°10'W and the distance of B from A is *22.2 km*.
5. *20 m*
6. *24.3m*

## 3.6 Summary of the Unit

1. A **rotation angle** is formed by rotating an **initial side** through an angle, about a fixed point called **vertex**, to terminal position called **terminal side.** Angle is positive if rotated in a counter clockwise direction and negative when rotated clockwise.

2. The amount we rotate the angle is called the measure of the angle and is measured in: degree or radian.

3. To convert degree measure to radian measure, multiply by $\frac{\pi \, radian}{180^0}$.

4. To convert radian measure to degree measure, multiply by $\frac{180^0}{\pi \, radian}$.

5. In a triangle whose hypotenuse *r*, the adjacent side $x$ and the opposite side $y$:

$$\sin\alpha = \frac{opposite\, side}{hypotenuse} = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\alpha = \frac{adjacent\, side}{hypotenuse} = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\tan\alpha = \frac{opposite\, side}{adjacent\, side} = \frac{y}{x} \qquad \cot\alpha = \frac{adjacent\, side}{opposite\, side} = \frac{x}{y}$$

$$\csc\alpha = \frac{hypotenuse}{opposite} = \frac{1}{\sin\alpha}, s\,ec\alpha = \frac{hypotenuse}{adjascent} = \frac{1}{\cos\alpha} \text{ and } \cot\alpha = \frac{adjacent}{opposite} = \frac{1}{\tan\alpha}$$

6. The table trigonometric number of remarkable angles

| $\alpha$ | $0^\circ$ | $30^\circ$ | $45^\circ$ | $60^\circ$ | $90^\circ$ | $180^\circ$ | $270^\circ$ | $360^\circ$ |
|---|---|---|---|---|---|---|---|---|
| $\sin\alpha$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | $-1$ | 0 |
| $\cos\alpha$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-1$ | 0 | 1 |
| $\tan\alpha$ | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | Does not exist | 0 | Does not exist | 0 |
| $\cot\alpha$ | Does not exist | $\sqrt{3}$ | 1 | $\frac{\sqrt{3}}{3}$ | 0 | Does not exist | 0 | Does not exist |

Trigonometric identities

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

7. Two angles are **equivalent** if their difference is $2k\pi,\ k \in \mathbb{Z}$ (or $360^\circ k,\ k \in \mathbb{Z}$ ). This means that the angle $\alpha$ and $\alpha + 2k\pi$ are equivalent angles.

$$\left.\begin{aligned}\sin\left(\alpha + 360^\circ k\right) &= \sin\alpha \\ \cos\left(\alpha + 360^\circ k\right) &= \cos\alpha \\ \tan\left(\alpha + 360^\circ k\right) &= \tan\alpha \\ \cot\left(\alpha + 360^\circ k\right) &= \cot\alpha\end{aligned}\right\} k \in \mathbb{Z}$$

8. Angle $-\alpha$ is **opposite** of the angle $\alpha$

$\sin(-\alpha) = -\sin\alpha$

$\cos(-\alpha) = \cos\alpha$

$\tan(-\alpha) = -\tan\alpha$

$\cot(-\alpha) = -\cot\alpha$

9. Two angles are said to be **complementary** if their sum is $90^\circ$ (or $\frac{\pi}{2}$). Note that $\alpha$ and $90^\circ - \alpha$ are complementary.

$\sin\left(90^\circ - \alpha\right) = \cos\alpha$

$\cos\left(90^\circ - \alpha\right) = \sin\alpha$

$\tan\left(90^\circ - \alpha\right) = \cot\alpha$

$\cot\left(90^\circ - \alpha\right) = \tan\alpha$

10. Two angles are said to be **supplementary** if their sum is $180^\circ$ (or $\pi$). It is easy to discover that $\alpha$ and $180^\circ - \alpha$ are supplementary.

$\sin(180^0-\alpha)=\sin\alpha$

$\cos(180^0-\alpha)=-\cos\alpha$

$\tan(180^0-\alpha)=-\tan\alpha$

$\cot(180^0-\alpha)=-\cot\alpha$

11. If $a,b,c$ are the lengths of the sides of a triangle and $A,B,C$ are the opposite angles respectively, then cosine law says that

$$\begin{cases} a^2=b^2+c^2-2bc\cos\widehat{A} \\ b^2=a^2+c^2-2ac\cos\widehat{B} \\ c^2=a^2+b^2-2ab\cos\widehat{C} \end{cases}$$

12. If $a,b,c$ are the lengths of the sides of a triangle and $A,B,C$ are the opposite angles respectively, then the sine law is

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

13. Applications

Many real situations involve right triangle. Using angles and trigonometric functions, we can solve problems involving right triangle like:

- Bearings and air navigation
- Angles of elevation and angle of depression
- Inclined plane

## 3.7. Additional information for the tutor

**Notice on reduction to a positive angle (reference angle)**

The values of the circular functions of an angle, if they exist, are the same, up to a sign, of the corresponding circular functions of its reference angle. More specifically, if α is the reference angle for θ, then:

| | | |
|---|---|---|
| $\sin\theta=\pm\sin\alpha$ | $\cos\theta=\pm\cos\alpha$ | $\tan\theta=\pm\tan\alpha$ |
| $\csc\theta=\pm\csc\alpha$ | $\sec\theta=\pm\sec\alpha$ | $\cot\theta=\pm\cot\alpha$ |

The choice of the (±) depends on the quadrant in which the terminal side of θ lies.

For example, $150^0$ is a II quadrant angle, the reference angle is $30^0$.

We find out that

$\sin 150^0 = \sin 30^0$ as the sine of any II quadrant angle is positive

$\cos 150^0 = -\cos 30^0$ as the cosine of any II quadrant angle is negative.

$\tan 150^0 = -\tan 30^0$ as the tangent of any II quadrant angle is negative.

## Notice on identities

- **The Pythagorean Identities:**

1. $\cos^2\theta + \sin^2\theta = 1$ Fundamental formula

Alternative Forms:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 - \cos^2\theta = \sin^2\theta$

$1 + \tan^2\theta = \sec^2\theta$, provided $\cos\theta \neq 0$.

Alternative Forms:

| | |
|---|---|
| • $\sec^2\theta - \tan^2\theta = 1$ | • $\sec^2\theta - 1 = \tan^2\theta$ |

$1 + \cot^2\theta = \csc^2\theta$, provided $\sin\theta \neq 0$.

Alternative Forms:

| | |
|---|---|
| • $\csc^2\theta - \cot^2\theta = 1$ | • $\csc^2\theta - \cot^2\theta = 1$ |

- **Pythagorean conjugates** are useful in proving trigonometric identities:

  - $(1 + \cos\theta)(1 - \cos\theta) = 1 - \cos^2\theta = \sin^2\theta$
  - $(1 + \sin\theta)(1 - \sin\theta) = 1 - \sin^2\theta = \cos^2\theta$
  - $(\sec\theta + 1)(\sec\theta - 1) = \sec^2\theta - 1 = \tan^2\theta$
  - $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$

- $(\csc\theta+1)(\csc\theta-1)=\csc^2\theta-1=\cot^2\theta$
- $(\csc\theta+\cot\theta)(\csc\theta-\cot\theta)=\csc^2\theta-\cot^2\theta=1$

## 3.8 End unit assessment

1. a. $\frac{\sin a(1+\cos a)}{(1-\cos a)(1+\cos a)}=\frac{\sin a(1+\cos a)}{\sin^2 a}$

$$=\frac{1+\cos a}{\sin a}$$

b. $\frac{1}{\cos^2 a}+\frac{1}{\sin^2 a}=\frac{\sin^2 a+\cos^2 a}{\cos^2 a\sin^2 a}$

$$=\frac{1}{\cos^2 a}\frac{1}{\sin^2 a}$$
$$=\sec^2 a\csc^2 a$$

c. $(\sec^2 a+\tan^2 a)(\sec^2 a-\tan^2 a)=(\sec^2 a+\tan^2 a)\left(\frac{1}{\cos^2 a}-\frac{\sin^2 a}{\cos^2 a}\right)$

$$=(\sec^2 a+\tan^2 a)\left(\frac{1-\sin^2 a}{\cos^2 a}\right)$$
$$=(\sec^2 a+\tan^2 a)\left(\frac{\cos^2 a}{\cos^2 a}\right)$$
$$=\sec^2 a+\tan^2 a$$

d. $\sqrt{\frac{(1-\cos a)(1-\cos a)}{(1+\cos a)(1-\cos a)}}=\sqrt{\frac{(1-\cos a)^2}{\sin^2 a}}=\frac{1-\cos a}{\sin a}$

2. $\tan\theta=3.18$

3. $\cos\theta=-0.8$; $\tan\theta=-0.75$

4. The height of the cliff is 107.96 m.

5. a) 400 are $140^0$ or $400^0$ i.e. its supplementary or equivalent angles.

b) $240^0$ or $660^0$ i.e. its supplementary or equivalent angles.

6. 11. 342 m

7. 205.26m

## 3.9 Additional activities

### *3.9.1. Remedial activities*

1. In the following Exercises, convert the angle from degree measure into radian measure, giving the exact value in terms of π.

a) $30^0$ b) $240^0$ c) $135^0$ d) $-270^0$ e) $-315^0$ f) $150^0$ g) $45^0$ h) $225^0$.

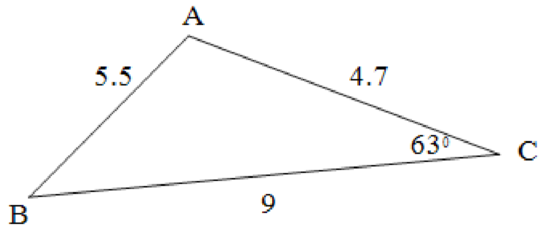
2. In the following Exercises, convert the angle from radian measure into degree measure.

a) $\frac{2\pi}{3}$ b) $-\frac{7\pi}{6}$ c) $\frac{11\pi}{6}$ d) $\frac{\pi}{3}$ e) $-\frac{5\pi}{3}$ f) $\frac{\pi}{6}$ g) $-\frac{\pi}{2}$.

3. Find the cosine and sine of the following angles.

a) $\theta = 270°$ b) $\theta = -\pi$ c) $\theta = 45°$ d) $\theta = \frac{\pi}{6}$ **e) $\theta = 60°$**

4. Using sine rule , find out the angle B



**Solution:**

1.

| Degrees | $30^0$ | $240^0$ | $135^0$ | $-270^0$ | $-315^0$ | $150^0$ | $45^0$ | $225^0$ |
|---|---|---|---|---|---|---|---|---|
| radians | $\frac{\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{3\pi}{4}$ | $-\frac{3\pi}{2}$ | $-\frac{7\pi}{4}$ | $\frac{5\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{5\pi}{4}$ |

| Radians | $\frac{2\pi}{3}$ | $-\frac{7\pi}{6}$ | $\frac{11\pi}{6}$ | $\frac{\pi}{3}$ | $-\frac{5\pi}{3}$ | $\frac{\pi}{6}$ | $-\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| Degrees | 120 | –210 | 330 | 60 | –300 | 30 | –90 |

3)

| Angle | $\theta = 270°$ | $\theta = -\pi$ | $\theta = 45°$ | $\theta = \frac{\pi}{6}$ | $\theta = 60°$ |
|---|---|---|---|---|---|
| Cosine | 0 | –1 | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ |
| Sine | –1 | 0 | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ |

4) $49.6^0$

### *3.9.2. Consolidation activities*

1. Using the given information about θ, find the indicated value

a) If θ lies in 2nd quadrant with $\sin\theta = \frac{3}{5}$, find $\cos\theta$.

b) If $\pi < \theta < \frac{3\pi}{2}$, with $\cos\theta = -\frac{\sqrt{5}}{5}$, find $\sin\theta$.

c) If $\sin\theta = 1$, find $\cos\theta$.

2. Before calculators became common classroom tools, they used trigonometric tables to find trigonometric ratios. Below is a simplified trigonometric table for angles between 40° and 56°. Without using a calculator, can you determine which column gives sine values, which gives cosine values, and which gives tangent values?

| Degrees | ? | ? | ? |
|---|---|---|---|
| 40 | 0.83909963 | 0.64278761 | 0.76604444 |
| 42 | 0.90040404 | 0.66913061 | 0.74314483 |
| 44 | 0.96568877 | 0.69465837 | 0.71933980 |
| 46 | 1.03553031 | 0.71933980 | 0.69465837 |
| 48 | 1.11061251 | 0.74314483 | 0.66913061 |
| 50 | 1.19175359 | 0.76604444 | 0.64278761 |
| 52 | 1.27994163 | 0.78801075 | 0.61566148 |
| 54 | 1.37638192 | 0.80901699 | 0.58778525 |
| 56 | 1.48256097 | 0.82903757 | 0.55919290 |

3. In order to determine the height of a tree in Nyungwe forest, two sightings from the ground, one 200m directly behind the other, are made. If the angles of inclination were $45^{000}$ and $30^0$, respectively, how tall is the tree to the nearest metre?

4. In an airport control tower A, 2 planes B and C are located at the same altitude on a radar screen. The range finder determines one plane to bear $N60^0E$ at 100km while the other bears $S50^0E$ at 150km. How far apart are the planes from each other?

5. Multiple choice

i). Which of the following trigonometric ratios could not be $\pi$?

(A) $\tan\theta$ (B) $\sec\theta$ (C) $\cot\theta$ (D) $\cos\theta$ (E) $\csc\theta$

ii). If a nonhorizontal line has slope $\sin\theta$, it will be perpendicular to a line with slope

(A) $\cos\theta$ (B) $-\cos\theta$ (C) $\csc\theta$ (D) $-\csc\theta$ (E) $-\sin\theta$

iii). If $\theta$ is the smallest angle in a $3-4-5$ right triangle, then $\sin\theta =$

(A) $\frac{4}{5}$ (B) $\frac{3}{5}$ (C) $\frac{5}{4}$ (D) $\frac{3}{4}$ (E) $\frac{5}{3}$

iv). Which of the following expressions does not represent a real number?

(A) sin 30° (B) tan 45° (C) cos 90° (D) csc 90° (E) sec 90°00ooooooo00

v). A central angle in a circle of radius r has a measure of $\theta$ radians. If the same central angle were drawn in a circle of radius 2r, its radian measure would be

(A) $\frac{\theta}{2}$ (B) $\frac{\theta}{2r}$ (C) $2\theta$ (D) $\theta$ (E) $2r\theta$

vi). If the perimeter of a sector is 4 times its radius, then the radian measure of the central angle of the sector is

(A) 2 (B) 4 (C) $\frac{4}{\pi}$ (D) $\frac{2}{\pi}$ (E) impossible to determine without knowing the radius.

vii) What is the radian measure of an angle of $x$ degrees?

(A) $\frac{x}{180}$ (B) $\pi x$ (C) $\frac{\pi x}{180}$ (D) $\frac{180}{x\pi}$ (E) $\frac{180x}{\pi}$

**Solution:**

1. a) From the Pythagorean identity, we get $\cos\theta = \pm\frac{4}{5}$, since θ is a Quadrant II angle, thus $\cos\theta = -\frac{4}{5}$.

b). From the Pythagorean identity, we get $\sin\theta = \pm\frac{2\sqrt{5}}{5}$, since we are given that $\pi < \theta < \frac{3\pi}{2}$, we note that θ is a Quadrant III angle, thus $\sin\theta = -\frac{2\sqrt{5}}{5}$.

c). When we substitute $\sin\theta = 1$ into $\cos^2\theta + \sin^2\theta = 1$, we get $\cos\theta = 0$.

2.

| Degrees | tangent | Sine | cosine |
|---|---|---|---|

3. The tree is approximately 273 m tall.

4. 157km

5. i) D ii) E iii) B iv) E v) D vi) A vii) C

### *3.9.3. Extended activities*

1. A 100-degree arc of a circle has a length of 7 cm. To the nearest centimeter, what is the radius of the circle?

2. When I stand 30 m away from a tree at home, the angle of elevation to the top of the tree is $50^\circ$ and the angle of depression to the base of the tree is $10^\circ$. What is the height of the tree? Round your answer to the nearest metre.

3. From the observation deck of the lighthouse at Rubavu Point 50 m above the surface of Lake Kivu, a lifeguard spots a boat out on the lake sailing directly toward the lighthouse. The first sighting had an angle of depression of $8.2^o$ and the second sighting had an angle of depression of $25.9^0$. How far had the boat traveled between the sightings?

4. The broadcast tower for radio station has two enormous flashing red lights on it: one at the very top and one a few metres below the top. From a point 5000 m away from the base of the tower on level ground the angle of elevation to the top light is $7.970^0$ and to the second light is $7.1250^0$. Find the distance between the lights to the nearest metres.

5. InatriangleABC,determinetheangleAforwhich $(a+b+c)(b+c-a)=3bc$

6. David is a cross-country skier and skis 10km in a direction $N40^0E$ of the ski lodge. At this point she turns and skis $S10^0E$ for 4km and arrives at a chalet. How far is David from the lodge?

**Answers:**

1. 4 cm

2. The tree is about 41m tall.

3. The boat has travelled about 244 m.

4. The lights are about 75 m apart.

5. $A=60^0$

6. 8 km

# UNIT 4

# POLYNOMIAL FUNCTIONS

## 4.1. Key Unit competence

Use concepts and definitions of functions to determine the domain of rational functions, represent them graphically in simple cases, and solve related problems.

## 4.2 Prerequisite

Student-teachers will perform better in this unit if:

- They have a good background on "linear functions" ,from unit6 of ordinary level, senior 3
- They have mastered "linear functions" studied in unit2 of Year 1
- They can convert easily a verbal problem into a mathematical statement

## 4.3 Cross-cutting issues to be addressed

- **Financial education**: Use examples of the application of functions in economics
- **Inclusive education** :Promote the participation of all student-teachers while teaching numerical functions by a proper distribution of activities;
- **Peace and value Education** :Ensure that the members of a group respect others' view and thoughts during group discussions, in the activities about polynomial, rational and irrational functions;
- **Gender** :When applicable, give equal opportunity for boys and girls to participate in a lesson about polynomial, rational and irrational functions ,ensure that the examples are chosen in such a way that neither boys nor girls are inconvenienced;

## 4.4 Guidance on introductory activity

This intends to help student-teachers to think about the general idea of the whole unit: definition, classification (polynomial, rational and irrational functions), operations (composite, inverse, etc), qualities (such as being odd, even, etc.), some specific sets (such as domain, range, etc.).

You can proceed as follows:

- Give clear instructions for students to form small groups and to work on the introductory activity;
- As they are discussing, circulate around to note the relevancy of the discussion and to provide guidance where necessary;
- Ensure that the learners have understood what the unit will be about and they are eager to learn; you can observe this through a clear and concise presentation of a group chosen randomly and the degree of attention other students are paying to the presentation;
- Sustain the curiosity of the learners by a proper management of your class;
- A prior knowledge of software such as geogebra and malmath would be very useful for you and for your students;

### Answers to "Introductory activity 4.0"

**a) i.** The use (what something is made for);

**ii.** Social event (a ceremony);

**iii.** A quantity whose value depends on the value of another quantity.

**b)** i; ii; iii

**c) . i.-** Independent: $x$;

-Dependent: $y$;

**ii.** - Independent: $r$;

-Dependent: $A$

**iii.** -Independent: $A$;

-dependent: $s$

d. Similarity: all of them are equalities

Difference: i. the denominator contains the independent variable: **rational function**

ii. the independent variable is neither in the denominator,

nor in the radicand: **polynomial function**

iii. the independent variable is in the radicand: **irrational function**

**e.** Irrational; rational; irrational; polynomial; rational

**f. i.** Domain: $Domf = ]-\infty;1[\cup]1;+\infty[$

Range: $\text{Im} f = ]-\infty;0[\cup]0;+\infty[$ : The function is equivalent to $y = \frac{4}{x-1}$ ;the only value that *y* cannot assume is *0;*

**ii.** Domain: $Domf = ]-\infty;+\infty[$ **:** as any expression, *r* can assume any value;

Range: $\text{Im} f = [0;+\infty[$ : the expression $A = \pi r^2 \geq 0$, for any value of *r*

**Note:** If we consider $A = \pi r^2$ as the formula for the area of a circle with radius *r,* then the domain is: $\text{Dom} f = ]0;+\infty[$, since, practically, the radius of a circle cannot be negative or zero. In this case, the range is $\text{Im} f = ]0;+\infty[$

**g.i.** For *f:* $\{0\}$ **:** only *0* is not in the domain**;** for *g:* $]-\infty;0[$ : all strictly negative real numbers are not in the domain; for *h*: $\Phi$ :no real number is not in the domain.

**ii.** For *f:* $\{0\}$ **:** only *0* is not in the range**;** for *g:* $]-\infty;0[$ : all strictly negative real numbers are not in the range; for *h*: $]-\infty;0[$ :all strictly negative numbers are not in the range.

## 4.5. List of lessons/sub-heading

| No | Lesson title | Learning objectives | Number of periods |
|---|---|---|---|
| 0 | Introductory activities | • Awaken the curiosity of student teachers on the content of unit 4 | **1** |
| 1 | Generalities on numerical functions and their operations | • Differentiate the types of functions<br>• Determine the sum, difference and product, quotient, composite and inverse of functions | **1** |
| 2 | Injective, surjective and bijective functions | • Determine whether a given function is Injective and/or surjective and bijective functions<br>• Find out any function which is Injective, surjective and bijective functions. | **1** |
| 3 | Existence condition for a given function | • Determine the intervals for which a given function is defined. | **1** |
| 4 | Domain and range of polynomials | • Determine the Domain and range of polynomial functions | **1** |
| 5 | Domain and range of rational function | • Determine the domain and range of simple rational functions<br>• Interpret graphs to find the domain and range of a function | **4** |
| 6 | Domain and range of irrational function | • Determine domain and range of simple irrational functions. | **3** |
| 7 | Composite functions | • Find the composite of functions and provide some real life examples of composition of functions.<br>• Define the composition of functions.<br>• Demonstrate understanding of composition of functions. | **2** |

| | | | |
|---|---|---|---|
| 8 | Inverse of a function | To determine whether a given function is invertible<br><br>To find the inverse of a function. | 1 |
| 9 | Even function and symmetry of a function | Analyze whether a function is even algebraically and from the graph | 2 |
| 10 | Odd function and symmetry | Analyze whether a function is odd algebraically and from the graph | 1 |
| 11 | Graph of linear and quadratic functions. | Interpret graphs of functions (linear and quadratic) related to practical context and make conclusions. | 2 |
| 12 | Application of functions in real life situation | Identify and explain different applications of functions in real life | 2 |
| | | Interpret graphs of functions (linear and quadratic) observed from practical context or journals and make conclusions. | 2 |
| | | Analyze, Apply, model, solve problems involving linear or quadratic functions, and interpret the results. | 2 |
| End unit assessment | | | 1 |
| Total number of periods | | | 27 |

## Lesson 1: **Types of numerical functions**

### a) Learning objective

Differentiate the types of functions

### b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

### c) Prerequisites/Revision/Introduction

Student-teachers will learn better this lesson if they have mastered:

- Numerical functions studied in ordinary level and in year 1;
- Operations on algebraic expressions studied in ordinary level and in year 1;
- Calculation of numerical values with or without a calculator.

### d) Learning activities

- Invite student-teachers to work in group and do the activity 4.1.1 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from a group to present his/her findings; then the teacher and the students will discuss the questions and answers. During the discussion, the teacher will monitor the participation of each member of the class to further comprehension in this lesson.
- As a tutor, harmonize the findings from presentation and guide them to explain why they take such type of function.
- Use different probing questions , guide them to explore the content and examples given in the student’s book ,lead them to be able to differentiate different types of functions: polynomial, rational and irrational functions.
- Throughout this time, the teacher will circulate in the room to ensure students understanding and to answer any questions they may have.
- The teacher will evaluate and determine whether more instruction is needed.
- Invite student-teachers to work individually the application activities 4.1.**1** for improving their skills

## Answer for activity 4.1.1

| Polynomial | Rational | Irrational |
|---|---|---|
| $f(x)=(x+1)^2$ | $h(x)=\dfrac{x^3+2x+1}{x-4}$ | $f(x)=\sqrt{x^2+x-2}$ |

## Answer of application activity 4.1.1

| Polynomial | Rational | Irrational |
|---|---|---|
| $f(x)=x^3+2x^2-2$ | $g(x)=\dfrac{x^3+2x^2-2}{x-5}$ | $h(x)=\sqrt{x^3+2x^2-2}$ |

## Lesson 2: Injective, surjective and bijective functions

### a) Learning objectives:

- Determine whether a given function is Injective function.
- Find out any function which is injective.

### b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

### c) Prerequisites

In this lesson, Student-teachers will perform better if they revise the content on relations and functions learnt in S2 and S3 including linear and quadratic functions learnt in TTC year1.

### d) Learning activities

- Invite student-teachers to discuss in small groups the activity 4.1.2
- Walk around each group and ask probing questions leading them to come up with idea about existence of a function.
- In the plenary class, invite representatives from some groups to present their findings;
- As a tutor, harmonize their answers and guide students to discover

that the function may or may not exist at a given value ( $x_0$ ).

- After this step, invite students to do the application activity 4.1.2 and evaluate whether lesson objectives are achieved.

## Answers for activity 4.1.2

**a. i.** No, there is no real number missing image;

**ii.** Yes f is a mapping

**iii.** Yes, for example, $f(2)=f(-2)=4$; *4* is image of more than one real number.

**iv.** f is not one-to-one, since some elements are sharing images

**v.** Yes, for example -*3* is not image; there is no real number *x* such that $x^2=-3$

**vi.** f is not onto.

**b.iii.** Yes, for example, $f(2)=f(-2)=4$; *4* is image of more than one real number.

**iv.** f is not one-to-one, since some elements are sharing images

**v.** No, all elements of $\mathbb{R}^+$ are images under *f.*

**vi.** f is onto.

**c. iii.** No, all elements of $\mathbb{R}^+$ are images of either one or zero element of $\mathbb{R}^+$ under function *f.*

**iv.** f is one-to-one, since no elements are sharing images

**v.** No, all elements of $\mathbb{R}^+$ are images under *f.*

**vi.** f is onto.

**d.** $A=[2;+\infty[$ and $A=]-\infty;4]$; or $A=]-\infty;2]$ and $A=]-\infty;4]$

## Answers to the application activity 4.1.2

1. $T$ is not a function since there is an element (2) which has two images. For a function one input cannot have many outputs.

$U$ is a function and $V$ is also a function.

2. i) $V$ is not injective since two elements from the domain are sharing image.

ii) $V$ is surjective (onto) since there no elements from B which is is not image.

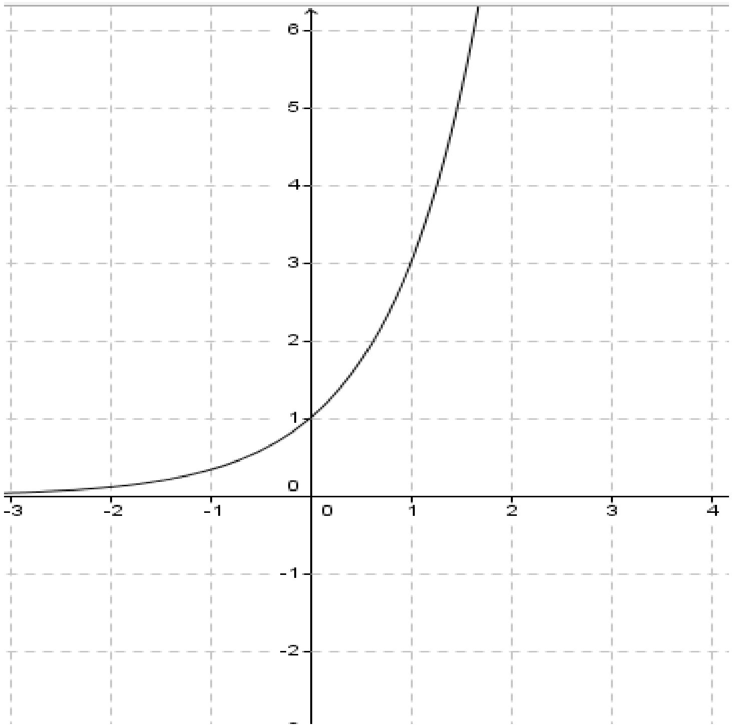
iii) No, A relation which is not one to one cannot have inverse. Only injective functions have inverses.

3. a) The relation from $\mathbb{N} \to \mathbb{N}$ defined by $x \to x+3$ is injective but not surjective. There is no element whose image is 1.

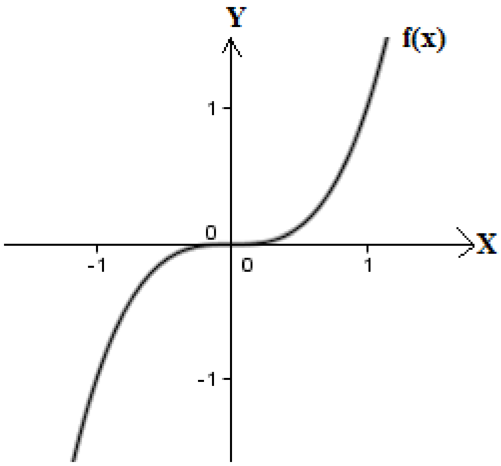
b) The relation from $\mathbb{N} \to \mathbb{N}$ defined by $x \to |x-5|$ is not injective but it is surjective since there are elements from the domain which are sharing the same image. Example is the images of 3 and 7 which are given by $=|3-5|=2$ and $|7-5|=2$ .It is surjective since any natural number is image under the function

4. The function defined from $x \to 3^x$ is injective but not surjective.

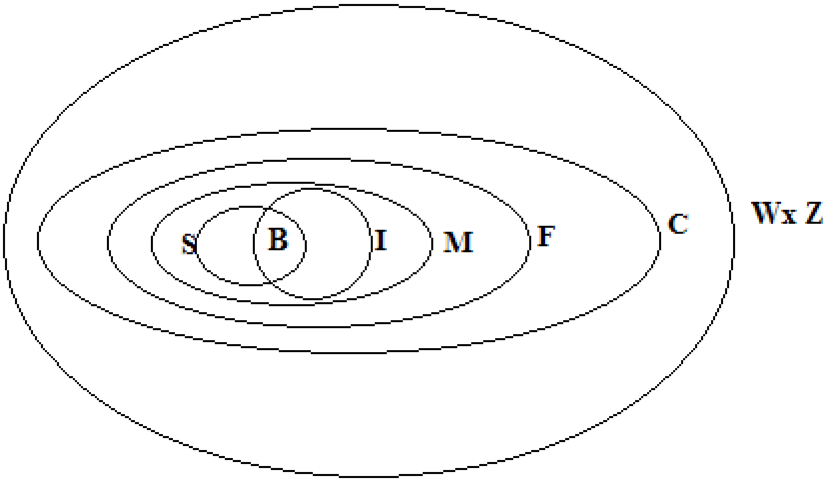
Reasons: all images are positive which means that there are negative elements from $\mathbb{R}$ which are not images



a) $g(x) = x^3$ is both injective and surjective. Hence Bijective ,see the diagram below:




5. Relationship between mappings, functions and correspondences




The following inclusion is true: $S \subset M \subset F \subset C \subset (W \times Z)$.

A bijection is a type of mapping which is at the same time surjective (S) and injective (I) mapping.

All mappings are functions *f(x)* for which every point *x* has an image.

All functions are types of correspondences for which you cannot observe a point which has more than one image.

**Very important note:**

Even though you can find that in the student's book the composition of functions and the inverse of a function come before the domain of definition of a function, it is better to start by teaching the domain of definition before teaching those concepts.

## Lesson 3: Existence condition for a given function

### a) Learning objective:

Determine the intervals for which a given function exists.

### b) Teaching resources: Manila papers, calculators.

### c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they have background knowledge on:

- Sets of numbers
- Solving equations and inequalities (unit 7) learnt in ordinary level,
- Propositional and predicate logic learnt in TTC year 1
- Calculation of numerical values with ,or without calculators

### d) Learning activities

- Invite students to discuss in small groups the activity 4.1.3
- Walk around each group and ask probing questions leading them to come up with idea about existence of a function.
- In the plenary class invite representatives from groups with different findings to present for a whole class , their results;
- As a tutor, harmonize their answers and guide students to discover that the function may or may not exist at a given x-value $x_0$.
- After this step, guide students to do the application activity 4.1.3 and evaluate whether lesson objectives were achieved.

## Answers to the activity 4.1.3

a) $f(x)=\frac{1}{x}$ at $x=0$ ,you would be dividing by 0. So $x\neq 0$

The numerical value of $f(x)$ does not exist when $x=0$.

b) $f(x)=\frac{2+x}{x-3}$ at $x=3$, you would be dividing by 0. So $x\neq 3$

$f(1)=\frac{-3}{2}$ , $f(2)=\frac{4}{-1}=-4$ , $f(3)$, the numerical value of $f(x)$ does not exist.

c) you would be dividing by 0. Therefore, $x\neq 1$

$f(x)=\frac{2(x-1)}{x-1}$ at $x=1$ $f(0)=2$, $f(1)$ does not exist.

d) $f(x)=\frac{x+1}{x^2-1}$ at $x=-1, x=1$, you would be dividing by 0. Therefore, $x\neq 1$ and $x\neq -1$ the function can be simplified to $f(x)=\frac{1}{x-1}$, $f(-1)$ and $f(1)$ do not exist , $f(2)=1$

e) For the function $f(x)=\frac{2(x-1)}{x^2+1}$ there is no restriction on the independent variable, even though there is a variable in the denominator. Since $x^2\geq 0, x^2+1$ can never be 0. The least it can be is 1.

$$f(-1)=\frac{2(-1-1)}{(-1)^2+1}=\frac{-4}{2}=-2$$

$f(-1)=-2$ ,f(1)=0 and f(2)=$\frac{2}{5}$

## Answers for the application activity 4.1.3

i) $x-4x^3\neq 0 \Leftrightarrow x(1-2x)(1+2x)\neq 0$

$$\Leftrightarrow x\neq 0; x\neq \frac{1}{2}; x\neq -\frac{1}{2}$$

ii) $4 - x^2 > 0 \Leftrightarrow (2 - x)(2 + x) > 0$

$$\Leftrightarrow -2 < x < 2$$

iii) $x \neq 0$

iv) No restrictions

## Lesson 4: Domain and range of polynomial functions

### a) Learning objective

Determine the domain and range of polynomial functions

### b) Teaching resources

Student -teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software such as Geogebra should be used etc.

### c) Prerequisites/Revision/Introduction

For learners to feel comfortable in this lesson, the following prerequisites are required:

- Polynomial functions, studied in lesson1 of this unit;
- Restrictions on an independent variable, and on the depended variable, studied in lesson3 of this unit;
- Calculation of numerical values, using a calculator or not

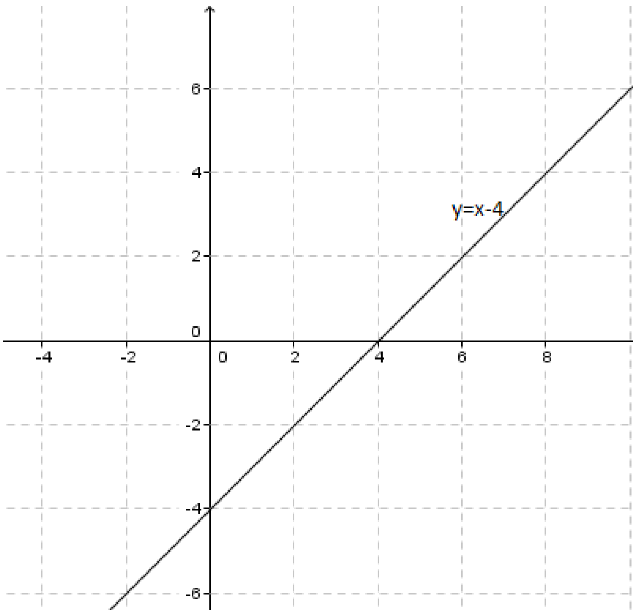
### d) Learning activities

- Invite student-teachers to work in pairs and do the activity 4.2.1 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Request two pairs to exchange their works and to discuss the set of values obtained;
- Verify and identify groups with different working steps;
- Invite one member to present the work of his/her group.
- As a tutor, harmonize the findings from presentation and guide them to explain why they took such values.

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to determine the domain and range for specified functions: Constant, linear, quadratic, Polynomial functions.
- After this step, guide students to do the application activity 4.2.1 and evaluate whether lesson objectives are achieved or not for eventual improvement for the following lessons.

### Answer for Activity 4.2.1

1. For any question you may graph the functions using geogebra (if possible) and projector.

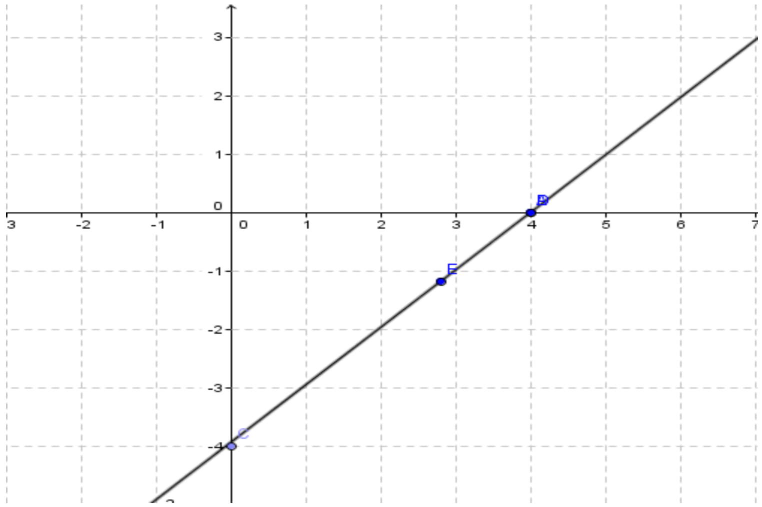



One end of the graph goes nonstop to negative infinity and another goes nonstop to plus infinity.

### Alternative way of plotting

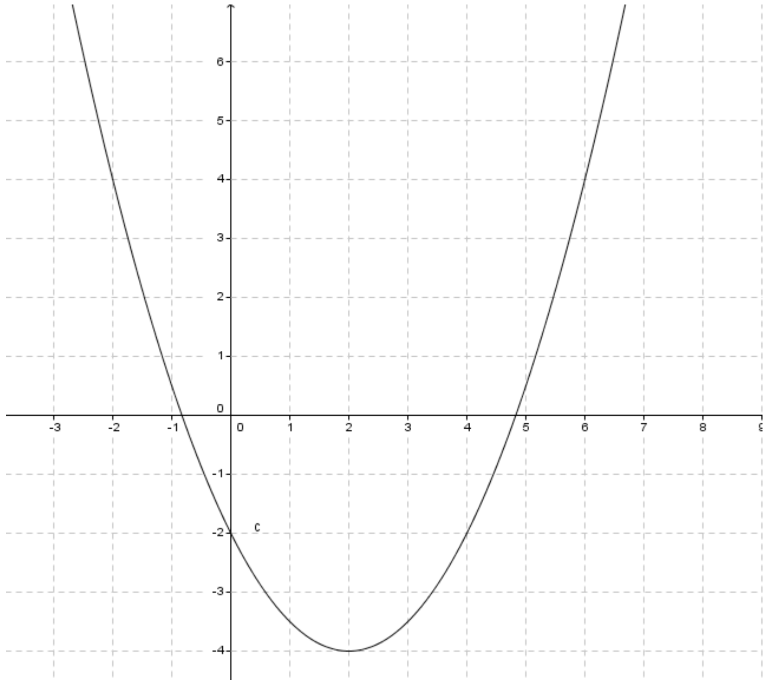
Two points are enough to plot a straight line.

| | | |
|---|---|---|
| $x$ | 0 | -4 |
| $y$ | 5 | 0 |




For each value of $x$ we can get the value of $y$. Observing the curve, it goes from negative infinity to plus infinity on both axes.

2. For any question of graphing geogebra or malmath may be used .

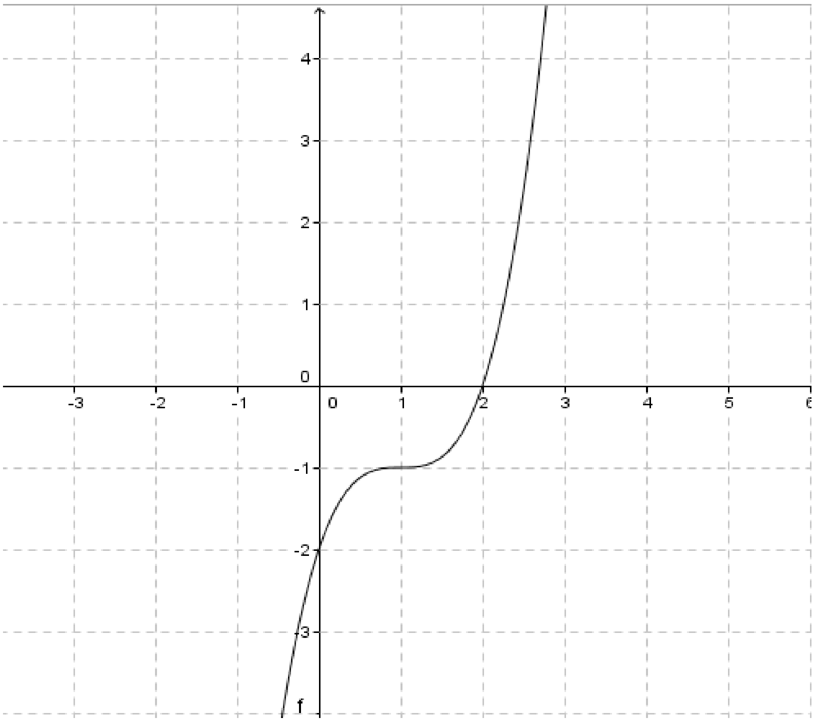



Using this graph, you can easily fill the table for values of $x$ such as

*x=-2;x=-1;x=0; x=1;x=2;x=3;x=4;x=5;...*

All x-values can get the outputs. But observing on the graph, the y-values lie in from -4 up to infinity.

3. Again using geogebra




Analysis of the graph:

Using this graph, you can easily fill the table for values of *x* such as *x=0; x=1;x=2;...*

All x-values are taken into consideration. The same as for y-values.

4. Any polynomial of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, where $a_n$ , $a_{n-1}, \ldots, a_2, a_1, a_0$ are real coefficients and $n = 0, 1, 2, 3, 4, \ldots$ is defined in the set of all real numbers.

## Answers for application activity 4.2.1

**a)** Domain: $Domf = ]-\infty; +\infty[$

Range: $\operatorname{Im} f = ]-\infty; +\infty[$

**b)** Domain: $Domf = ]-\infty; +\infty[$

Range: $\operatorname{Im} f = ]-\infty; +\infty[$

**c)** Domain: $Domf = ]-\infty;+\infty[$

Range: $\operatorname{Im} f = ]-\infty;+\infty[$

## Lesson 5: Domain and range of rational function

### a) Learning objectives

- Determine the Domain and range rational functions
- Interpret graphs to find the domain and range of rational functions

### b) Teaching resources

Student -teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software and math applications such as Geogebra, malmath, photomath should be used etc.

### c) Prerequisites/Revision/Introduction

For learners to feel comfortable in this lesson, the following prerequisites are necessary:

- Solving equations studied in ordinary level and in year one;
- Polynomial functions, studied in lesson1 of this unit;
- Restrictions on an independent variable, and on the depended variable, studied in lesson3 of this unit;
- Calculation of numerical values, using a calculator or not

### d) Learning activities

- Invite student-teachers to work in small groups the activity 4.2.2 found in their Mathematics books;
- Give them time to explore and workout the activity and move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work.
- As a tutor, harmonize the findings from presentation and guide them to explain why they took such values.
- Use different probing questions and guide student-teachers to explore

the content and examples given in the student's book and lead them to determine the domain of rational functions

**Note:** that the range will be determined only for elementary functions.

- After this step, guide students to do the application activity **4.2.2** and evaluate whether lesson objectives were achieved.

## Answers learning activity 4.2.2

a) For $f(x)=\frac{1}{x}$ and if $x=0$, we would be dividing by zero; therefore, x=0 is not in the domain

b) The function $f(x)=\frac{x}{(x-1)(x+3)}$ is not defined for $x=1$ and $x=-3$

## Answers for the application activity 4.2.2

1) a) Condition of existence is that $x+2\geq 0 \Leftrightarrow x\geq -2$,

and $x^2-9\neq 0 \Leftrightarrow x\neq -3; x\neq 3$ then the domain is $domf=[-2,+\infty[-\{3\}$

b) Condition of existence is that $6x^2-x-2\neq 0 \Rightarrow domf=\mathbb{R}\setminus\left\{\frac{2}{3},-\frac{1}{2}\right\}$

c) Condition of existence is that $25x^2-4\neq 0 \Rightarrow domf=\mathbb{R}\setminus\left\{-\frac{2}{5},\frac{2}{5}\right\}$.

d) Condition of existence is that $x^3+2x^2-8x\neq 0 \Rightarrow x\neq 0, x\neq -2$ or $x\neq 4$ therefore $domf=\mathbb{R}\setminus\{-2,0,4\}$.

e) Condition of existence is that $x-2\neq 0 \Rightarrow x\neq 2$ therefore $domf=\mathbb{R}\setminus\{2\}$.

2) Observing the graph we can see that there is a jump on $x=-1$ and $x=1$, therefore this function is not defined on these values. Therefore,

$domf=\{x\in\mathbb{R}\setminus\{-1,1\}\}$ and range is $\{y\in\,]-\infty,-1]\cup[1,+\infty[\}$

**Note:** To find the range, just read the interval in which the curve is lying on y-axis. If the graph is not given graph it using different methods ( geogebra or Photomath), hence read the intervals in which it is lying.

## Lesson 6: Domain and range of irrational function

### a) Learning objectives

- Determine the Domain and range irrational functions
- Interpret graphs to find the domain and range of irrational functions

### b) Teaching resources

Student -teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software and applications such as Geogebra, malmath, photomathshould be used etc.

### c) Prerequisites/Revision/Introduction

For learners to feel comfortable in this lesson, the following prerequisites are required:

- Solving equations studied in ordinary level and in year one;
- Solving inequalities, studied in ordinary level and inn year one;
- Polynomial functions, studied in lesson1 of this unit;
- Restrictions on an independent variable, and on the depended variable, studied in lesson3 of this unit;
- Calculation of numerical values, using a calculator or not

### d) Learning activities

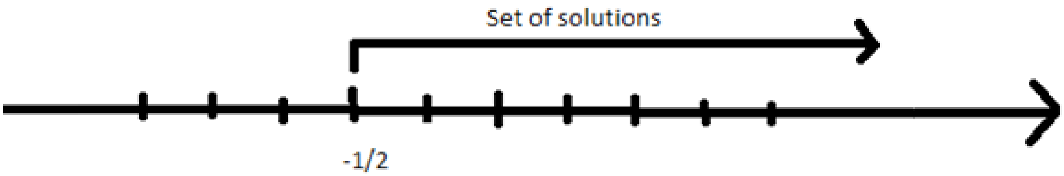
- Invite student-teachers to work in small groups the activity 4.2.3 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from a group to present the work.
- As a tutor, harmonize the findings from presentation and guide them to explain why they took such values for which they think may cause the function undefined.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to determine the domain and range of irrational functions.

**Note:** that the range will be determined only for elementary functions basically using drawn graphs.

- After this step, guide student-teachers to do the application activity 4.2.3 and evaluate whether lesson objectives were achieved.

## Answers to activity 4.2.3

For this activity, by simple guided questions, the tutor will remind students that the square root of a negative number does not exist in $\mathbb{R}$. From this information, the condition of existence will be set up. For the function $f(x)=\sqrt{2x+1}$, the condition of existence is that $2x+1\geq 0 \Rightarrow x\geq -\frac{1}{2}$. This is shown on a number line as follow :




The set of solutions is written as interval as follow: $\left[\frac{-1}{2},+\infty\right[$

Therefore, the function is not defined for $x\in ]-\infty;-\frac{1}{2}[$

For this activity, by simple guided questions, the tutor will remind students that a radical with an odd index is the same as a polynomial.

Therefore the function $f(x)=\sqrt[3]{x^2+x-2}$ is defined for every value in the set of real numbers; the function is not defined for the empty set.

The function is $g(x)=\sqrt{\frac{x-2}{x+1}}$ is not defined for the values of x such that

$\frac{x-2}{x+1}<0$ or $x$ = -1

To identify this interval, one can solve $\frac{x-2}{x+1}<0$ and $x=-1$ or use the table of signs.

| $x$ | $-\infty$ -1 2 $+\infty$ |
|---|---|
| $x-2$ | - - - - - - 0 + + + |
| $x+1$ | - - - 0 + + + + + + + + + + + + |
| $\frac{x-2}{x+1}$ | + + + + undefined - - 0 + + + + + + |

If $x\in]-1;2[$ the function is undefined.

**Note:** For $\frac{x-2}{x+1}\le 0$, ensure that the students avoid the errors or wrong calculations consisting of cross-multiplying when solving an inequality. The use of a table is better to provide the all desired interval.

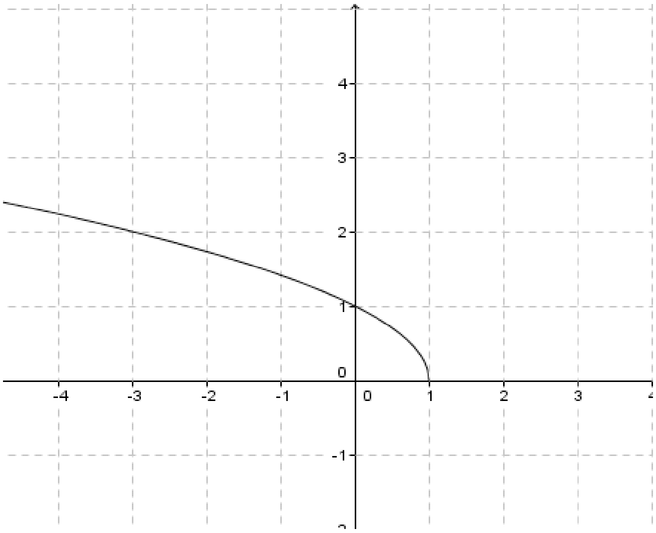
## Answers for the application activity 4.2.3

1.

| **Function** | **Domain of definition.** |
|---|---|
| $f(x)=\sqrt{4x-8}$ | $[2,+\infty[$ |
| $g(x)=\sqrt{x^2+5x-6}$ | $]-\infty,-6]\cup[1,+\infty[$ |
| $h(x)=\frac{x^3+2x^2-2}{\sqrt[3]{x+4}}$ | $]-\infty;-4[\cup]-4;+\infty[$ |
| $f(x)=\frac{x-2}{\sqrt[4]{x^2-25}}$ | $]-\infty,-5[\cup]5,+\infty[$ |
| $f(x)=\sqrt{\frac{(x-1)^2}{x+4}}$ | $]-4,+\infty[$ |
| $h(x)=\sqrt{\frac{(x-1)(x+3)}{8-2x}}$ | $]-\infty;-3]\cup[1;4[$ |

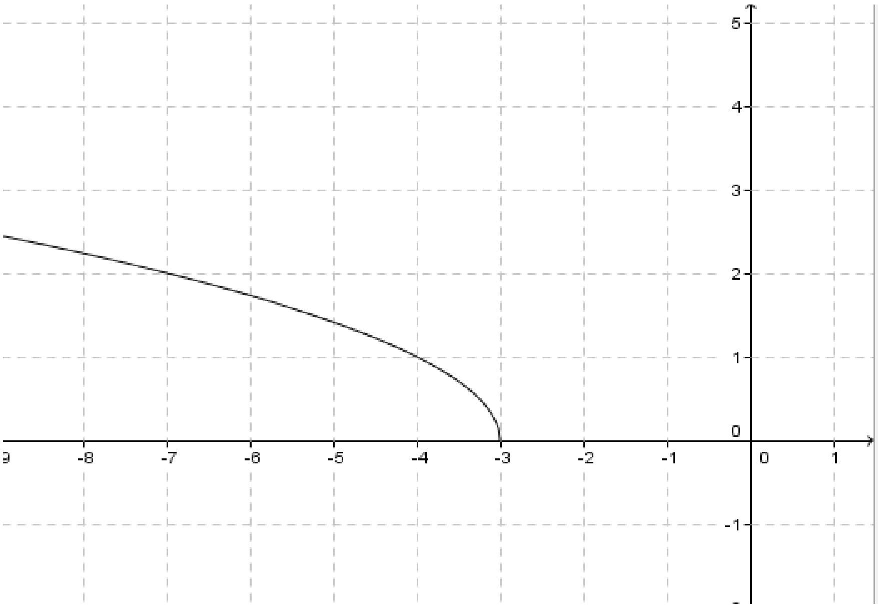
| | |
|---|---|
| $f(x)=\dfrac{x-1}{\sqrt{2-x}}$ | $]-\infty,2[$ |
| $f(x)=\sqrt{4-x^2}$ | $[-2,2]$ |

a) Graph of the function $f(x)=\sqrt{1-x}$




The domain of $f(x)=\sqrt{1-x}$ is $Domf=]-\infty;1]$ and the range is $\text{Imf}=[0;+\infty[$

b) Graph of the function $f(x)=\sqrt{-x-3}$



The domain is $Domf = ]-\infty;-3]$ and the range is $\text{Imf} = [0;+\infty[$ range

## Lesson 7: Composition of Functions

### a) Learning objectives:

- Define the composition of functions.
- Find the composite of functions
- Demonstrate understanding of composition of functions.

### b) Teaching resources:

Online resources, books, manila papers,

### c) Prerequisites

Student-teachers will understand better this lesson if they have mastered:

- Composition of functions studied in ordinary level;
- Operations on polynomials studied in year one,unit4;
- Polynomial functions studied in year one, unit11.

### d) Learning activities

- Invite student-teachers to work in groups and do the activity 4.3 found in their Mathematics books;
- Move around for facilitating students where necessary and ask some challenging questions to lead them to find correct answer for the activity;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance concept of composition of functions.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the content developed

- Let students ask any remaining question to help them for better understanding.
- Invite students to work on the application activity 4.3 and evaluate whether lesson objectives were achieved.

### Answer for activity 4.3

a) $18x^2+30x+7$, yes the function is in terms of x.

b) $6x^2+6x-13$, the function obtained is new function.

c) The functions $f[g(x)]$ and $g[f(x)]$ are different.

### Answers to the Application activity 4.3

a. $f[g(x)]=x^2+2x+2$ b. $g[f(x)]=x^2+2$ c. $fog(1)=5$

d. $gof(1)=3$ e. $fog(x^2)=x^4+2x^2+2$ f. $gof(\sqrt{x})=|x|+2$

## Lesson 8: inverse function

### a) Learning objectives:

- Determine whether a given function is invertible
- Find the inverse of a function.

### b) Teaching resources:

Digital materials including calculator, sticks, manila papers, markers, Mathematics book , etc

### c) Prerequisites:

The student-teachers will feel comfortable in this lesson if they have mastered:

- Inverse function, studied in ordinary level;
- Bijective functions studied in lesson 2 of this unit;
- Making a letter the subject of a formula, studied in ordinary level;

## d) Learning activities.

- Invite student-teachers to work in groups and do the activity **4.4** found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance concept of inverse of functions.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the content developed
- Let students any remaining question to help them for better understanding.
- Invite students to work on the application activity **4.4** and evaluate whether lesson objectives were achieved.

### Answers to the Activity 4.4

a) $y = 4x + 6 \Rightarrow x = \dfrac{y-6}{4}$

b) $f^{-1}(x) = \dfrac{x-6}{4}$

c) $f\left[f^{-1}(x)\right] = x$ and $f^{-1}\left[f(x)\right] = x$. Therefore , $f^{-1}\left[f(x)\right] = f\left[f^{-1}(x)\right]$

### Answers for the application activity 4.4

1. a) $x = \dfrac{14.75 - p}{0.01p}$

b) $x = \dfrac{14.75 - p}{0.01p} \Rightarrow x = \dfrac{14.75 - 10}{0.01(10)} = 47.5$ units.

2. a) let us find $f\left[g(x)\right]=5\left(\frac{x-1}{5}\right)+1=x$ and $g\left[f(x)\right]=\frac{(5x+1)-1}{5}=x$

Hence, the two functions are inverse to each other since the condition is verified.

b) We need to show that $f\left[g(x)\right]=g\left[f(x)\right]=x$.

$f\left[g(x)\right]=9-\left(\sqrt{9-x}\right)^2=x$ And $g\left[f(x)\right]=\sqrt{9-\left(9-x^2\right)}=x$ ,provided $9-x\geq 0$

Hence the two functions are inverse to each other.

c) We are supposed to show that $f\left[g(x)\right]=g\left[f(x)\right]=x$

$f\left[g(x)\right]=1-\left(\sqrt[3]{1-x}\right)^3=x$ and $g\left[f(x)\right]=\sqrt[3]{1-\left(1-x^3\right)}=x$, then the two functions are inverse to each other.

## Lesson 9: Even function and symmetry

### a) Learning objective

Analyze whether a function is even, algebraically and from the graph

### b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, etc.

### c) Prerequisites/Revision/Introduction

Student-teachers will participate fully in this lesson, if:

- they have mastered reflection and symmetry, studied in ordinary level;
- powers of negative numbers , studied in ordinary level;
- calculation of numerical values;
- Plotting points and drawing figures in the Cartesian plane;

### d) Learning activities

- Invite student-teachers to work in groups and do the activity **4.5.1** found in their Mathematics books;
- Move around in the class for facilitating students where necessary and

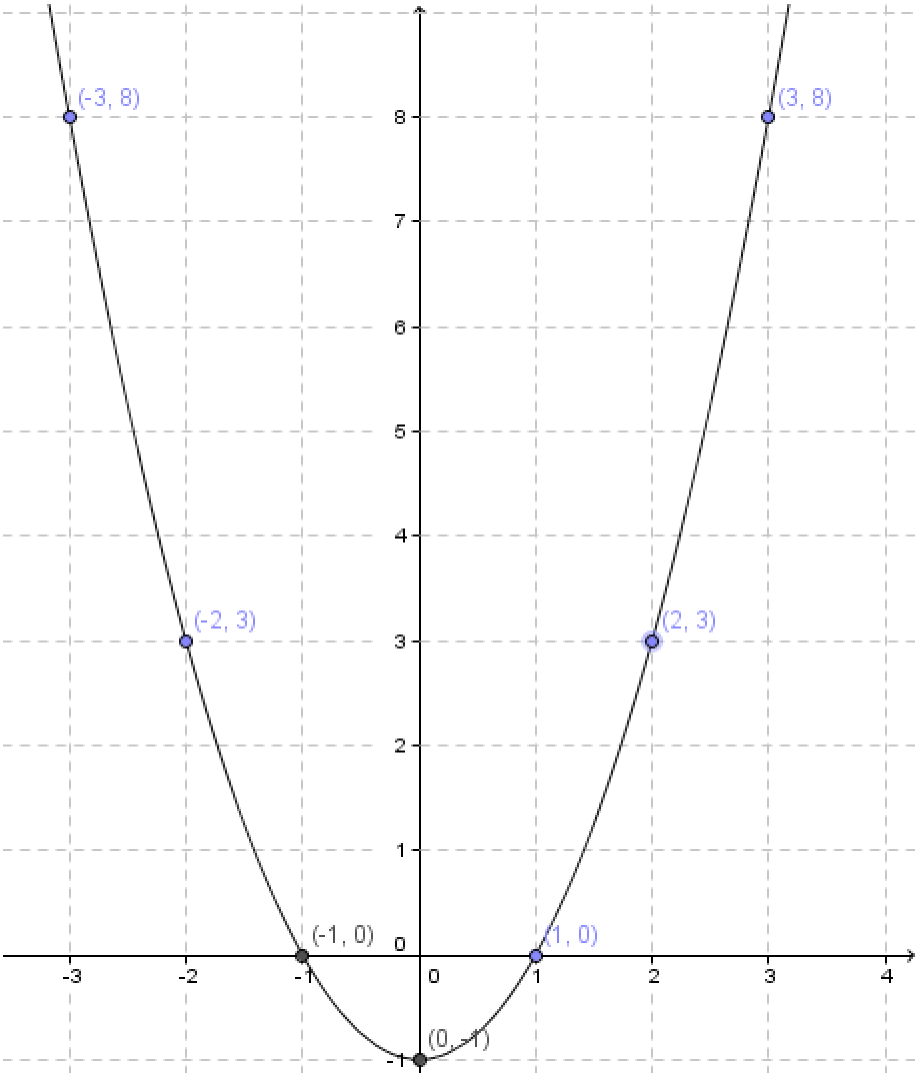
give more clarification on eventual challenges they may face during their work;

- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance the characteristics of even functions and odd functions.
- Use graphs for simple functions to illustrate the characteristics of even functions: The graph of even function is symmetric about the vertical axis (the line $x = 0$ is the axis of symmetry).
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to verify the parity of different functions.
- After this step, guide students to do the application activity 4.5.1 and evaluate whether lesson objectives were achieved.

## Answers for the activity 4.5.1

1.

| $x$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|
| $y = x^2 - 1$ | 8 | 3 | 0 | -1 | 0 | 3 | 8 | 15 | 24 |




2. The negative and positive values have the same images examples: image of -1 and 1 is 0.The line $x = 0$ divides the graph into two equal parts which are symmetric to each other.

### Answers to the application activity 4.5.1

1) The function $f(x)=\frac{x^2+1}{x^4+3}$ will be even if $f(x)=f(-x)$

But $f(-x)=\frac{(-x)^2+1}{(-x)^4+3}=\frac{x^2+1}{x^4+3}$ ,Hence, the function is even, since $f(x)=f(-x)$

2) Not even because $f(x)\neq f(-x)$, this means

$$f(-x)=\sqrt[3]{x^2}(-x-4)=-\sqrt[3]{x^2}(x+4)$$

3) Not even because $f(x)\neq f(-x)$ , this means $f(-x)=-x\sqrt{9+x}$

## Lesson 10: Odd function and symmetry

### a) Learning objective

Analyze whether a function is odd algebraically and from the graph

### b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, etc.

### c) Prerequisites/Revision/Introduction

Student-teachers will participate fully in this lesson, if:

- they have mastered symmetry, studied in ordinary level;
- powers of negative numbers , studied in ordinary level;
- calculation of numerical values;
- Plotting points and drawing figures in the Cartesian plane;

### d) Learning activities

- Invite student-teachers to work in groups and do the activity 4.5.2.found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;

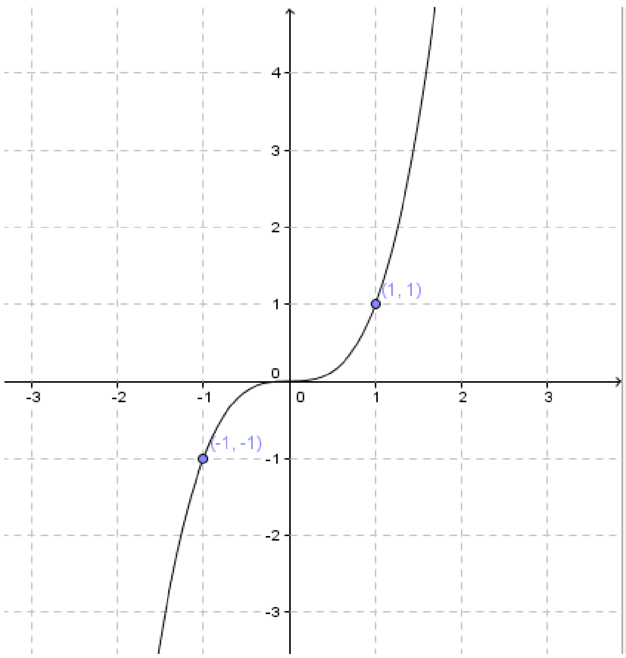
- Invite groups to present their work and as a tutor, harmonize the findings from presentation and guide students to enhance the characteristics of even functions and odd functions.
- Use graphs for simple functions to illustrate the characteristics of odd functions: the graph of odd function looks the same when rotated through half a revolution about the origin point $(0,0)$ is the centre of symmetry for its part.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead students to verify the parity of different functions.
- After this step, guide students to do the application activity 4.5.2.and evaluate whether lesson objectives were achieved.

## Answers to the activity 4.5.2.

For this activity, use a calculator to find f(x) or the geogebra to plot graph

| $x$ | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y=x^3$ | -125 | -64 | -27 | -8 | -1 | 0 | 1 | 8 | 27 | 64 | 125 |

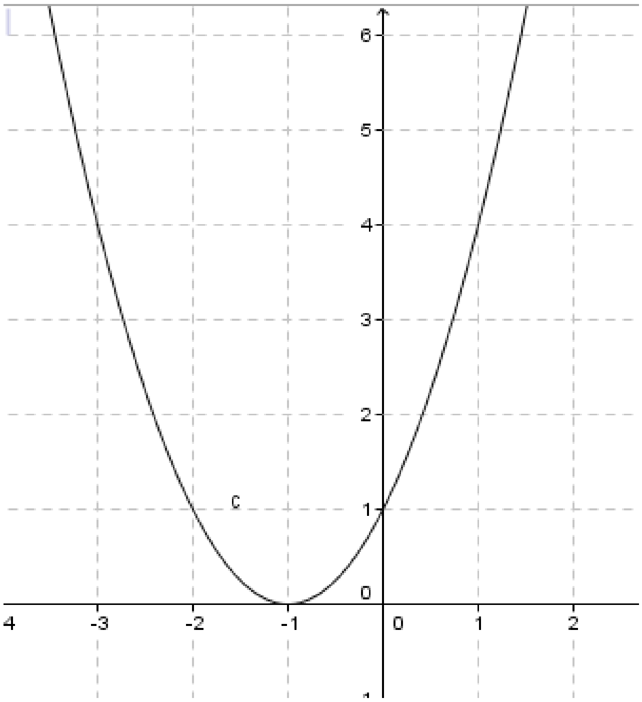
**Graph of the function** $f(x)=x^3$



The images of negative and positive x-values are symmetric to each other compare to the point $(0,0)$.

The point $(0,0)$ is the center of symmetry of the graph. Therefore, the function is odd.

**Graph of the function** $f(x)=x^2+2x+1$




The graph is not symmetric about the origin (0,0) nor symmetric about the y-axis. Therefore, this function is neither even nor odd.

## Answers to the application activity 4.5.2.

1) Condition of existence for $f(x)=\frac{x^3}{9-x^2}$ is $9-x^2\neq 0$ therefore,

$domf=]-\infty,-3[\cup]-3,3[\cup]3,+\infty[$

For this question the students-teacher will verify whether,

$f(x)=f(-x)$ or whether $f(-x)=-f(x)$, the conclusion is that the function is odd because $f(-x)=-f(x)$.

2) The function $f(x)=\sqrt{x^2+5x+6}$ is valid if $x^2+5x+6\geq 0$. the student-teacher can determine the exact interval in which this function is positive. can determine `The students will verify whether $f(x)=f(-x)$ or whether $f(-x)=-f(x)$ the conclusion is that the function is neither even nor odd.

## Lesson 11: Graph of linear and quadratic functions

### a) Learning objective

Interpret graphs of functions (linear and quadratic) related to practical context and make conclusions.

### b) Teaching resources

Student -teacher's book and other Reference Textbooks to facilitate learning process, Mathematical set, Digital materials including calculator, sticks, manila papers. Mathematics book Calculator, Markers, Pens and Pencils, Online resources

**Note:** Where it is possible, the tutor helps student-teachers to plot graphs using appropriate software such as Geogebra etc.

### c) Prerequisites/Revision/Introduction

In this lesson, Student-teachers must be skilled in **Unit 3** of **S1** and **Unit 6** of **S3.**

### d) Learning activities

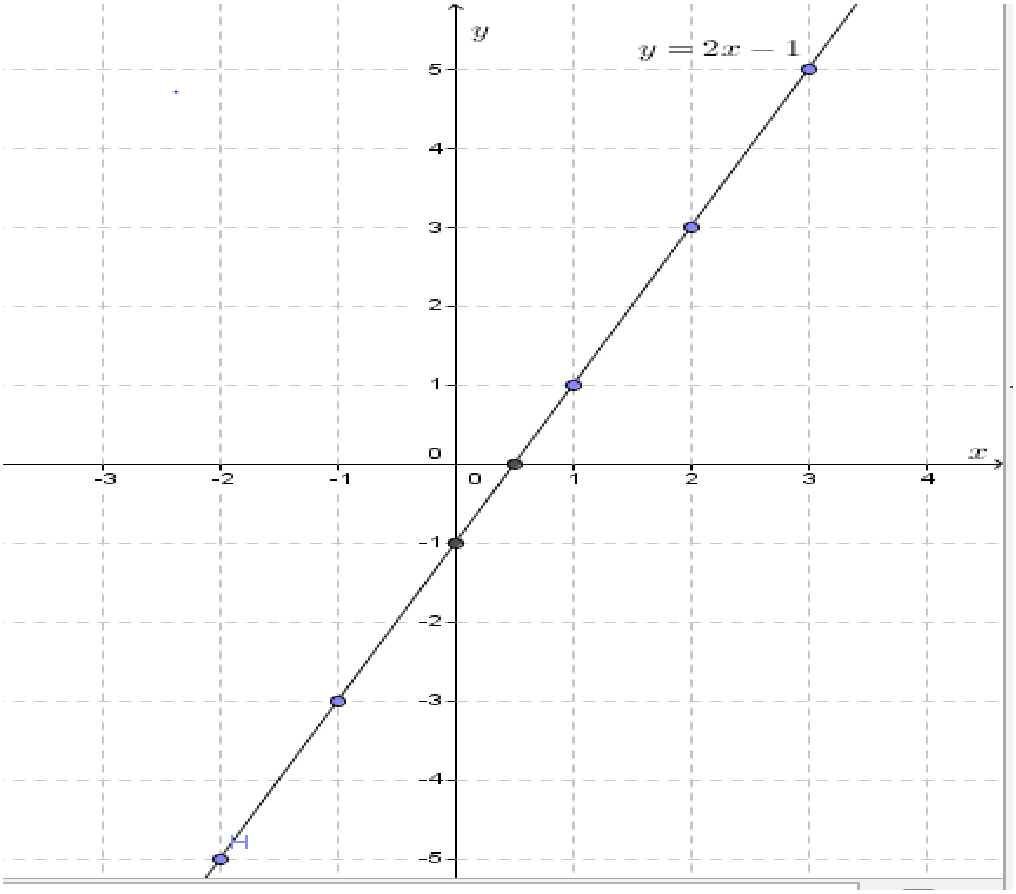
- Ask student-teachers in pairs to read and discuss on the **activity 4.6**
- Make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling student-teachers.
- Facilitate the use of geometric materials to make accurate graphs.
- Make sure that the notebooks of student-teachers include squared papers/graph papers.
- Call student-teachers to present the findings and promote gender where possible.
- Help them to harmonise the answers.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead students to verify the parity of different functions.

- After this step, guide students to do the **application activity 4.6** and evaluate whether lesson objectives were achieved.

## Answer of activity 4.6

1. $y = 2x - 1$ for $-3 \leq x \leq 3$

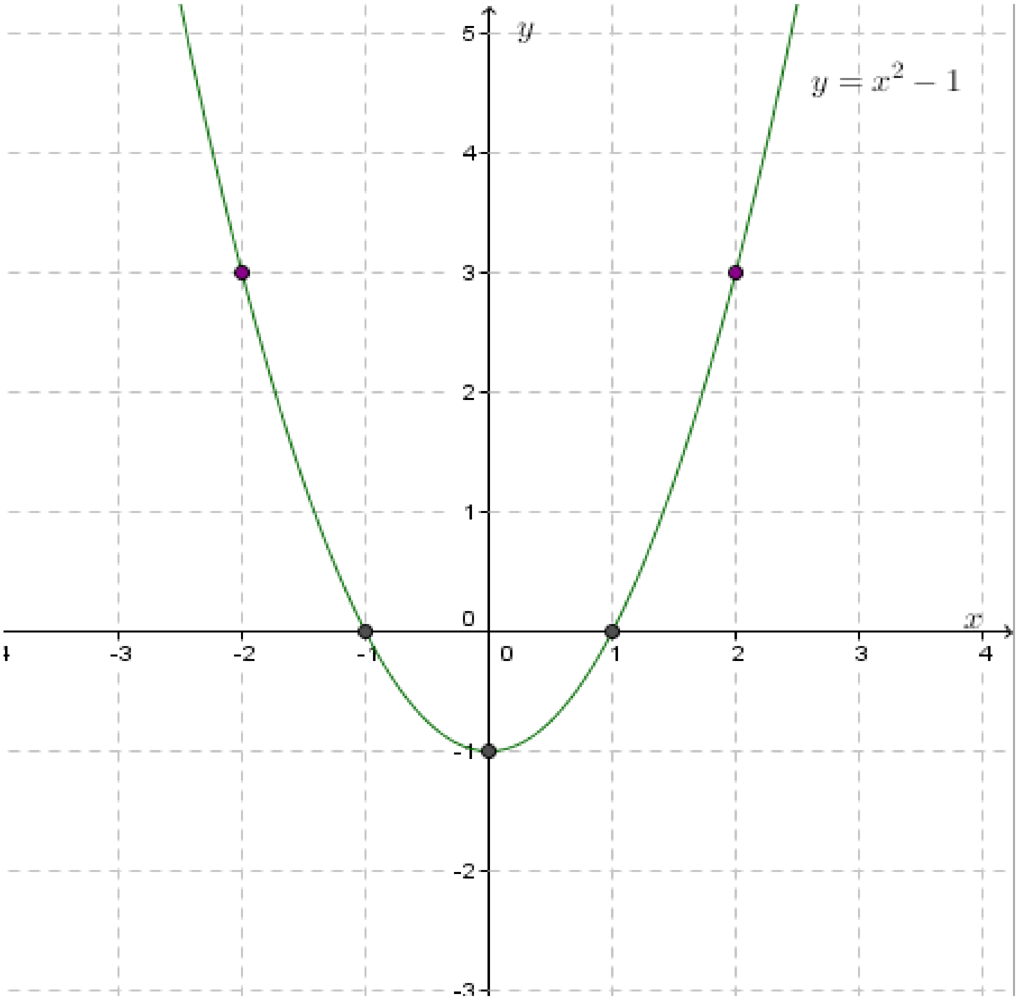
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = 2x - 1$ | -7 | -5 | -3 | -1 | 1 | 3 | 5 |



**This is linear function**

2. $y = x^2 - 1$ for $-3 \le x \le 3$

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y = x^2 - 1$ | -8 | -3 | 0 | -1 | 0 | 3 | 8 |

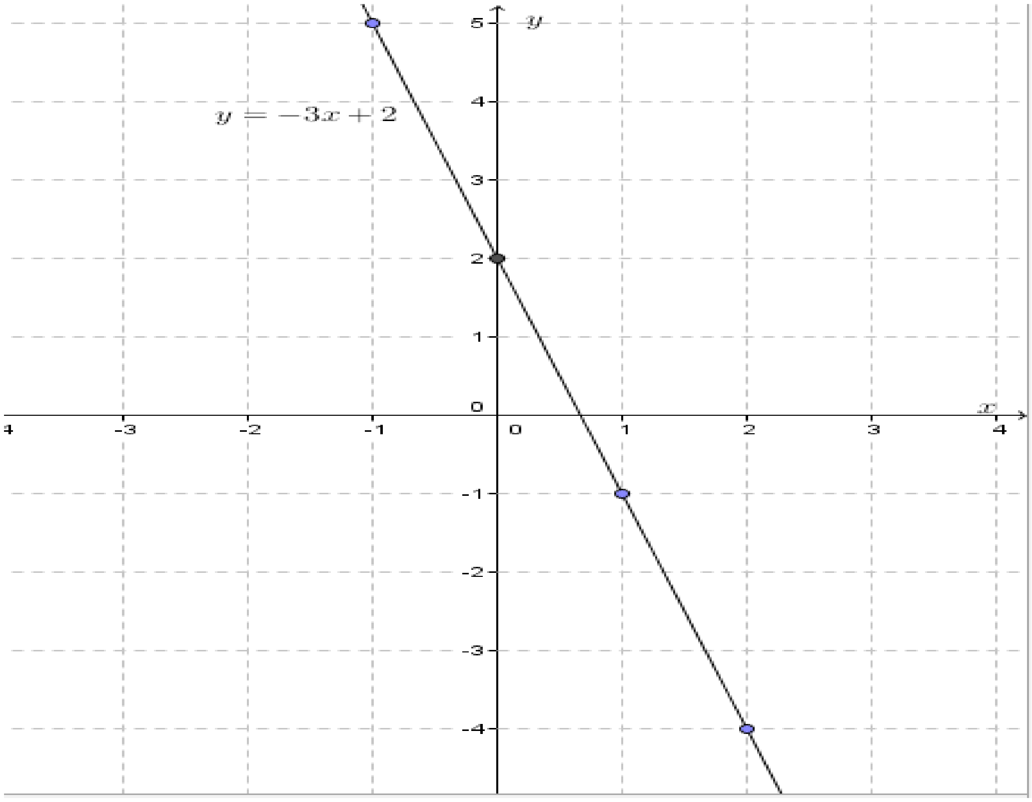



This is quadratic function

## Answer to application activity 4.6

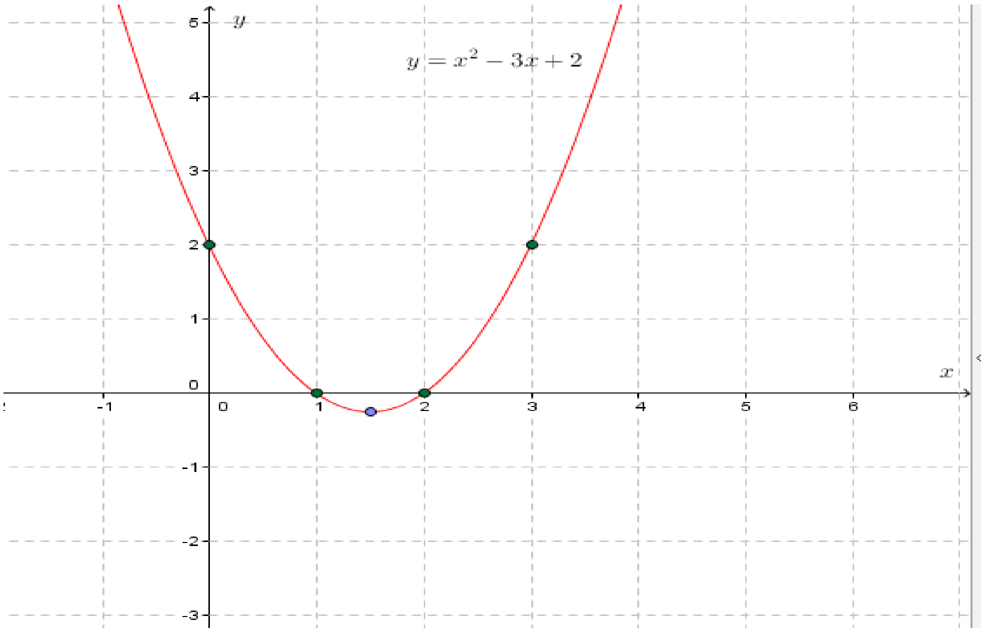
1. a) $y = -3x + 2$

| x | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| $y = -3x + 2$ | 5 | -2 | -1 | -4 |

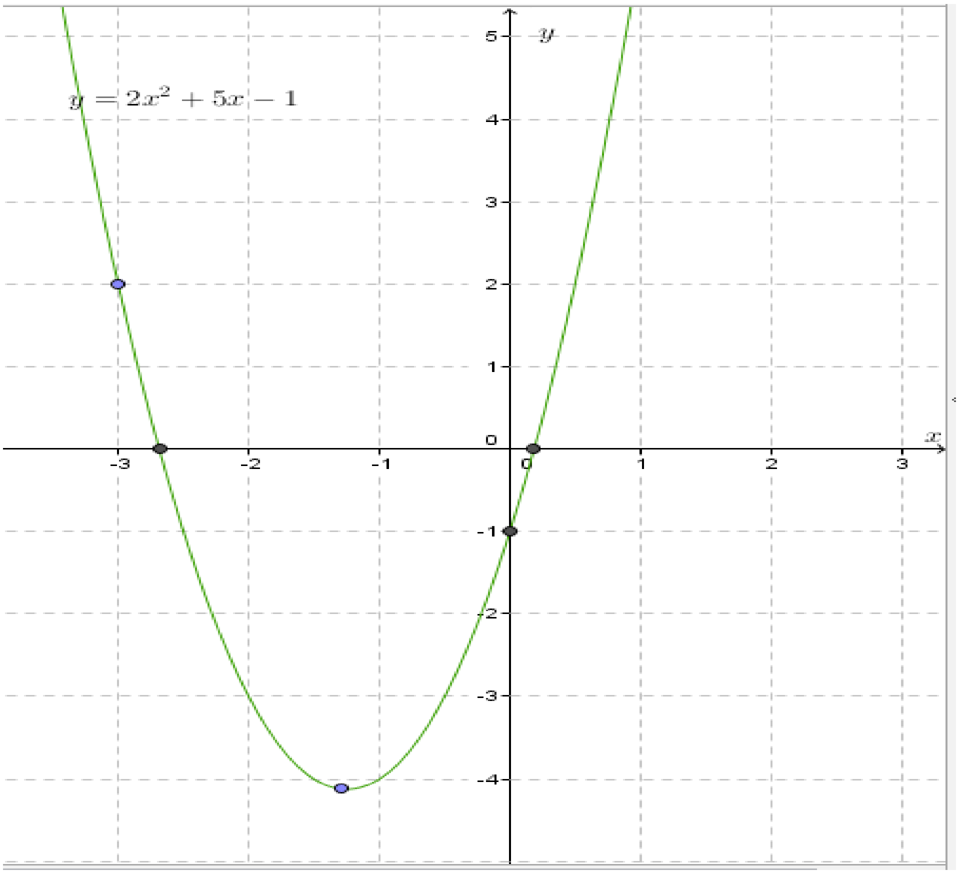



b) $y = x^2 - 3x + 2$

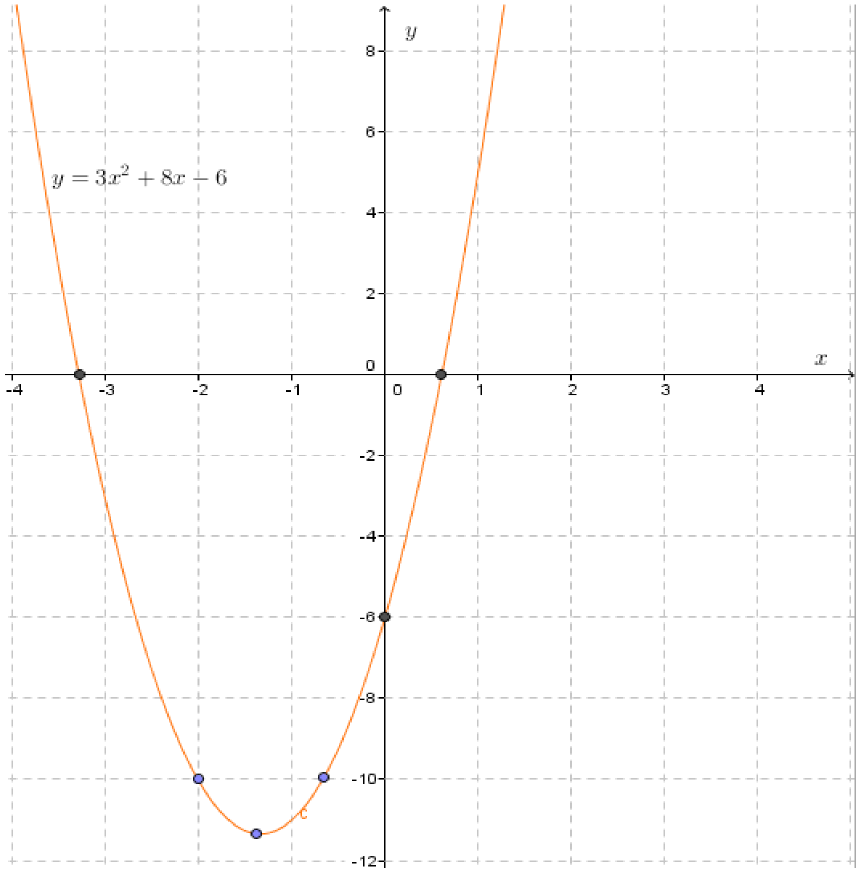
| x | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $y = x^2 - 3x + 2$ | 6 | 2 | 0 | 0 | 2 |




2. a)



b) $y=3x^2+8x-6$




## Lesson 12: Application of functions in real life situation

### a) Learning objective

- Identify and explain different applications of functions in real life
- Interpret graphs of functions (linear and quadratic) observed from practical context or journals and make conclusions.
- Analyze, apply, model, solve problems involving linear or quadratic functions, and interpret the results.

### b) Teaching resources

Student -teacher's book and other Reference Textbooks to facilitate learning process, Mathematical set, Digital materials including calculator, sticks, manila papers.

Mathematics book Calculator, Markers, Pens and Pencils, Online resources.

**Note:** Where it is possible, the tutor helps student-teachers to plot graphs using appropriate software such as Geogebra etc.

## c) Prerequisites/Revision/Introduction

In this lesson, student-teachers must refer to previous lessons of this unit.

## d) Learning activities

- Invite student-teachers to work in group and do the **activity 4.7** found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from each group to present their findings; then the teacher and the students will discuss the question one and provide answers for the question two.
- Facilitate working, especially straggling student-teachers.
- As a tutor, harmonize the findings from presentation and guide students to discuss and keep in mind the positive beliefs on the applications of functions in real life;
- Use different probing questions and guide them to explore the content and examples on the application of functions given in the student's book.
- Invite student-teachers to work individually the **application activities 4.7** and evaluate whether lesson objectives were achieved.

### Answer of activity 4.7

1) Here the answer may vary. Namely, we can say that functions are used in our everyday life, in economics, in physics, in Biology, in Chemistry, in History, in Geography,…

   Student teachers will provide different examples, try to analyse them and highlight some of them.

2) i). , For $x = 10, \mathbf{C}(\mathbf{10}) = \mathbf{80}(\mathbf{10}) + \mathbf{150} = \mathbf{800} + \mathbf{150} = \mathbf{950}$

   Therefore, the cost of producing 10 units is 950 FRW.

ii). $C(x)=15,000$, then

$15,000=\mathbf{80}x+\mathbf{150}$

$$x=\frac{15,000-150}{80}$$

$$x=185.625$$

Therefore, with 15,000 FRW, the company can produce 185 units.

iii). The restriction on the domain x ≥ 0 is necessary because it makes no sense when the number of products produced is negative.

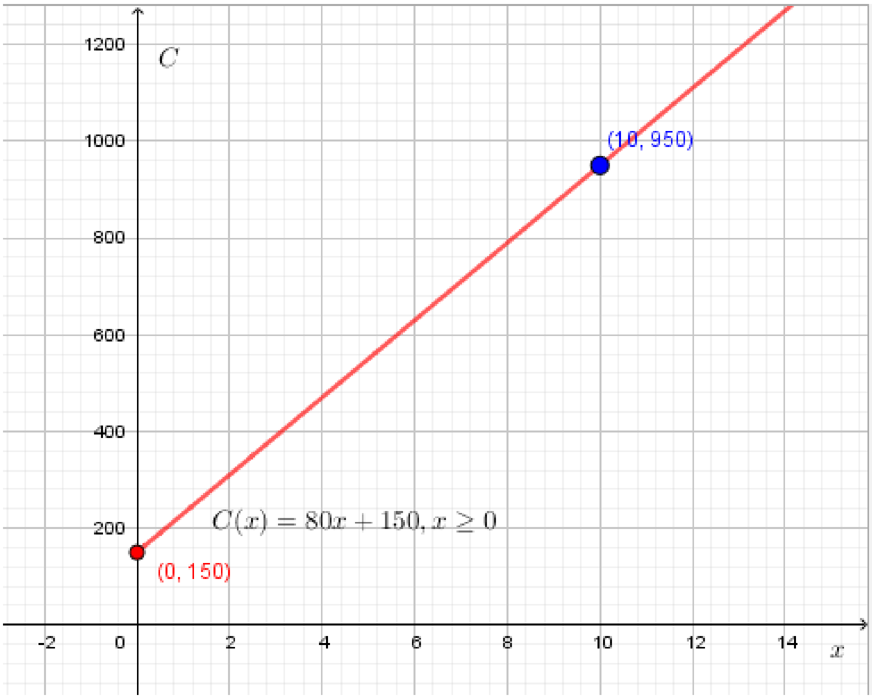
iv) We calculate C(0), or for $x=0$,

$$f(0)=80(0)+150=150$$

It has no meaning to produce zero units, but this amount of 150FRW is considered as the fixed or start-up of the venture which represents the cost for the initial activities done before the production.

e). We recognize that the slope is defined by $m=\frac{\Delta y}{\Delta x}=\frac{\Delta C}{\Delta x}=\frac{80}{1}=80$.

This represents the variation of the cost when the production varies at one unit.



**Answer of application activity 4.7**

**1) a).** Consider two points formed by corresponding price and products. These points are respectively $(20, 220)$ *and* $(40,190)$.

Use the point-slope form to find equation of the linear function, where

$$m = \frac{190-220}{40-20} = -\frac{3}{2}$$

Then, use $(20, 220)$ and $-\frac{3}{2}$;

$$y - 220 = -\frac{3}{2}(x - 20)$$

$$y = -\frac{3}{2}x + 250$$

$\therefore \quad P(x) = -\frac{3}{2}x + 250$ this is the linear function.

**b)** To determine the applied domain, we look at the physical constraints of the products.

Certainly, we can't sell negative number of products, so $x \geq 0$.

**c)** Since the slope is negative ( $m = -\frac{3}{2} = -1.5$ ), we have that the price is decreasing at a rate of \$1.5 per a product sold(said differently, we can sell one more product for every \$1.5 drop in price )

**d)** $P(150) = -\frac{3}{2}(150) + 250 = 25$ Therefore, the price of 150 products is \$25.

**e)** When the price \$150, then

$$150 = -\frac{3}{2}x + 250$$

$$300 = -3x + 500$$

$$x = \frac{500-300}{3} = \frac{200}{3} = 66.6$$

This means we would be able to sell 66 products a week, if the price were \$150 per unit or item.

2) Average rate is given by

$$\frac{\Delta C}{\Delta x}=\frac{C(5000)-C(3000)}{5000-3000}$$

$$C(5000)=(5000)^2-10(5000)+27=24,950,027$$

$$C(3000)=(3000)^2-10(3000)+27=8,970,027$$

$$\therefore Rate=\frac{24,950,027-8,970,027}{5000-3000}=\frac{15980000}{2000}=7990$$

A hundred is cost 7990FRW, it means one pen costing 79.9FRW.

3) Solution is optional.

## 4.6. Summary of the unit

In this unit, we dealt with numerical functions: **polynomials, rational and irrational** functions, how to determine whether a given function is a polynomial, rational or irrational function.

We focused on:

- Some particular properties that a numerical function can possess or not, such as**: "being one-to-one", being onto", being bijective", "being even", "being odd".**In each case, we were able to determine the property algebraically and using graph.
- Some particular subsets of the sets between which the function is established: the **domain and the range**; in each case, we saw how to use equations and inequalities to determine the domain , and how to use graphs to determine the domain and the range;
- Some operations that can be performed with numerical functions: **composition** and **inverse** of functions.

## 4.7. Additional Information for Tutors

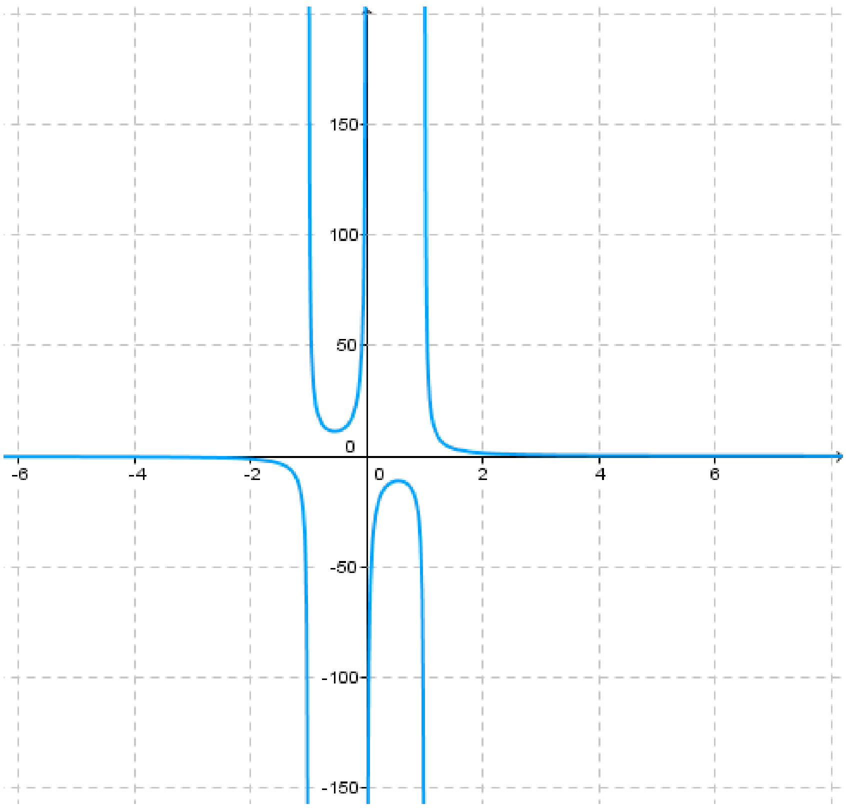
- The teacher must be equipped with skills about using Geogebra if possible install it the smart classroom computers to help learners for participation.
- It could be better to let learners graph and explore by themselves the graphs of functions in that case in smart classroom.

- Some concepts are tricky and you need to be aware so that your lesson does not turn to a mess.
- Emphasize the use of graph paper/gridded paper while student-teachers draw the graphs.
- Emphasize and facilitate students to use geometric materials to ameliorate the quality of graphs.
- Remind students to name axes (x-axis and y-axis).
- Recall them to mention/highlight the origin/intersection point of axes by (0.0).

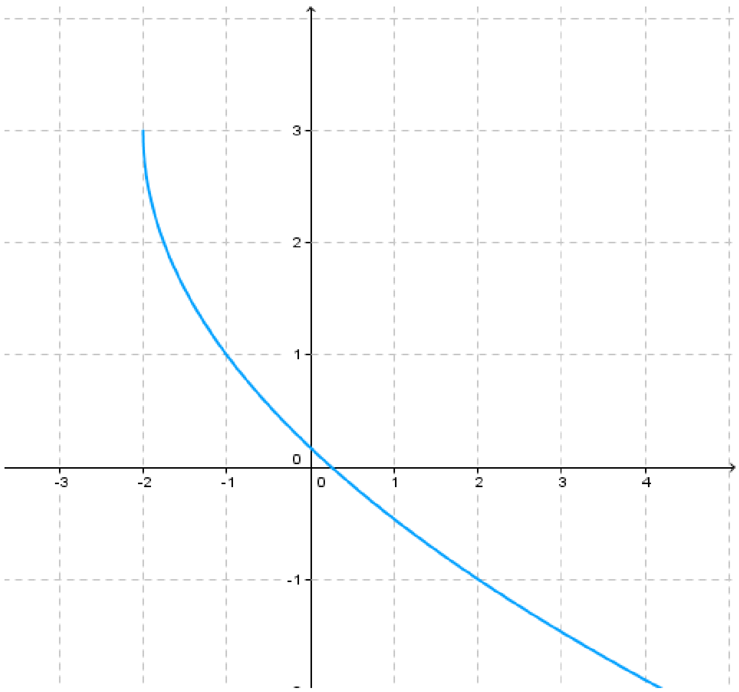
## 4.7 Answers for end unit assessment.

1) The students will verify whether $f(x) = f(-x)$ or whether $f(-x) = -f(x)$

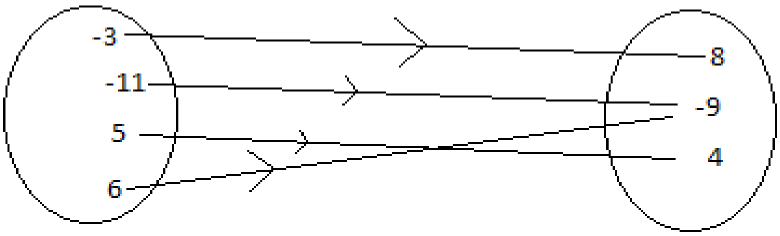
a) The function is odd. There, the graph is symmetric to the origin (see the graph)



b) The function is neither even nor odd. There, the graph is neither symmetric to the origin nor to the axis.




**2. Hint:** if drawn, using venn diagrams, the learners will easier get the point.




This function is subjective but not injective since elements are sharing the same image.

3. If the graph is needed use geogebra.

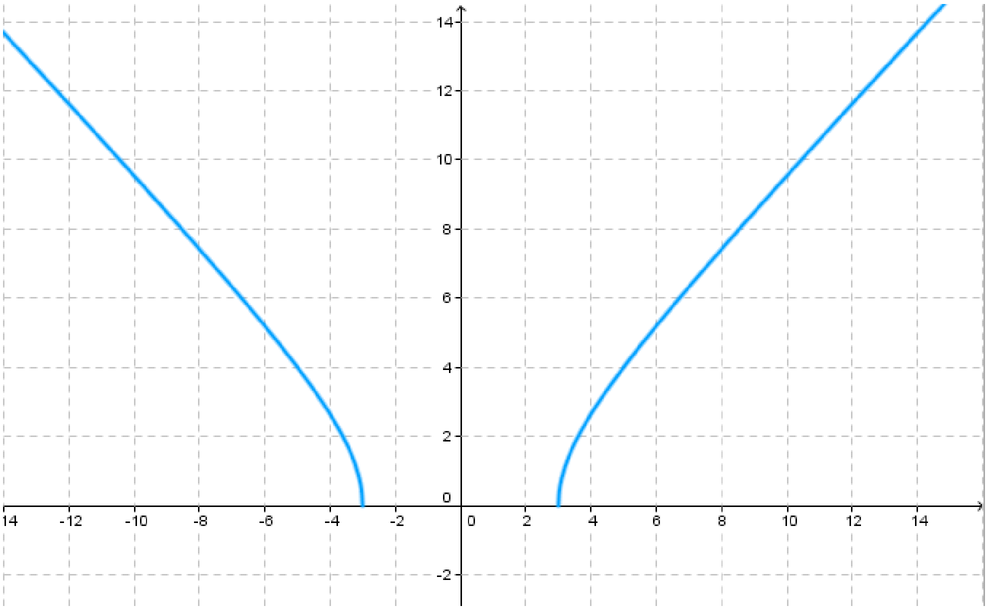
i. $h(5)=\sqrt{5^2-9}=4$

ii. $h(x)=2 \Rightarrow \sqrt{x^2-9}=2$

$x^2 = 11 \Rightarrow \begin{cases} x = \sqrt{11} \\ x = -\sqrt{11} \end{cases}$ Remember to verify the restrictions.

iii. The domain is $]-\infty, -3] \cup [3, +\infty[$

Can be obtained from the graph of the function $h(x) = \sqrt{x^2 - 9}$




The range can be seen from the graph $[0; +\infty[$

4.

i. The idea is just to replace as usual. $h(-4) - f(-1) = 0$

ii. If $g(x) = 0$ that is $\frac{x^2}{x-3} = 0 \Rightarrow x = 0$

iii. Existence condition is required for this sub-question.

- For $g(x)$ to be defined, $x - 3 \neq 0 \Rightarrow$ domg$= ]-\infty, 3[ \cup ]3, +\infty[$
- For $h(x)$ to be defined, $5 - x \geq 0 \Rightarrow domh = ]-\infty, 5]$

5.

i. Let $y = 3x - 5$ if we undo each operation and solve for $x$, we get $x = \frac{y+5}{3}$

Therefore $f^{-1}(x) = \frac{x+5}{3}$

ii. Let $y=\dfrac{2}{x-4}\Leftrightarrow y(x-4)=2$

$xy-4y=2$ Remember we are solving for $x$

$xy=4x+2\Rightarrow x=\dfrac{4x+2}{y}$ or $f^{-1}(x)=\dfrac{4x+2}{x}$

6. $f^{-1}(x)=x-2=g(x)$

7. The functions are inverse to each other if $f[g(x)]=g[f(x)]=x$

$f[g(x)]=(x-2)+2=x$ And $g[f(x)]=(x+2)-2=x$

Hence the two functions are inverse to each other.

| Rules for Excluding Numbers from the Domain of $f[g(x)]$ | Applying the Rules to $f(x)=\dfrac{2}{x-1}$ and $g(x)=\dfrac{3}{x}$ |
|---|---|
| If $x$ is not in the domain of $g(x)$ it must not be in the domain of $f[g(x)]$ | Because $g(x)=\dfrac{3}{x}$, 0 is not in the domain of $g(x)$. Thus, 0 must be excluded from the domain of $f[g(x)]$. |
| Any $x$ for which $g(x)$ is not in the domain of $f(x)$ must not be in he domain of $f[g(x)]$ | Because $f[g(x)]=\dfrac{2}{g(x)-1}$, we must exclude from the domain of $f[g(x)]$ any $x$ for which $g(x)$ is equal to 1. $\dfrac{3}{x}=1\Rightarrow x=3$ 3 must be excluded from the domain of $f[g(x)]$ Hence, the domain of $f[g(x)]$ is $]-\infty,0[\cup]0,3[\cup]3,+\infty[$ |

8) a) The model $f(x)=x-400$ for the price of the computer, means that the reserved price of the computer is 400$. A buyer to buy a computer must at least have more than 400$.

The model $g(x)=0.75x$ means that the price has been discounted by 25% of its original price.

b) $f\left[g(x)\right]=0.75x-400$ $. There is a discount of 25% but whatever the case, the cost price cannot go below 400$.

## 4.8 Additional activities

### *4.8.1 Remedial activities*

1. Let the function $f(x)=-2x^2+6x-3$ find f(-2)

**Solution:**

f(-2) = - 23

2.Considerthefunction $f(x)=-7x-5$ and g(x)=10x-12 find $g\left[f(x)\right]$ and $f\left[g(x)\right]$

**Solution**

$g[f(x)]=g(-7x-5)=10(-7x-5)-12=-70x-62$

$f[g(x)]=f(10x-12)=-7(10x-12)-5=-70x+79$

3. Use a dictionary or research on internet to define the following terms in mathematics context: Relations, functions, mappings, injection, surjection and bijection.

4. Find the domain of the real valued function h defined by $h(x)=\sqrt{x-2}$

### *4.8.2 Consolidation activities*

1. Find the domain of the function defined by $f(x)=\sqrt{-x^2+9}+\dfrac{1}{x-1}$

**Solution:**

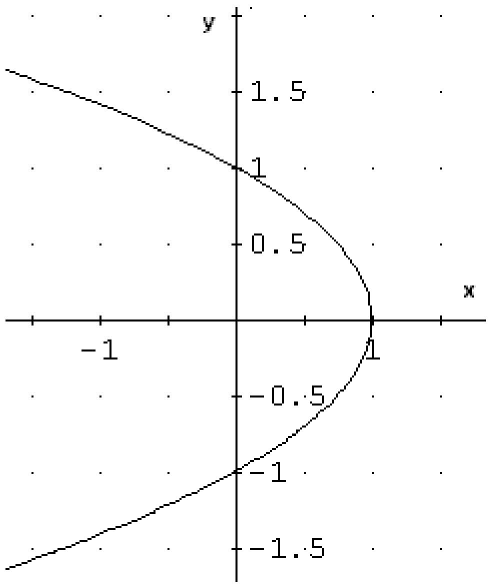
domf= $\left[-3,1\right[\cup\left]1,+3\right]$

2. Discuss whether the function defined from $\mathbb{R}\rightarrow\mathbb{R}$ by $f(x)=x^2$ is injective, surjective or neither.

**Solution:**

- Using counter example $f(-2)=4$ and $f(2)=4$, Hence the function is not injective since two elements from the domain are sharing the same image.
- The function $f(x)=x^2$ is always positive for any value of x that we can take. For that, the function is not surjective since negative y-values are missing their antecedents.

3. Does the graph below represents a function? Explain.




**Solution:**

No, the graph is not a function since for one input example (0) there are two y-values(vertical line test).

### *4.9.3 Extended activities*

1. A small ball is projected vertically upward from the top of a building with the initial velocity $v_0 = 144\ m/\sec$. Its distance $s(t)$ in meter above the ground after t seconds is given by the equation $s(t) = -16t^2 + 144t + 100$.

   a) What is the distance $s(t)$ at the initial time when $t = 0$?

   b) Make a table of values for $s(t)$ to show the distance from the initial time t=0 to t=10 seconds.

c) Use the table to draw the graph of $s(t)$ and show the position of the ball at t = 5 seconds

d) Discuss the parity of the function $s(t)$.

**Solution:**

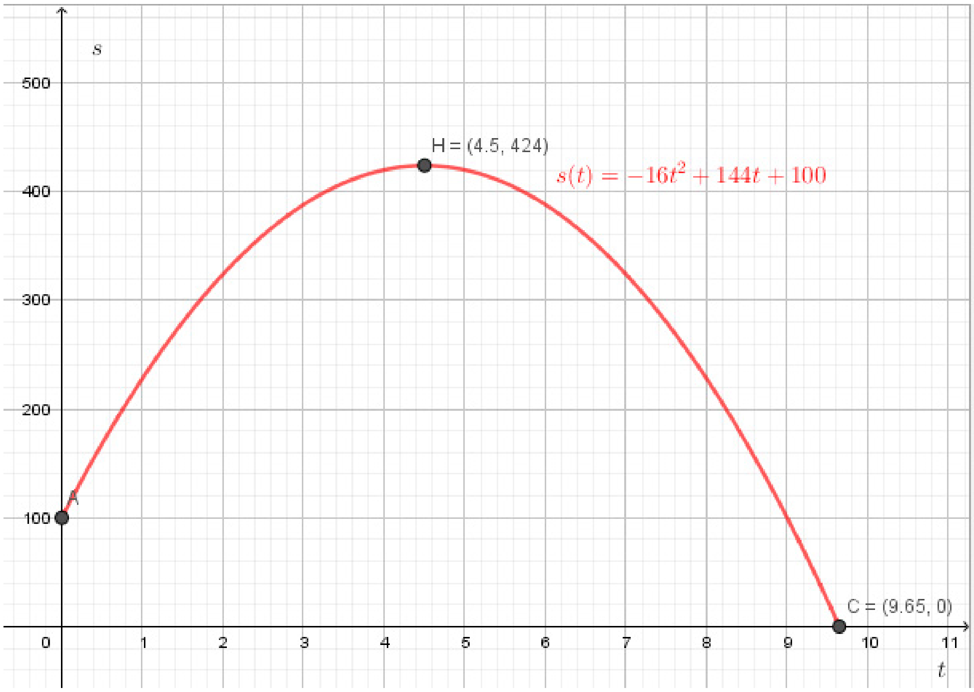
$s(t)=-16t^2+144t+100$ represents the height or the vertical distance above the ground.

a) At the initial time $t=0$, then the distance is $s(0)=-16(0)+144(0)+100=100$

This distance is 100m.

b)

| T | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\frac{483}{50}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $s(t)$ | 100 | 228 | 324 | 388 | 420 | 420 | 388 | 324 | 328 | 100 | 0 |




The graph shows that the ball falls on the ground between $t=9$ and $t=10$.

To find this time, you must solve the equation $s(t) = -16t^2 + 144t + 100 = 0$.

$$t = \frac{483}{50}$$

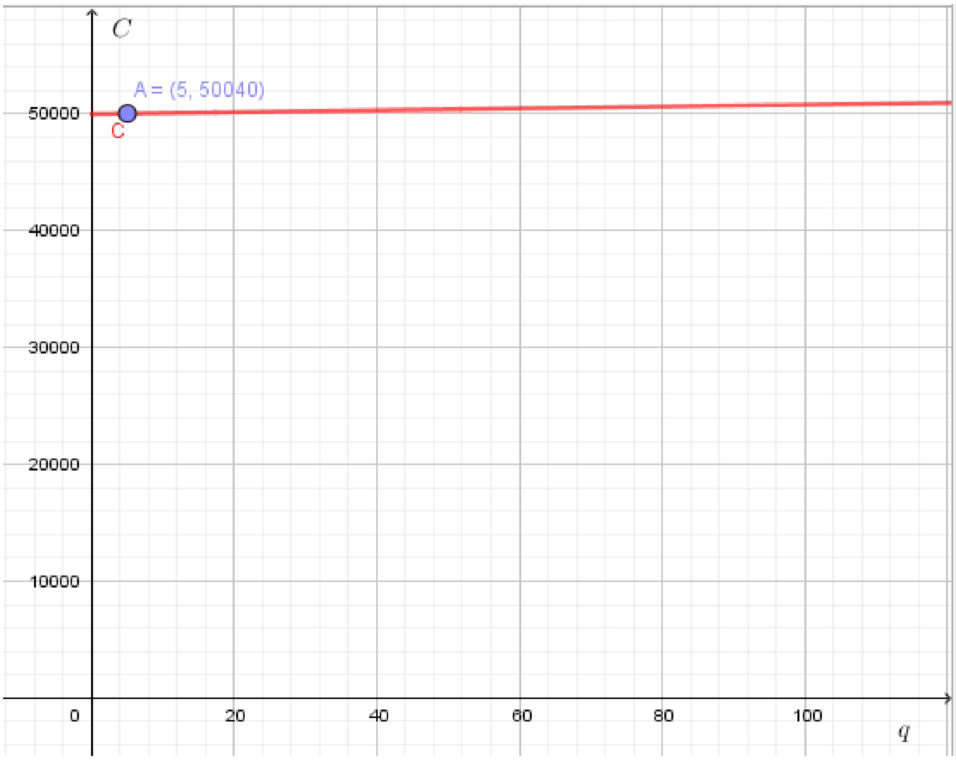
2. The total cost C for units produced by a company is given by $C(q) = 50000 + 7q$ where q is the number of units produced.

**Solution:**

a) The amount 50000 represents the fixed cost;

b) the number 8 represents the the marginal cost(cost of a unit of product);

c) The graph for $C(q) = 50000 + 7q$ is the following:




d) The real domain of C that corresponds to q which is positive is $[0;\infty[$. The range is $[50000;\infty[$

e) $C(q)$ is not an odd function because $C(-q) \neq -C(q)$.

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