

MATHEMATICS FOR TTCs

TUTOR'S GUIDE

YEAR

2

OPTION:

SCIENCE AND MATHEMATICS EDUCATION (SME)

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FOREWORD

Dear Tutor,

Rwanda Education Board is honoured to present the tutor's guide for Mathematics in the option of Science and mathematics Education (SME) which serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the mathematics content. The Rwandan educational philosophy is to ensure that learners achieve full potential at every level of education which will prepare them to be well integrated in society and exploit employment opportunities.

Specifically, TTC curriculum was reviewed to train quality teachers who will confidently and efficiently implement the Competence Based Curriculum in pre-primary and primary education. The rationale of the changes is to ensure that TTC leavers are qualified for job opportunities and further studies in Higher Education in different programs under education career advancement.

In line with efforts to improve the quality of education, the government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate their learning process. Many factors influence what they learn, how well they learn and the competences they acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials.

The ambition to develop a knowledge-based society and the growth of regional and global competition in the jobs market has necessitated the shift to a competence-based curriculum. After a successful shift from knowledge to a competence based curriculum in general education, TTC textbooks also were elaborated to align them to the new curriculum.

The book provides active teaching and learning techniques that engage student teachers to develop competences. In view of this, your role is to:

- Plan your lessons and prepare appropriate teaching materials.
- Organize group discussions for students considering the importance of social constructivism suggesting that learning occurs more effectively when the students works collaboratively with more knowledgeable

and experienced people.

- Engage students through active learning methods such as inquiry methods, group discussions, research, investigative activities and group and individual work activities.
- Provide supervised opportunities for students to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.
- Support and facilitate the learning process by valuing students' contributions in the class activities.
- Guide students towards the harmonization of their findings.
- Encourage individual, peer and group evaluation of the work done in the classroom and use appropriate competence-based assessment approaches and methods.

To facilitate you in your teaching activities, the content of this book is self-explanatory so that you can easily use it. It is divided in 3 parts:

The part I explain the structure of this book and give you the methodological guidance;

The part II gives a sample lesson plan;

The part III details the teaching guidance for each concept given in the student book.

Even though this Teacher's guide contains the guidance on solutions for all activities given in the learner's book, you are requested to work through each question before judging student's findings.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, UR-CE Lecturers, TTC Tutors, Teachers from general education and Educate! for their technical support. A word of gratitude goes also to the Head Teachers and TTCs principals who availed their staff for various activities.

Dr. NDAYAMBAJE Irénée

Director General, REB

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I wish to express my appreciation to the people who played a major role in the development of this tutor`s guide for Mathematics in the option of Science and Mathematics Education (SME). It would not have been successful without active participation of different education stakeholders.

I owe gratitude to different universities and schools in Rwanda that allowed their staff to work with REB in the in-house textbooks production initiative.

I wish to extend my sincere gratitude to lecturers and teachers whose efforts during writing exercise of this tutor`s guide were very much valuable.

Finally, my word of gratitude goes to the Rwanda Education Board staffs who were involved in the whole process of in-house textbook writing.

Joan MURUNGI

Head of CTRLR Department

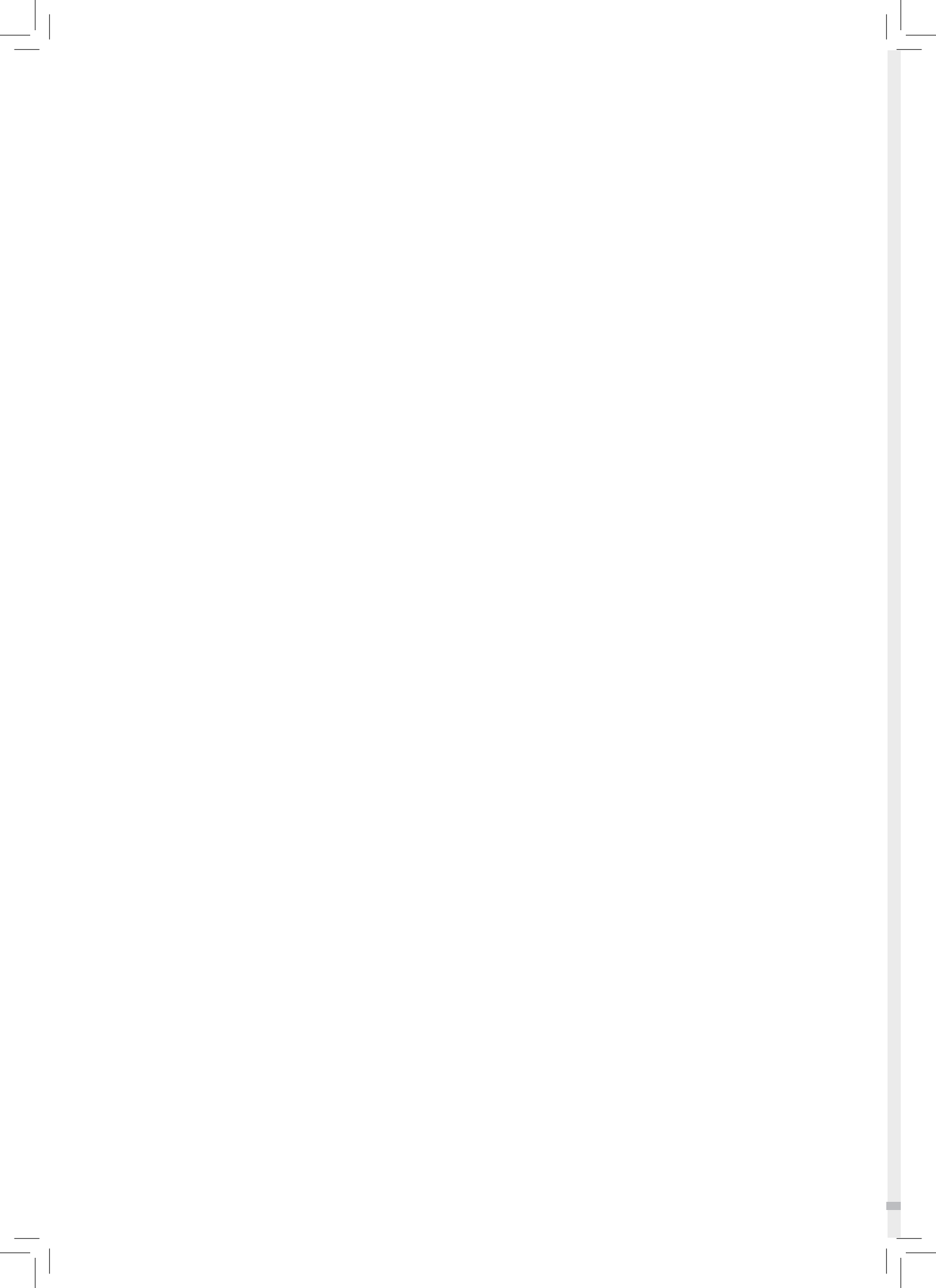
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PART I. GENERAL INTRODUCTION

1.1 The structure of the guide

The tutor's guide of Mathematics is composed of three parts:

The Part I concerns general introduction that discusses methodological guidance on how best to teach and learn Mathematics, developing competences in teaching and learning, addressing cross-cutting issues in teaching and learning and Guidance on assessment.

Part II presents a sample lesson plan. This lesson plan serves to guide the teacher on how to prepare a lesson in Mathematics.

The Part III is about the structure of a unit and the structure of a lesson. This includes information related to the different components of the unit and these components are the same for all units. This part provides information and guidelines on how to facilitate student teachers while working on learning activities. More over, all application activities from the textbook have answers in this part.

1.2 Methodological guidance

1.2.1 Developing competences

Since 2015 Rwanda shifted from a knowledge based to a competency-based curriculum for pre-primary, primary, secondary education and recently the TTC curriculum. This called for changing the way of learning by shifting from teacher centred to a learner centred approach. Teachers are not only responsible for knowledge transfer but also for fostering learners' learning achievement and creating safe and supportive learning environment. It implies also that learners have to demonstrate what they are able to transfer the acquired knowledge, skills, values and attitude to new situations.

The competence-based curriculum employs an approach of teaching and learning based on discrete skills rather than dwelling on only knowledge or the cognitive domain of learning. It focuses on what learner can do rather than what learner knows. Learners develop competences through subject unit with specific learning objectives broken down into knowledge, skills and attitudes through learning activities.

In addition to the competences related to Mathematics, student teachers also develop generic competences which should promote the development of the higher order thinking skills and professional skills in Mathematics teaching. Generic competences are developed throughout all units of Mathematics as follows:

Generic competences	Ways of developing generic competences
Critical thinking	All activities that require learners to calculate, convert, interpret, analyze, compare and contrast, etc have a common factor of developing critical thinking into learners
Creativity and innovation	All activities that require learners to plot a graph of a given algebraic data, to organize and interpret statistical data collected and to apply skills in solving problems of economics have a common character of developing creativity into learners
Research and problem solving	All activities that require learners to make a research and apply their knowledge to solve problems from the real-life situation have a character of developing research and problem solving into learners.
Communication	During Mathematics class, all activities that require learners to discuss either in groups or in the whole class, present findings, debate ...have a common character of developing communication skills into learners.
Co-operation, interpersonal relations and life skills	All activities that require learners to work in pairs or in groups have character of developing cooperation and life skills among learners.
Lifelong learning	All activities that are connected with research have a common character of developing into learners a curiosity of applying the knowledge learnt in a range of situations. The purpose of such kind of activities is for enabling learners to become life-long learners who can adapt to the fast-changing world and the uncertain future by taking initiative to update knowledge and skills with minimum external support.

Professional skills	Specific instructional activities and procedures that a teacher may use in the class room to facilitate, directly or indirectly, students to be engaged in learning activities. These include a range of teaching skills: the skill of questioning, reinforcement, probing, explaining, stimulus variation, introducing a lesson; illustrating with examples, using blackboard, silence and nonverbal cues, using audio – visual aids, recognizing attending behaviour and the skill of achieving closure.
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The generic competences help learners deepen their understanding of Mathematics and apply their knowledge in a range of situations. As students develop generic competences they also acquire the set of skills that employers look for in their employees, and so the generic competences prepare students for the world of work.

1.2.2 Addressing cross cutting issues

Among the changes brought by the competence-based curriculum is the integration of cross cutting issues as an integral part of the teaching learning process-as they relate to and must be considered within all subjects to be appropriately addressed. The eight cross cutting issues identified in the national curriculum framework are: *Comprehensive Sexuality Education, Environment and Sustainability, Financial Education, Genocide studies, Gender, Inclusive Education, Peace and Values Education, and Standardization Culture.*

Some cross-cutting issues may seem specific to particular learning areas/ subjects but the teacher need to address all of them whenever an opportunity arises. In addition, learners should always be given an opportunity during the learning process to address these cross-cutting issues both within and out of the classroom.

Below are examples of how crosscutting issues can be addressed:

Cross-Cutting Issue	Ways of addressing cross-cutting issues
<p>Comprehensive Sexuality Education: The primary goal of introducing Comprehensive Sexuality Education program in schools is to equip children, adolescents, and young people with knowledge, skills and values in an age appropriate and culturally gender sensitive manner so as to enable them to make responsible choices about their sexual and social relationships, explain and clarify feelings, values and attitudes, and promote and sustain risk reducing behaviour.</p>	<p>Using different charts and their interpretation, Mathematics tutor should lead students to discuss the following situations: “Alcohol abuse and unwanted pregnancies” and advise student teachers on how they can instil learners to fight those abuses.</p> <p>Some examples can be given when learning statistics, powers, logarithms and their properties.</p>
<p>Environment and Sustainability: Integration of Environment, Climate Change and Sustainability in the curriculum focuses on and advocates for the need to balance economic growth, society well-being and ecological systems. Learners need basic knowledge from the natural sciences, social sciences, and humanities to understand to interpret principles of sustainability.</p>	<p>Using Real life models or students’ experience, Mathematics Tutor should lead student teachers to illustrate the situation of “population growth” and discuss its effects on the environment and sustainability.</p>

<p>Financial Education:</p> <p>The integration of Financial Education into the curriculum is aimed at a comprehensive Financial Education program as a precondition for achieving financial inclusion targets and improving the financial capability of Rwandans so that they can make appropriate financial decisions that best fit the circumstances of one's life.</p>	<p>Through different examples and calculations on interest rate problems, total revenue and total cost, Mathematics Tutor can lead student teachers to discuss how to make appropriate financial decisions.</p>
<p>Gender: At school, gender will be understood as family complementarities, gender roles and responsibilities, the need for gender equality and equity, gender stereotypes, gender sensitivity, etc.</p>	<p>Mathematics Tutor should address gender as cross-cutting issue through assigning leading roles in the management of groups to both girls and boys and providing equal opportunity in the lesson participation and avoid any gender stereotype in the whole teaching and learning process.</p>
<p>Inclusive Education: Inclusion is based on the right of all learners to a quality and equitable education that meets their basic learning needs and understands the diversity of backgrounds and abilities as a learning opportunity.</p>	<p>Firstly, Mathematics Tutors need to identify/recognize students with special needs. Then by using adapted teaching and learning resources while conducting a lesson and setting appropriate tasks to the level of students, they can cater for students with special education needs. They must create opportunity where student teachers can discuss how to cater for learners with special educational needs.</p>

<p>Peace and Values Education: Peace and Values Education (PVE) is defined as education that promotes social cohesion, positive values, including pluralism and personal responsibility, empathy, critical thinking and action in order to build a more peaceful society.</p>	<ul style="list-style-type: none"> • Through a given lesson, a tutor should: • Set a learning objective which is addressing positive attitudes and values, • Encourage students to develop the culture of tolerance during discussion and to be able to instil it in colleagues and cohabitants; • Encourage students to respect ideas for others.
<p>Standardization Culture: Standardization Culture in Rwanda will be promoted through formal education and plays a vital role in terms of health improvement, economic growth, industrialization, trade and general welfare of the people through the effective implementation of Standardization, Quality Assurance, Metrology and Testing.</p>	<p>With different word problems related to the effective implementation of Standardization, Quality Assurance, Metrology and Testing, students can be motivated to be aware of health improvement, economic growth, industrialization, trade and general welfare of the people.</p>

1.2.3 Guidance on how to help students with special education needs in classroom

In the classroom, students learn in different way depending to their learning pace, needs or any other special problem they might have. However, the teacher has the responsibility to know how to adopt his/her methodologies and approaches in order to meet the learning need of each student in the classroom. Also teachers need to understand that student with special needs, need to be taught differently or need some accommodations to enhance the learning environment. This will be done depending to the subject and the nature of the lesson.

In order to create a well-rounded learning atmosphere, teachers need to:

- Remember that learners learn in different ways so they have to offer a variety of activities (e.g. role-play, music and singing, word games and quizzes, and outdoor activities);
- Maintain an organized classroom and limits distraction. This will help learners with special needs to stay on track during lesson and follow instruction easily;
- Vary the pace of teaching to meet the needs of each child. Some learners process information and learn more slowly than others;
- Break down instructions into smaller, manageable tasks. Learners with special needs often have difficulty understanding long-winded or several instructions at once. It is better to use simple, concrete sentences in order to facilitate them understand what you are asking.
- Use clear consistent language to explain the meaning (and demonstrate or show pictures) if you introduce new words or concepts;
- Make full use of facial expressions, gestures and body language;
- Pair a learner who has a disability with a friend. Let them do things together and learn from each other. Make sure the friend is not over protective and does not do everything for the one with disability. Both learners will benefit from this strategy;
- Use multi-sensory strategies. As all learners learn in different ways, it is important to make every lesson as multi-sensory as possible. Learners with learning disabilities might have difficulty in one area, while they might excel in another. For example, use both visual and auditory cues.

Below are general strategies related to each main category of disabilities and how to deal with every situation that may arise in the classroom. However, the list is not exhaustive because each child is unique with different needs and that should be handled differently.

Strategy to help learners with developmental impairment:

- Use simple words and sentences when giving instructions;
- Use real objects that learners can feel and handle. Rather than just working abstractly with pen and paper;
- Break a task down into small steps or learning objectives. The learner should start with an activity that she/he can do already before moving on to something that is more difficult;

- Gradually give the learner less help;
- Let the learner with disability work in the same group with those without disability.

Strategy to help learners with visual impairment:

- Help learners to use their other senses (hearing, touch, smell and taste) and carry out activities that will promote their learning and development;
- Use simple, clear and consistent language;
- Use tactile objects to help explain a concept;
- If the learner has some sight, ask him/her what he/she can see;
- Make sure the learner has a group of friends who are helpful and who allow him/her to be as independent as possible;
- Plan activities so that learners work in pairs or groups whenever possible;

Strategy to help learners with hearing disabilities or communication difficulties

- Always get the learner's attention before you begin to speak;
- Encourage the learner to look at your face;
- Use gestures, body language and facial expressions;
- Use pictures and objects as much as possible.
- Keep background noise to a minimum.

Strategies to help learners with physical disabilities or mobility difficulties:

- Adapt activities so that learners who use wheelchairs or other mobility aids, can participate.
- Ask parents/caregivers to assist with adapting furniture e.g. the height of a table may need to be changed to make it easier for a learner to reach it or fit their legs or wheelchair under;
- Encourage peer support when needed;
- Get advice from parents or a health professional about assistive devices if the learner has one.

Adaptation of assessment strategies:

At the end of each unit, the tutor is advised to provide additional activities to help students achieve the key unit competence. These assessment activities are for remedial, consolidation and extension designed to cater for the needs of all categories of students; slow, average and gifted students respectively. Therefore, the tutor is expected to do assessment that fits individual student.

Remedial activities	After evaluation, slow students are provided with lower order thinking activities related to the concepts learnt to facilitate them in their learning. These activities can also be given to assist deepening knowledge acquired through the learning activities for slow students.
Consolidation activities	After introduction of any concept, a range number of activities can be provided to all students to enhance/reinforce learning.
Extended activities	After evaluation, gifted and talented students can be provided with high order thinking activities related to the concepts learnt to make them think deeply and critically. These activities can be assigned to gifted and talented students to keep them working while other students are getting up to required level of knowledge through the learning activity.

1.2.4. Guidance on assessment

Assessment is an integral part of teaching and learning process. The main purpose of assessment is for improvement of learning outcomes. Assessment for learning/ Continuous/ formative assessment intends to improve students' learning and tutor's teaching whereas assessment of learning/summative assessment intends to improve the entire school's performance and education system in general.

Continuous/ formative assessment

It is an on-going process that arises during the teaching and learning process. It includes lesson evaluation and end of sub unit assessment. This formative

assessment should play a big role in teaching and learning process. The teacher should encourage individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.

Formative assessment is used to:

- Determine the extent to which learning objectives are being achieved and competences are being acquired and to identify which students need remedial interventions, reinforcement as well as extended activities. The application activities are developed in the learner book and they are designed to be given as remedial, reinforcement, end lesson assessment, homework or assignment
- Motivate students to learn and succeed by encouraging students to read, or learn more, revise, etc.
- Check effectiveness of teaching methods in terms of variety, appropriateness, relevance, or need for new approaches and strategies. Mathematics tutors need to consider various aspects of the instructional process including appropriate language levels, meaningful examples, suitable methods and teaching aids/ materials, etc.
- Help students to take control of their own learning.

In teaching Mathematics, formative or continuous assessment should compare performance against instructional objectives. Formative assessment should measure the student's ability with respect to a criterion or standard. For this reason, it is used to determine what students can do, rather than how much they know.

Summative assessment

The assessment can serve as summative and informative depending to its purpose. The end unit assessment will be considered summative when it is done at end of unit and want to start a new one.

It will be formative assessment, when it is done in order to give information on the progress of learners and from there decide what adjustments need to be done.

The assessment done at the end of the term, end of year, is considered as summative assessment so that the teacher, school and parents are informed of the achievement of educational objective and think of improvement strategies.

There is also end of level/ cycle assessment in form of national examinations.

When carrying out assessment?

Assessment should be clearly visible in lesson, unit, term and yearly plans.

- **Before learning (diagnostic):** At the beginning of a new unit or a section of work; assessment can be organized to find out what students already know / can do, and to check whether the students are at the same level.
- **During learning (formative/continuous):** When students appear to be having difficulty with some of the work, by using on-going assessment (continuous). The assessment aims at giving students support and feedback.
- **After learning (summative):** At the end of a section of work or a learning unit, the Mathematics Tutor has to assess after the learning. This is also known as Assessment of Learning to establish and record overall progress of students towards full achievement. Summative assessment in Rwandan schools mainly takes the form of written tests at the end of a learning unit or end of the month, and examinations at the end of a term, school year or cycle.

Instruments used in assessment.

- **Observation:** This is where the Mathematics tutor gathers information by watching students interacting, conversing, working, playing, etc. A tutor can use observations to collect data on behaviours that are difficult to assess by other methods such as attitudes, values, and generic competences and intellectual skills. It is very important because it is used before the lesson begins and throughout the lesson since the tutor has to continue observing each and every activity.
- **Questioning**
 - (a) Oral questioning: a process which requires a student to respond verbally to questions
 - (b) Class activities/ exercises: tasks that are given during the learning/ teaching process
 - (c) Short and informal questions usually asked during a lesson
 - (d) Homework and assignments: tasks assigned to students by their tutors to be completed outside of class.

Homework assignments, portfolio, project work, interview, debate, science fair, Mathematics projects and Mathematics competitions are also the different forms/instruments of assessment.

1.2.5. Teaching methods and techniques that promote active learning

The different learning styles for students can be catered for, if the teacher uses active learning whereby learners are really engaged in the learning process.

The main teaching methods used in mathematics are the following:

- **Dogmatic method** (the teacher tells the students what to do, What to observe, How to attempt, How to conclude)
- **Inductive-deductive method:** Inductive method is to move from specific examples to generalization and deductive method is to move from generalization to specific examples.
- **Analytic-synthetic method:** Analytic method proceeds from unknown to known, 'Analysis' means 'breaking up' of the problem in hand so that it ultimately gets connected with something obvious or already known. Synthetic method is the opposite of the analytic method. Here one proceeds from known to unknown.
- **Skills Laboratory method:** Laboratory method is based on the maxim "learning by doing." It is a procedure for stimulating the activities of the students and to encourage them to make discoveries through practical activities.
- **Problem solving method, Project method and Seminar Method.**

The following are some active techniques to be used in Mathematics:

- Group work
- Research
- Probing questions
- Practical activities (drawing, plotting, interpreting graphs)
- Modelling
- Brainstorming
- Quiz Technique
- Discussion Technique
- Scenario building Technique

What is Active learning?

Active learning is a pedagogical approach that engages learners in doing things and thinking about the things they are doing. Learners play the key role in the active learning process. They are not empty vessels to fill but people with ideas, capacity and skills to build on for effective learning. Thus, in active learning, learners are encouraged to bring their own experience and knowledge into the learning process.

The role of the teacher in active learning	The role of learners in active learning
<ul style="list-style-type: none">• The teacher engages learners through active learning methods such as inquiry methods, group discussions, research, investigative activities, group and individual work activities.• He/she encourages individual, peer and group evaluation of the work done in the classroom and uses appropriate competence-based assessment approaches and methods.• He provides supervised opportunities for learners to develop different competences by giving tasks which enhance critical thinking, problem solving, research, creativity and innovation, communication and cooperation.• Teacher supports and facilitates the learning process by valuing learners' contributions in the class activities.	<p>A learner engaged in active learning:</p> <ul style="list-style-type: none">• Communicates and shares relevant information with peers through presentations, discussions, group work and other learner-centred activities (role play, case studies, project work, research and investigation);• Actively participates and takes responsibility for his/her own learning;• Develops knowledge and skills in active ways;• Carries out research/investigation by consulting print/online documents and resourceful people, and presents their findings;• Ensures the effective contribution of each group member in assigned tasks through clear explanation and arguments, critical thinking, responsibility and confidence in public speaking• Draws conclusions based on the findings from the learning activities.

Main steps for a lesson in active learning approach

All the principles and characteristics of the active learning process highlighted above are reflected in steps of a lesson as displayed below. Generally, the lesson is divided into three main parts whereby each one is divided into smaller steps to make sure that learners are involved in the learning process. Below are those main part and their small steps:

1) Introduction

Introduction is a part where the teacher makes connection between the current and previous lesson through appropriate technique. The teacher opens short discussions to encourage learners to think about the previous learning experience and connect it with the current instructional objective. The teacher reviews the prior knowledge, skills and attitudes which have a link with the new concepts to create good foundation and logical sequencings.

2) Development of the new lesson

The development of a lesson that introduces a new concept will go through the following small steps: discovery activities, presentation of learners' findings, exploitation, synthesis/summary and exercises/application activities.

- **Discovery activity**

Step 1

- The teacher discusses convincingly with learners to take responsibility of their learning
- He/she distributes the task/activity and gives instructions related to the tasks (working in groups, pairs, or individual to instigate collaborative learning, to discover knowledge to be learned)

Step 2

- The teacher let learners work collaboratively on the task;
- During this period the teacher refrains to intervene directly on the knowledge;
- He/she then monitors how the learners are progressing towards the knowledge to be learned and boosts those who are still behind (but without communicating to them the knowledge).

- **Presentation of learners' findings/productions**
 - In this episode, the teacher invites representatives of groups to present their productions/findings.
 - After three/four or an acceptable number of presentations, the teacher decides to engage the class into exploitation of learners' productions.
- **Exploitation of learner's findings/ productions**
 - The teacher asks learners to evaluate the productions: which ones are correct, incomplete or false
 - Then the teacher judges the logic of the learners' products, corrects those which are false, completes those which are incomplete, and confirms those which are correct.
- **Institutionalization or harmonization (summary/conclusion/ and examples)**
 - The teacher summarizes the learned knowledge and gives examples which illustrate the learned content.
- **Application activities**
 - Exercises of applying processes and products/objects related to learned unit/sub-unit
 - Exercises in real life contexts
 - Teacher guides learners to make the connection of what they learnt to real life situations. At this level, the role of teacher is to monitor the fixation of process and product/object being learned.

3) Assessment

In this step the teacher asks some questions to assess achievement of instructional objective. During assessment activity, learners work individually on the task/activity. The teacher avoids intervening directly. In fact, results from this assessment inform the teacher on next steps for the whole class and individuals. In some cases, the teacher can end with a homework/ assignment. Doing this will allow learners to relay their understanding on the concepts covered that day. Teacher leads them not to wait until the last minute for doing the homework as this often results in an incomplete homework set and/or an incomplete understanding of the concept.

PART II: SAMPLE LESSON PLAN

School Name:

Teacher's name:

Term	Date	Subject	Class	Unit No	Lesson No	Duration	Class size
I /...../2020	Mathematics	Year2 SME	2	9 of 11	40 min	...
Type of Special Educational Needs to be catered for in this lesson and number of learners in each category				3 slow learners and 2 low vision learners:			
Unit title		Points, straight lines and planes in 3D.					
Key unit competency:		Extend understanding and use geometric presentations of lines and planes, locate points and determine equation of a line and plane in 3D.					
Title of the lesson		Distance from a point to a plane					
Instructional Objective		Using a T-square and a ruler, learners will be able to describe how to accurately measure the distance from a point to a plane and measure practically the distance between a point on the floor and the ceiling of a house					
Plan for this Class (location: in / outside)		The lesson is held indoors, the class is organized into groups, 3 slow learners are scattered in different groups, and 2 low vision learners seat on the front desks near the blackboard in order to see and participate fully in all activities					
Learning Materials (for ALL learners)		Textbooks, the classroom in which the learners are studying					
References		<ul style="list-style-type: none"> • TTC syllabus, • Mathematics Year2 Mathematics textbook and Teacher's guide • S5 Subsidiary Mathematics textbook and Teacher's guide 					

Timing for each step	Description of teaching and learning activity		Generic competences and cross cutting issues to be addressed + a short explanation
	Teacher activities	Learner activities	
	<ul style="list-style-type: none"> Learners work individually the questions in the introduction, and the correction is done on the chalk board by two learners, one after another under the guidance of the teacher. Then they discuss in groups the discovery activity, followed by the presentation by a sample group, interaction of learners and harmonization of the results under the facilitation of the teacher. Next, they discuss in pairs the solved example and compare their results with the answer proposed in the book Finally, the learners are assigned individual tasks, and the correction is done on the chalk board, and the teacher winds up the lesson by assigning homework activities to the learners. 		
Introduction: 5 minutes	The teacher asks learners to work individually: Find the distance between: <ul style="list-style-type: none"> two points (see 2.1.2) a point and a line (see 2.2.3) The teacher links the introduction to the lesson of the day 	<ul style="list-style-type: none"> Learners work individually. Two learners, one after another, write the answers on the chalkboard: 	Communication skills developed through the presentation and sharing ideas

2. Development of the lesson			
<p>2.1 Discovery activity:</p> <p>10 minutes</p>	<ul style="list-style-type: none"> • The teacher organizes the learners into groups • Teacher gives learners activity 2.3.2 to discuss in groups and gives instructions related to the task • Teacher goes round to monitor the work of each group and provide assistance where needed 	<ul style="list-style-type: none"> • Learners form groups • Each group analyzes and discuss the activity 2.3.2 under the direction of the task manager of the group • Learners present to the teacher their eventual problems 	<ul style="list-style-type: none"> • Cooperation and communication skills through discussions • Peace and values education; Cooperation , mutual respect, tolerance through discussions with people with different views and respect one's views
<p>2.2 Presentation of learner's findings and exploitation:</p> <p>15 minutes</p>	<ul style="list-style-type: none"> • Teacher invites the reporter of a sample group to present the findings of the group • The teacher encourages learners to follow attentively • Teacher takes notes on key points from learners' presentation. • The teacher asks learners to amend the presentation and to evaluate their work 	<ul style="list-style-type: none"> • The reporter presents the work on the behalf of the group. <p>Expected answers</p> <p>(Refer to solution of activity 2.3.2, in TG)</p> <ul style="list-style-type: none"> • Learners follow the presentation • Learners evaluate the findings of other learners • Learners evaluate their own findings 	<ul style="list-style-type: none"> • Cooperation and communication/ attentive listening during presentations and group discussions • Critical thinking through evaluating other's findings

<p>2.3.Conclusion/ Summary: 5 minutes</p> <p>Assessment 5 minutes</p>	<ul style="list-style-type: none"> • Teacher facilitates the learners to elaborate the summary of the presentation • Teacher requests learners to write down the main points in their books • Teacher asks learners to individually work out the application activity 2.3.2 • Teacher assigns homework activities from TG (additional activity) to learners 	<ul style="list-style-type: none"> • The learners come to the main point: <p>The distance from point $P(x_0, y_0, z_0)$ to plane π :</p> $ax + by + cz + d = 0$ <p>is given by</p> $\frac{ ax_0 + by_0 + cz_0 + d }{\sqrt{a^2 + b^2 + c^2}}$ <ul style="list-style-type: none"> • Learners take notes in their books • Individually learners work out the application activity 2.3.2. and finally they make a correction on the chalk board. <p>Expected answers</p> <p>(Refer to solution of application activity 2.3.2, in TG)</p> <ul style="list-style-type: none"> • Learners individually write down the assigned homework activities 	<ul style="list-style-type: none"> • Critical thinking and problem solving skills are developed through analyzing and solving real life Mathematical problem: e.g. finding the distance from the floor to the ceiling of a house • Financial education is addressed through good management of the money for transport by choosing the shortest distance
<p>Observation on lesson delivery</p>	<p>To be completed after receiving the feed-back from the learners (what did the learners like, what challenged them...)</p>		

PART III: UNIT DEVELOPMENT

UNIT 1

SEQUENCES AND SERIES

1.1 Key unit competence:

Apply arithmetic and geometric sequences to solve problems in financial mathematics.

1.2 Prerequisites

Student-teachers will easily learn this unit, if they have a good background on Arithmetic (Unit 1 Year 1), Equations and inequalities (unit 2 Year 1), and on limits of functions (unit 4 Year 1).

1.3 Cross-cutting issues to be addressed

Inclusive education (promote education for all while teaching)

Peace and value Education (respect others' view and thoughts during class discussions)

Gender (provide equal opportunity to boys and girls in the lesson)

1.4 Guidance on introductory activity

- Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 1.0 found in unit 1 of student's book;
- Guide students to read and analyse the questions insisting on the analysis of the given data and to determine the number of insects that will be there in second, third, fourth,...nth generation.
- Invite some group members to present groups' findings, then try to harmonize their answers; try to insist on the list formed by the number of insects at any generation and the generalisation (number of insects at nth generation).
- Basing on student-teachers' experience, prior knowledge and abilities

shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be learnt in this unit.

Answer for introductory activity:

Number of insects in is given by:

$$1^{\text{st}} \text{ generation} \rightarrow 126, \quad 2^{\text{nd}} \text{ generation} \rightarrow 126 \times 2 = 252,$$

$$3^{\text{rd}} \text{ generation} \rightarrow 252 \times 2 = 504, \quad 4^{\text{th}} \text{ generation} \rightarrow 1008,$$

$$\text{At } n^{\text{th}} \text{ generation} \rightarrow 126 \times 2^{(n-1)} = 126.n^{(n-1)}$$

$$5^{\text{th}} \text{ generation} \rightarrow 2016, \quad 6^{\text{th}} \text{ generation} \rightarrow 4032,$$

$$7^{\text{th}} \text{ generation} \rightarrow 8064, \quad 8^{\text{th}} \text{ generation} \rightarrow 16128,$$

$$9^{\text{th}} \text{ generation} \rightarrow 32256, \quad 10^{\text{th}} \text{ generation} \rightarrow 64512$$

1.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 1.	1
1	Generalities on sequences and series	To define sequences and series, infinite sequence and to determine terms of a sequence	2
2	Convergent and divergent sequence	Explore sequences and differentiate the convergent from a divergent sequence	1
3	Monotone sequences	Use basic concepts and formulas of sequences to identify the Monotonic sequences	1
4	Arithmetic sequence and its general term	Use basic concepts of sequences to determine terms of an arithmetic sequence	2

5	Arithmetic means	Use basic concepts of sequences to find an arithmetic mean of two numbers	1
5	Arithmetic series	Use basic concepts of sequences to calculate the sum of the first " n " terms of an arithmetic sequence	2
7	Harmonic sequences and its general term	Use basic concepts and formulas of sequences to find the terms of an harmonic sequence	1
8	Generalities on Geometric sequence and its general term	Use basic concepts and formulas of sequences to determine terms of a geometric sequence	2
	Geometric means	Use basic concepts and formulas of sequences to find a Geometric mean of two numbers	1
9	Geometric series	Use basic concepts and formulas of sequences to calculate the sum of the first " n " term of a Geometric sequence	2
11	Infinity Geometric series	Explain infinity Geometric series and determine their sum.	1
12	Application of sequences on Economics	Apply the concepts of sequences and series to solve problems related to finance or Economics.	3
13	End unit assessment		1
	Total		21

Lesson 1: Generalities on sequences and series

a) Learning objectives

Define sequences and series, infinite sequence and to determine terms of a sequence

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the arithmetic and on equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.1 in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification as they have to give the fraction that represents the part they see when they fold a paper n times;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the fraction that represents the part they see when they fold the paper n times;
- Use different probing questions and guide students to explore the content and examples given in the student's book and lead them to discover how to define sequences, series, infinite sequence, and to determine terms of a given sequence.
- After this step, guide students to do the application activity 1.1 and evaluate whether lesson objectives were achieved.

Answer for activity 1.1

When they fold once they see $\frac{1}{2}$; When they fold twice they see $\frac{1}{2^2}$; When they fold 3 times they see $\frac{1}{2^3}$; When they fold n times they see $\frac{1}{2^n}$; ... When they fold 10 times they see $\frac{1}{2^{10}}$; ...

The list of the fractions obtained is: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$

Answer for application activity 1.1

$$\text{Given } \{u_n\}: \begin{cases} u_0 = 1 \\ u_n = \frac{2n^2}{n^2 + 1} \end{cases}$$

$$1) u_1 = \frac{2 \times 1^2}{1^2 + 1} = 1; u_2 = \frac{2 \times 2^2}{2^2 + 1} = \frac{8}{5}; u_3 = \frac{2 \times 3^2}{3^2 + 1} = \frac{18}{10}$$

$$2) \text{ The five first terms of } \{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{+\infty} \text{ are: } \sqrt{2} - 1, \sqrt{2} - \sqrt{3}, 2 - \sqrt{3}, \sqrt{5} - 2, \sqrt{6} - \sqrt{5}$$

$$3) \{2n-1\}_{n=1}^{+\infty} \text{ or } \{2n+1\}_{n=0}^{+\infty}$$

Lesson 2: Convergent or divergent sequences

a) Learning objectives

Explore sequences and differentiate the convergent from a divergent sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the Arithmetic and on equations and inequalities learnt in Year 1 Unit 1 & Unit 4.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.2 in their Mathematics books, they have to determine value of a sequence as n approaches to $+\infty$;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to use the limit of the sequence at the infinity;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to differentiate the convergent from a divergent sequence.
- After this step, guide students to do the application activity 1.2 and evaluate whether lesson objectives were achieved.

Answer for activity 1.2

$$1) \lim_{n \rightarrow \infty} \left(\frac{3n^2 - 1}{n^2} \right) = 3$$

$$2) \lim_{n \rightarrow \infty} n^2 = +\infty$$

Answer for application activity 1.2

1. $\{2 + (0.1)^n\}$ converges to 2

2. $\left\{ \frac{1-2n}{1+2n} \right\}$ converge to -1

3. $\left\{ \frac{1-5n^4}{n^4 + 8n^3} \right\}$ converges to -5

4. $\{-1^n\}$ diverge

5. $\left\{ \frac{2n}{\sqrt{3n+1}} \right\}$ converges to $\frac{2}{\sqrt{3}}$

6. $\frac{\sqrt{7n^2+2}}{n^3+8}$ converge to 0.

Lesson 3: Monotonic sequences

a) Learning objectives

Use basic concepts and formulas of sequences to identify the Monotonic sequences

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on the convergent or divergence sequence (lesson 2 of this unit).

d) Learning activities:

- Invite student teachers to work in groups and do the activity 1.3 in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the increasing sequence, the decreasing sequence, or non-increasing sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts and formulas of sequences to identify the monotonic sequences as increasing sequences or the decreasing sequences.
- After this step, guide students to do the application activity 1.3 and evaluate whether lesson objectives were achieved.

Answer for activity 1.3

1. 1, 2, 3, 4, 5, 6, ... is an ascending
2. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ is a descending
3. 1, -1, 1, -1, 1, ... it is both, not monotonic
4. 2, 2, 2, 2, 2, 2, ... is neither, it is stationary.

Answer for application activity 1.3

- 1) 1, 2, 3, ..., n, ... is increasing
- 2) $\left\{ \frac{n}{n+1} = 1 - \frac{1}{n+1} \right\}$ is decreasing.
- 3) $\left\{ \frac{1}{2^n} \right\}$ is decreasing
- 4) 3, 3, 3, 3, ... non increasing
- 5) 1, -1, 1, -1, ... not monotonic.

Lesson 4: Arithmetic sequence and its general term

a) Learning objectives

Use basic concepts of sequences to determine terms of an arithmetic sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson 1, lesson 2 and the lesson 3 of this unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.4 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to identify the common difference of an arithmetic sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to determine the general term and different terms of an arithmetic sequence.
- After this step, guide students to do the application activity 1.4 and evaluate whether lesson objectives were achieved.

Answer for activity 1.4

- a) $\{u_n\} : 5, 5+3, 5+3+3, 5+3 \times 3, \dots, 5+3n, \dots$ This constant is $d = 3$
- b) $\{v_n\} = 26; 26+5; 26+(5 \times 2), 26+(5 \times 3), \dots$ This constant is $d = 5$
- c) $\{w_n\} : 20, 20-2, 20-2 \cdot 2, 20-2 \cdot 3, \dots, 20-2 \cdot n, \dots, 0$. This constant is $d = -2$

Answer for application activity 1.4

1) If $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, then

$$\frac{2}{a+c} = \frac{1}{a+b} + \frac{1}{b+c}$$

$$\Leftrightarrow \frac{2}{a+c} = \frac{2b+c+a}{(a+b)(b+c)}$$

$$\Leftrightarrow 2(ab+ac+b^2+bc) = (a+c)(2b+a+c)$$

$$\Leftrightarrow 2ab+2ac+2b^2+2bc = 2ab+a^2+ac+2bc+ac+c^2$$

$$\Leftrightarrow 2b^2 = a^2 + c^2$$

Also a^2, b^2, c^2 are 3 consecutive terms of an arithmetic progression if $2b^2 = a^2 + c^2$.

Thus, if $\frac{1}{a+b}, \frac{1}{a+c}, \frac{1}{b+c}$ are 3 consecutive terms of an arithmetic progression, it will be the same for a^2, b^2, c^2 .

2) Let the second term be x . The first term is $x - d$ and the third term is $x + d$ where d is the common difference.

$$\text{Now, } x - d + x + x + d = 30 \Rightarrow 3x = 30 \text{ or } x = 10$$

$$\text{Also, } (x - d)^2 + x^2 + (x + d)^2 = 332$$

$$\text{Or } (10 - d)^2 + 100 + (10 + d)^2 = 332$$

$$\text{Or } 2d^2 = 32 \Rightarrow d = \pm 4$$

i) Therefore, the progression is 6, 10, 14 or 14, 10, 6

3) We need to find x such that $(1 + x)^2$, $(q + x)^2$, and $(q^2 + x)^2$ form an arithmetic progression.

$$2(q + x)^2 = (1 + x)^2 + (q^2 + x)^2$$

$$\Leftrightarrow 2(q^2 + 2qx + x^2) = 1 + 2x + x^2 + q^4 + 2xq^2 + x^2$$

$$\Leftrightarrow 2q^2 + 4qx + 2x^2 = 1 + 2x + x^2 + q^4 + 2xq^2 + x^2$$

$$\Leftrightarrow 2q^2 + 4qx = 1 + 2x + q^4 + 2xq^2$$

$$\Leftrightarrow 4qx - 2x - 2xq^2 = 1 - 2q^2 + q^4$$

$$\Leftrightarrow x(4q - 2 - 2q^2) = (1 - q^2)^2$$

$$\Leftrightarrow x = \frac{(1 - q^2)^2}{-2(1 - 2q + q^2)}$$

$$\Leftrightarrow x = \frac{(1 - q)^2(1 + q)^2}{-2(1 - q)^2}$$

$$\Leftrightarrow x = \frac{(1 + q)^2}{-2}$$

$$\text{Thus, } x = \frac{-(1 + q)^2}{2}$$

Lesson 5: Arithmetic Means of an arithmetic sequences

a) Learning objectives

Use basic concepts of sequences to find an arithmetic mean of two numbers.

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to the lesson 1, lesson 2 , the lesson 3 and the lesson 4 of this unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.5 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to explain the arithmetic means of such sequences;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to find arithmetic means of two terms of an arithmetic sequences.
- After this step, guide students to do the application activity 1.5 and evaluate whether lesson objectives were achieved.

Answer for activity 1.5

$$u_1 = 2, u_7 = 20$$

$$u_n = u_1 + (n-1)d \Rightarrow u_7 = u_1 + 6d$$

$$\Rightarrow 20 = 2 + 6d$$

$$\Rightarrow d = 3$$

$$u_1, A = u_2 = 2 + 3 = 5, B = u_3 = 2 + 6 = 7, C = u_4 = 2 + 9 = 11, D = u_5 = 2 + 12 = 14$$

$$E = u_6 = 2 + 15 = 17, u_7 = 2 + 18 = 20.$$

Answer for application activity 1.5

1. $-3, -1, 1, 3, 5, 7$
2. $2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32$
3. 16
4. 14
5. 0

Lesson 6: Arithmetic Series

a) Learning objectives

Use basic concepts of sequences to calculate the sum of the first " n " terms of an arithmetic sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 5 of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.6 in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;

- As a tutor, harmonize the findings from presentation and guide students to establish how to determine the sum s_n for the first n terms of the arithmetic sequence $\{u_n\}$;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to use basic concepts of sequences to calculate the sum of the first " n " terms of different arithmetic sequences.
- After this step, guide students to do the application activity 1.6 and evaluate whether lesson objectives were achieved.

Answer for activity 1. 6

The sequence 2,5,8,11,14,....

a) The first term is 2, $d = 3$, the general term $u_n = 2 + 3(n - 1)$

b) $S_6 = 2 + (2 + 3) + (2 + 2.3) + .. + (2 + 5.3) = 6.2 + 3(1 + 2 + 3 + 4 + 5) = 6.2 + 3 \frac{6(5)}{2}$

as we apply the sum of all first positive integers not greater than n .

c) $s_n = 2n + 3(1 + 2 + 3 + \dots + n) = 2n + 3 \frac{n(n-1)}{2}$.

Answer for application activity 1. 6

1) $2n(n+3)$

2) 860

3) 11

4) The bottom row requires 100 tiles and the top row, 50 tiles. Since each successive row requires two less tiles, the total number of tiles required is

$$S = 100 + 98 + 96 + \dots + (100 - 2 \cdot (n - 1)) + \dots + 50$$

As $100 - 2 \cdot (n - 1) = 50$, we have:

$$100 - 2n + 2 = 50 \text{ Which gives } n = 26.$$

This is the sum of an arithmetic sequence; the common difference is -2 . The number of terms to be added is $n = 21$ with the first term $u_1 = 100$ and the last term $u_n = 50$

The sum S is $S = \frac{n}{2}(u_1 + u_n) = \frac{6}{2}(100 + 6) = 1950$.
In all, 1950 tiles will be required.

Lesson 7: Harmonic sequences and its general terms

a) Learning objectives

Use basic concepts and formulas of sequences to find the terms of an harmonic sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this unit (from the lesson 1 to lesson 6 of this unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.7 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation
- After this step, guide students to do the application activity 1.7 and evaluate whether lesson objectives were achieved. and guide students to discover that a harmonic sequence is made by the reciprocals of terms of the arithmetic sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find the terms of a harmonic sequence

Answer for activity 1.7

a) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots, \frac{1}{2n}, \dots$

b) Its first term is $\frac{1}{2}$, the third is $\frac{1}{6}$, the general term is $\frac{1}{2n}$.

The relationship between two consecutive terms is that

$$u_{n+1} - u_n = \frac{1}{2n+2} - \frac{1}{2n} = \frac{-1}{(2n+2)(n)}$$

$$n(u_{n+1} - u_n) = u_{n+1} \Leftrightarrow u_n = \left(1 - \frac{1}{n}\right)u_{n+1}$$

Answer for application activity 1.7

1. The sequence is $6, 4, 3, \frac{12}{5}, 2, \frac{12}{7}, \frac{3}{2}, \frac{4}{3}$. The 4th term is $\frac{12}{5}$, 8th term is $\frac{4}{3}$

2. $3, \frac{90}{23}, \frac{90}{16}, 10$

4. $\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{4}, \frac{1}{\sqrt{5}}, \dots, \frac{\sqrt{5}}{13}$

5. 6 and 2

6. $\frac{60}{16-n}$

Lesson 8: Generalities on Geometric sequences and their general terms

a) Learning objectives

Use basic concepts and formulas of sequences to determine terms of a geometric sequence.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 7 of this unit) and to the equations and

inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.8 in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to make a geometric sequence of numbers, its general term and the common ratio;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine terms of a geometric sequence.
- After this step, guide students to do the application activity 1.8 and evaluate whether lesson objectives were achieved.

Answer for activity 1.8

Learners will take a piece of paper and cut it into two equal parts. Take one part and cut it again into two equal parts. When they continue in this manner the fraction corresponding to the obtained parts according to the original piece of paper are as follows:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$$

Answer for application activity 1.8

1. 98304

2. $\frac{\sqrt[5]{16}}{4}$

3. -21.87

4. $\frac{1}{16}$

$$5. \quad (u_n): u_n = \frac{1}{2} \left(\frac{3}{2} \right)^{n-1}, u_8 = \frac{2187}{256}$$

$$6. \quad p = 5$$

Lesson 9: Geometric means of a geometric sequence

a) Learning objectives

Use basic concepts and formulas of sequences to find geometric means of two numbers.

b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and the equations and inequalities learnt in Year 1 Unit 1 & Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.9 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarifications on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to explain geometric means of two terms in a geometric sequence;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to find geometric mean of two numbers.
- After this step, guide students to do the application activity 1.9 and evaluate whether lesson objectives were achieved.

Answer for activity 1.9

$$u_1 = 1, u_6 = 243$$

$$u_n = u_1 \cdot r^{n-1} \Rightarrow u_6 = u_1 \cdot r^5$$

$$\Rightarrow 243 = r^5$$

$$\Rightarrow 3^5 = r^5$$

$$\Rightarrow r = 3$$

The sequence is 1, 3, 9, 27, 81, 243

Answer for application activity 1.9

$$1) \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$$

$$2) 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \frac{2}{243}, \frac{2}{729}$$

$$3) \text{ a) } 14 \quad \text{ b) } \frac{9}{2}$$

$$4) 12 \text{ and } 108$$

$$5) 64 \text{ and } 4$$

Lesson 10: Geometric Series

a) Learning objectives

Use basic concepts and formulas of sequences to calculate the sum of the first "n" terms of a Geometric sequence

b) Teaching resources

Learner's book and other reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1 & Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.10 in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation and guide students to determine the sum of n terms of the sequence they found;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to Use basic concepts and formulas of sequences to calculate the sum of the first " n " term of a Geometric sequence.
- After this step, guide students to do the application activity 1.10 and evaluate whether lesson objectives were achieved.

Answer for activity 1.10

a)

Place	Money gotten
1 st	$u_1 = 100,000Frw$
2 nd	$u_2 = \frac{1}{2}(100,000Frw) = 50,000Frw$
3 rd	$u_3 = \frac{1}{2}(50,000Frw) = 25,000Frw$
4 th	$u_4 = \frac{1}{2}(25,000Frw) = 12,500Frw$
5 th	$u_5 = \frac{1}{2}(12,500Frw) = 6,250Frw$

b) The total of their money is

$$\begin{aligned} & u_1 + u_2 + u_3 + u_4 + u_5 \\ & = 100,000 + \frac{1}{2}(100,000) + \frac{1}{2^2}100,000 + \frac{1}{2^3}100,000 + \frac{1}{2^4}100,000 \end{aligned}$$

c) The money for the first is $100,000Frw$, this is greater than the money for the fifth student which is $6,250Frw$. When you win at the first place, the

d) The money for the student who passed at the n^{th} place is $u_n = \frac{1}{2^{(n-1)}}100,000Frw$

To determine the total amount of money for n students, students teachers will make the sum from $u_1 = 100,000Frw$ to $u_n = \frac{1}{2^{(n-1)}}100,000Frw$ and they will find

$$S_n = 100000 \times \frac{\left(1 - \left(\frac{1}{2}\right)^n\right)}{\left(1 - \frac{1}{2}\right)}$$

Answer for application activity 1.10

1) 21.25

2) 39.1

3) $1, \frac{5}{4}$

4) -32

5) The interest is compounded each quarter. So $n=1$ and the interest rate per period is $\frac{6\%}{4}$ or 1.5%. The common ratio r for the geometric series is then $(1+0.015)$ or 1.015

The first term u_1 in this series is the account balance at the end of the first quarter. Thus, $u_1 = 500(1.015)$ or 507.05.

Apply the formula for the sum of a geometric series $S_n = \frac{u_1 - u_1 r^n}{1 - r}$, we have:

$$S_n = \frac{507.5 - 507.5(1.015)^4}{1 - 1.015}; n = 4; r = 1.015$$

$$S_4 = 2076.13$$

Alexis's account balance at the end of one year is \$2076.13.

Lesson 11: Infinity geometric series and its convergence

a) Learning objectives

Explain infinity geometric series and determine their sum.

b) Teaching resources

Student's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.11 in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group with different working steps to present their work;
- As a tutor, harmonize the findings from presentation and guide students to determine the limit of a given series;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to explain infinity Geometric series and determine the their sum.
- After this step, guide students to do the application activity 1.11 and evaluate whether lesson objectives were achieved.

Answer for activity 1.11

$$\text{a) } \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{5 \left(1 - \left(\frac{1}{2} \right)^n \right)}{\left(1 - \left(\frac{1}{2} \right) \right)} \right] = 10$$

b) If $-1 < r < 1$, $-1 < r < 1$, thus $\lim_{n \rightarrow \infty} \frac{u_1(1-r^n)}{1-r} = \frac{u_1}{1-r}$

Answer for application activity 1.11

1. a) $\sum_{n=1}^{\infty} 10 \left(1 - \frac{3x}{2}\right)^n$ this exists if $\left|1 - \frac{3x}{2}\right| < 1$. Solving this inequality

we find $0 < x < \frac{4}{3}$

b) If $x = 1.3$, the sum can be $\frac{u_1}{1-r} = \frac{10}{1 - \left(1 - \frac{3(1.3)}{2}\right)} = \frac{200}{39}$

2) The decimal $0.999\dots = 0.9 + 0.09 + 0.009 + \dots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots$ is an

infinitegeometric series. We will write it in the form $\sum_{k=1}^{\infty} a_1 r^{k-1}$, then

$$0.999\dots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots = \sum_{k=1}^{\infty} \frac{9}{10^k} = \sum_{k=1}^{\infty} \frac{9}{10 \cdot 10^{k-1}} = \sum_{k=1}^{\infty} \frac{9}{10} \left(\frac{1}{10}\right)^{k-1}$$

Now we can compare this series to $\sum_{k=1}^{\infty} a_1 r^{k-1}$ and conclude that $a_1 = \frac{9}{10}$ and

$$r = \frac{1}{10} \text{ Since } |r| < 1, \text{ the series converges and its sum is } 0.999\dots = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

The repeating decimal $0.999\dots$ equal 1.

3) $\sum_{n=1}^{\infty} 2 \left(\frac{2}{3}\right)^{n-1} = 2 + \frac{4}{3} + \frac{8}{9} + \dots$ This geometric series converges as $\frac{2}{3} < 1$. Its sum

is its limit $\frac{2}{1 - \frac{2}{3}} = 6$

Lesson 12: Application of sequences in real life

a) Learning objectives

Apply the concepts of sequences and series to solve problems related to finance or Economics.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they refer to all previous lessons of this Unit (from the lesson 1 to lesson 8 of this unit) and on the equations and inequalities learnt in Year 1 Unit 1& Unit 4 respectively.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 1.12 found in their Mathematics books;
- Visit each group for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to solve problems related to finance or Economics.
- After this step, guide students to do the application activity 1.12 and evaluate whether lesson objectives were achieved.

Answer for activity 1.12

Refer to the student's book to highlight some applications of sequences and series. They are also used to determine *the monthly payments made to pay off an automobile or home loan with interest portion, the list of maximum daily temperatures in one area for a month, etc.* Sequences are used in calculating *interest, population growth, half-life and decay in radioactivity.*

Answer for application activity 1.1 2

1) $P = 1300, r = 7\% = 0.07, k = 1$

$$A = 1300 \left(1 + \frac{0.07}{1} \right)^{1 \times 17} = 4106.46$$

The account will contain \$4,106.46.

2) We apply the formula $P = P_0 e^{rt}$ With initial population $P_0 = 153,800$ rate of growth $r = 0.05$, and time $t = 2000 - 1970 = 30 \text{ years}$. Thus, a prediction for the population of the city in the year 2000 is $153,800 e^{(0.05)(30)} = 153,800 e^{1.5} \approx 689,284$

3) This is an ordinary annuity with $n = 30$ annual deposits of $P = \$2000$.

The rate of interest per payment period is $i = \frac{0.04}{1} = 0.04$.

The amount A of the annuity after 30 deposits is

$$A = P \cdot \left[\frac{(1+i)^n - 1}{i} \right] = 2000 \left[\frac{\left((1+0.04)^{30} - 1 \right)}{0.04} \right] = 112,169.88.$$

4) This is an example of a sinking fund. The payment P required twice a year to accumulate 4,000,000 Frw in 12 years (24 payments at a rate of interest of $i = \frac{0.04}{2} = 0.02$ per payment period) obeys:

$$A = P \cdot \left[\frac{(1+i)^n - 1}{i} \right]$$

$$4,000,000 = P \left[\frac{\left((1+0.02)^{24} - 1 \right)}{0.02} \right]$$

$$4,000,000 = P(30.4218)$$

$$P = 131,484.39 \text{ Frw}$$

The school leader will need to make a payment of \$131,484.39 every 6 months to redeem the bonds in 12 years.

1.6 Unit summary

1. Numbers in sequence are denoted $u_1, u_2, u_3, \dots, u_{n-1}, u_n, \dots$ and shortly $\{u_n\}$.

The natural number n is called **term number** and value u_n is called a **general term** of a sequence and the term u_1 is the **initial term**.

2. As a sequence continues indefinitely, it can be denoted as $\{u_n\}_{n=1}^{+\infty}$.

3. A sequence $\{u_n\}$ is said to be

- increasing if $u_1 < u_2 < u_3 < \dots < u_n < \dots$
- non decreasing if $u_1 \leq u_2 \leq u_3 \leq \dots \leq u_n \leq \dots$
- decreasing if $u_1 > u_2 > u_3 > \dots > u_n > \dots$
- non increasing $u_1 \geq u_2 \geq u_3 \geq \dots \geq u_n \geq \dots$

4. A numerical sequence is said to be **convergent** if the limit exist whereas if the limit does not exist (or is infinity) the sequence is said to be **divergent**. A number L is called a **limit** of a numerical sequence $\{u_n\}$ if $\lim_{n \rightarrow \infty} u_n = L$

5. One of the most famous and important of all diverging series is the **harmonic series**,

$$\sum_{k=1}^{+\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

6. Sequences of numbers that follow a pattern of adding a fixed number from one term to the next are called **arithmetic sequences** or **arithmetic progressions**.

7. For an arithmetic sequence u_{n-1}, u_n, u_{n+1} , we have $2u_n = u_{n-1} + u_{n+1}$.

8. If u_p is any p^{th} term of a sequence then the n^{th} term is given by $u_n = u_p + (n - p)d$

9. The sum of first n terms of a finite arithmetic sequence with initial term u_1 is given by $s_n = \frac{n}{2}[u_1 + u_n]$

10. Sequences of numbers that follow a pattern of multiplying a fixed number from one term to the next are called **geometric sequences**.

11. For a geometric sequence u_{n-1}, u_n, u_{n+1} , we have $u_n^2 = u_{n-1} \cdot u_{n+1}$

12. The n^{th} term, u_n , of a geometric sequence $\{u_n\}$ with common ratio r and initial term u_1 is given by $u_n = u_1 r^{n-1}$

13. The sum of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by: $s_n = \frac{u_1(1-r^n)}{1-r}$ with $r \neq 1$

14. Also, the product of first n terms of a geometric sequence with initial term u_1 and common ratio r is given by $P_n = (u_1)^n r^{\frac{n(n-1)}{2}}$

15. For the formula $s_n = \frac{u_1(1-r^n)}{1-r}$

If $-1 < r < 1$, $S_\infty = \frac{u_1}{1-r}$

Sequences are used in calculating interest, population growth, half-life and decay in radioactivity.

1.7. Additional information for the teacher

For the educative action of the teacher to be effective (in order to respond to all aspects of the learners' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding that is required from the learner. It is therefore, it is imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

Here the tutor has to emphasize the application of sequences and series in solving problems related to finance or Economics in real life situations.

1.8 Answer for end unit assessment

1) $0, -\frac{1}{4}, -\frac{2}{9}, -\frac{3}{16}$ b) $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}$ c) $1, 3, 1, 3$

2) a) $(-1)^n, n = 0, 1, 2, \dots$ b) $n^2 - 1, n = 1, 2, 3, \dots$ c) $4n - 3, n = 1, 2, 3, \dots$

3) a) converges to $\sqrt{2}$ b) Converges to 0 c) Converges to 1

4) £11 million

5) 2048000

6) $99.8^\circ F$

7) 1800

1.9 Additional activities

1.9.1 Remedial activities

1. Find the 20th term of the following arithmetic progressions and calculate the sum of first 20 terms

a. $2, 6, 10, 14, \dots$

b. $-5, -3.5, -2, -0.5, \dots$

Solution:

a) $u_{20} = 78, S_{20} = 800$ b. $u_{20} = 23.5, S_{20} = 185$

2. In an arithmetic progression, the sum of the 8th and 14th terms is 50. The 5th term is equal to 13. Find that progression.

Solution:

$5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, \dots$

1.9.2 Consolidation activities

1) Find x consecutive integers numbers known that the first number is 8 and their sum is x^3 .

Solution: 8, 9, 10

2) In a geometric progression, we have

a. $u_1 = 3, r = 4, n = 5$; find u_n and sum of terms.

b. $u_n = \frac{3}{64}, u_1 = 12, n = 9$; find r and sum of terms.

Solution: a. $u_5 = 768, S_5 = 341$ b. $r = \frac{1}{2}, S_n = \frac{1533}{64}$

1.9.3 Extended activities

1. In an arithmetic progression, we have

a. $u_1 = 4, d = 2, n = 8$; find u_n and sum of terms

b. $d = 4, u_n = 39, n = 10$; find u_1 and sum of terms

c. $u_1 = 3, u_n = 21, S_n = 120$; find n and d

d. $u_n = 199, n = 100, S_n = 10000$; find u_1 and d .

Solution:

a) $u_8 = 18, S_8 = 88$ b) $u_1 = 3, S_{10} = 210$ c) $n = 10, d = 2$ d. $u_1 = 1, d = 2$

2. A mine worker discovers an ore sample containing 500 mg of radioactive material. It is discovered that the radioactive material has a half life of 1 day. Find the amount of radioactive material in the sample at the beginning of the 7th day.

Solution:

Half-life of one-day means that the half of the amount remains after 1 day.

Begin of day 1:	Begin of day 2:	Begin of day 3:	...
500 mg	250 mg	125 mg	

End of day 1:	End of day 2:	End of day 3:	...
250 mg	125 mg	62.5 mg	

Decide to either work with the “beginning” of each day, or the “end” of each day, as each can yield the answer. Only the starting value and number of terms will differ. We will use “beginning”: The beginning of the 7th day corresponds to

$$u_7 = 500 \left(\frac{1}{2} \right)^{7-1} = 7.8125 \text{ mg} .$$

UNIT 2

POINTS, STRAIGHT LINES AND PLANES IN 3D

2.1 Key unit competence

Extend understanding and use geometric presentations of lines and planes, locate points and determine equation of a line and plane in 3D.

2.2 Prerequisite

Student-teachers will perform better in this unit if:

- They have a good background on “Points and straight lines” (unit9 of year1)
- They can represent easily shapes of three dimensions, such as cuboids (Ordinary level mathematics syllabus, unit7)
- Perform numerical calculations correctly, mentally or using a calculator.

2.3 Cross-cutting issues to be addressed

- **Inclusive education:** promote the participation of all student-teachers while teaching
- **Peace and value Education:** During group activities, the teacher will encourage student teachers discuss without confrontation their points of view
- **Cooperation:** the teacher will encourage student teachers to help each other and to respect opinions of colleagues.
- **Gender:** Give equal opportunities to all learners (girls and boys) to participate actively in all learning activities from the beginning to the end of the lesson.

2.4 Guidance on introductory activity

- Guide the student-teachers to form small groups and to work on the introductory activity;

- Through class discussions, let student-teachers discover the geometric concepts involved in the activity and the relationship between them;
- Walk around to monitor the work of each group and to assist any group in need;
- After a given time, invite student teachers to present their findings and harmonize them.
- Ensure that the learners eager to study the unit and sustain their curiosity

Answer for introductory activity

Answers will vary from one group to another. Learners may observe the faces of the building or the sides of the bed can illustrate plans, the edges, where two faces intersect may illustrate lines and the intersection of edges may illustrate points.

2.5. List of lessons and sub-heading

No	Lesson title	Learning objectives	Periods
0	Introduction activity	To arouse the curiosity of student teachers on the content of unit 2	1
1	Cartesian coordinates of a point and its presentation in 3D	Define the position of a point in 3D by its coordinates	2
2	Distance between 2 points in 3D	Determine the distance between two points in 3D	2
3	Mid-point of a line segment	Locate the mid - point of a segment in 3D	2
4	Vector in 3D and vector product	Describe a vector in 3 D and determine the vector product of 2 vectors	2
5	Equation of Line in 3D (Vector equation, parametric equations, Cartesian equation)	Determine equations of a straight line (vector equation, parametric equation, Cartesian equation).	3
6	Distance from a point to a line	Calculate accurately the distance from a point to a line	2

7	Distance between two lines	Calculate accurately the distance between two lines	2
8	Equation of a Plane in 3D (Vector equation, Parametric equations, Cartesian equation)	Equation of Line in 3D	4
9	Distance between a point and a plane, line and plane and between two planes	Calculate accurately the distance between a point and a plane, line and plane and between two planes	3
		Practical activity: to measure the distance between the floor and the ceiling of a house	1
10	Angle (between two lines, between a line and a plane between two planes)	Calculate the angle (between two lines, between a line and a plane or Between two planes) using dot and vector product	2
11	End unit assessment		2
Total number of periods in this unit			28

Lesson 1: Cartesian coordinates of a point and its presentation in 3D

a. Learning objective:

Define the position of a point in 3D by its coordinates

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other Reference textbooks to facilitate research.

c. Prerequisites/Revision/Introduction:

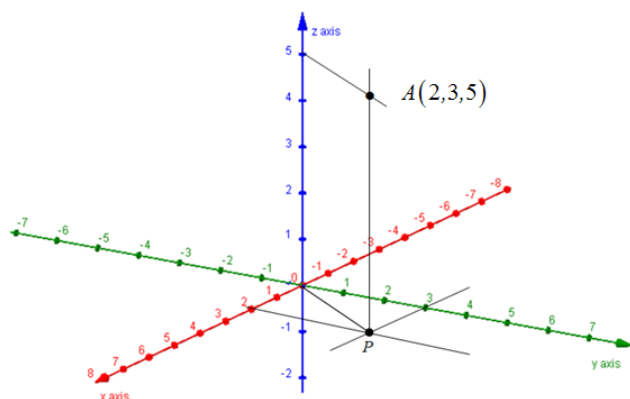
Student-teachers will learn better this lesson if they have a good back ground on:

- The representation, in the plane, of three dimensional shapes, such as parallelepiped, studied in Ordinary level, year one, unit7.
- The description of the position of an object with respect to the observer(left-right; below-above; in front-behind)

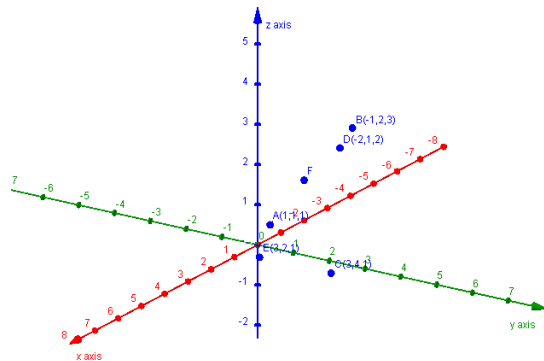
d. Learning activities

- Have the learners organized themselves into small groups under your management;
- Ask them to discuss the activity2.1.1 from student-teacher's book;
- After a given time, choose randomly a group to present his/her findings to the whole class and note how the members are organizing the presentation;
- During the harmonization, help student-teachers discover that the coordinates of a point describe the way to reach the point starting from the origin and moving successively x units in the direction of OX , then y units in the direction of OY and z units in the direction of OZ ;
- Guide students to explore examples and the content given in the student's book to enhance how to represent points in 3D;
- Request learners to practice individually through application activity 2.1.1 and go round to provide assistance where needed;
- Assess the learners' competences and evaluate whether lesson objectives were achieved.

Answers to activity 2.1.1



Answer to application activity 2.1.1



Lesson 2: Distance between two points in 3D

a. Learning objective:

Determine the distance between two points in 3D

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other Reference textbooks to facilitate research

c. Prerequisites/Revision/Introduction:

Student-teachers will learn better this lesson if they have mastered:

- The representation of points of 3D in a plane, as covered in lesson 1 of this unit;
- The use of a ruler to measure the distance between two points as studied in ordinary level senior one, unit 7;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Start by requesting learners to organize themselves into small groups from given criteria;
- Ask them to discuss the activity 2.1.2 from student-teacher's book for a while;
- Let a group present his/her findings to the whole class, after a proper sitting arrangement of the learners, and after ensuring their readiness to follow;

- During the harmonization, help student-teachers to discover that the distance between two points is a measure of how far is a point from another and how it is calculated;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Have learners practice individually, by solving application activity 2.1.2 and provide assistance for learners in need;
- Assess whether learning has occurred on not to feed forwards.

Answers to activity 2.1.2

- B is closer to A than to C
- The concept used is "Distance"

Answers to application Activity 2.1.2

a. $\sqrt{5}; \sqrt{5}$

b. Isosceles triangle

Lesson 3: Midpoint of a line segment

a. Learning objective:

Locate the mid - point of a line segment in 3D

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other Reference textbooks to facilitate research

c. Prerequisites/Revision/Introduction:

Student-teachers will learn better this lesson if they have mastered:

- The representation of points of 3D in a plane;
- The distance between two points, as covered in lessons 1 and 2 of this unit;
- The use of a pair of compasses and a ruler to plot the midpoint of two points as studied in Ordinary level, unit7, year1;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Ensure that the learners are organized into small groups;
- Ask them to discuss the activity 2.1.3 from student-teacher's book for a while;
- Let a group present his/her findings to the whole class, after a proper sitting arrangement of the learners and their readiness to follow;
- During the harmonization, help student-teachers to discover that the midpoint is the point half way between the two points and how its coordinates are obtained;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Have learners practice individually, by solving application activity 2.1.3 and provide assistance for learners in need;
- Assess whether learning has occurred or not to feed forwards

Answer to activity 2.1.3

I is equally distant to A and to B

Answer to application Activity 2.1.3

a. $\left(0, \frac{7}{2}, \frac{11}{2}\right)$ b. $\left(6, \frac{5}{2}, -\frac{1}{2}\right)$ c. $\left(-\frac{7}{2}, \frac{11}{2}, 5\right)$

Lesson 4: Vectors in 3D and vector product

a. Learning objective:

Describe a vector in 3 D and determine the vector product of 2 vectors

b. Teaching resources:

Student-teacher's book and other Reference textbooks containing the concept of vector product

c. Prerequisites/Revision/Introduction:

For student-teachers to master well the concepts in this lesson, the following prerequisites are required:

- Differentiating between a scalar quantity and a vector quantity;
- The distance between two points, as covered in lessons 1 and 2 of this unit;
- The right-hand rule: the axes are such that the index lies on the OX-axis, the middle finger lies on the OY-axis and the thumb lies on the OZ-axis.
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Instruct the learners to form small groups;
- Ask them to discuss the activity 2.2.1 from student-teacher's book for a while;
- Choose at random a group to present his/her findings to the whole class while other learners are following attentively;
- Have them notice that cross product is performed in three dimensions ONLY;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Through well-chosen questions, bring learners to summarize the main points of the lesson, ensure that learners mentioned how to calculate the dot product, and the cross product using determinants of order 2 as means;
- Let learners practice individually, by solving application activity 2.2.1. As they are working go around different learners and provide assistance where necessary;
- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.

Answers to activity 2.2.1

a. Scalars have only magnitude; vectors have both magnitude and direction

b. i. vector

ii. Vector

iii. Scalar

c. i. 0 ; right angled triangle at C

ii. $6(4\vec{i} + \vec{j} + 5\vec{k})$; $3\sqrt{42}$

Answers to application activity 2.2.1

a. $-2\vec{i} - 5\vec{j} - 4\vec{k}$

b. -37

Lesson 5: Equations of Lines in 3D

a. Learning objective:

Determine equations of a straight line (vector equation, parametric equation, Cartesian equation).

b. Teaching resources:

Student-teacher's book and other Reference textbooks containing Space Geometry of straight lines

c. Prerequisites/Revision/Introduction:

For better understanding and mastering of the concepts involved in the lesson, the following background points are needed:

- Points and straight lines, from year 1: unit9;
- Operations on vectors in 3D, from lesson4 of this unit;
- Calculation of determinant of order two, from lesson4 of this unit;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Set criteria for formation of small groups(ensure that not the same members are in the same group all the time, and not the same the same members for the same task) ;
- Let them discuss the activity2.2.2 from student-teacher's book for a while; as they are discussing, monitor the contribution of each member;
- Vary the technique of choosing the group for presentation (for example, taking into account the gender, where applicable) of the group work to the whole class

- Be vigilant on the sitting arrangement of the learners and on the effectiveness of the discipline during the presentation;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Ask clear and concise questions to learners to capture the main points of the lesson, ensure that learners mention the different equations of a straight line;
- Give learners time to practice individually, by solving application activity 2.2.1 .As they are working check the effectiveness of the work and give further explanation for learners in need;
- Get a clear picture of the lesson you gave to notice the strong points to keep up , so as to improve your next lesson.

Answers to activity 2.2.2

a.-A point and a non-zero vector

-Two distinct points

b. $\vec{AP} = t\vec{u}$, for some $t \in \mathbb{R}$

c. $\vec{r} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + t(a\vec{i} + b\vec{j} + c\vec{k})$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Answers to application activity 2.2.2

a. $\vec{r} = 2\vec{i} + \vec{j} + 4\vec{k} + t(\vec{i} - 3\vec{k})$

$$\begin{cases} x = 2 + t \\ y = 1 \\ z = 4 - 3t \end{cases}$$

$$\frac{x-2}{1} = \frac{z-4}{-3}; y=1$$

$$\text{b. } \vec{r} = \vec{i} + \vec{j} + 3\vec{k} + t(\vec{i} + 4\vec{j} + \vec{k})$$

$$\begin{cases} x = 1 + t \\ y = 1 + 4t \\ z = 3 + t \end{cases}$$

$$\frac{x-1}{1} = \frac{y-1}{4} = \frac{z-3}{1}$$

$$\text{c. } \vec{r} = 2\vec{i} + \vec{j} + 4\vec{k} + t(4\vec{i} + 2\vec{j} - 2\vec{k})$$

$$\begin{cases} x = 2 + 2t \\ y = 1 + 1t \\ z = 4 - t \end{cases}$$

$$\frac{x-2}{2} = \frac{y-1}{1} = \frac{z-4}{-1}$$

$$\text{d. } \vec{r} = \vec{i} + \vec{j} + \vec{k} + t(3\vec{i} + 4\vec{j} + 5\vec{k})$$

$$\begin{cases} x = 1 + 3t \\ y = 1 + 4t \\ z = 1 + 5t \end{cases}$$

$$\frac{x-1}{3} = \frac{y-1}{4} = \frac{z-1}{5}$$

Lesson 6: Distance from a point to a line

a. Learning objective:

Calculate accurately the distance from a point to a line

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other Reference textbooks containing Space Geometry

c. Prerequisites/Revision/Introduction:

For better understanding and mastering of the concepts involved in the lesson, the following background points are needed:

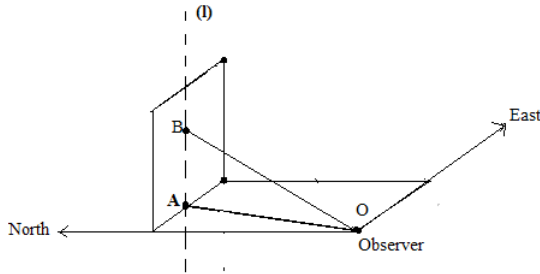
- Distance from a point to a line, from year 1: unit9;
- Distance between two points in 3D, from lesson2 of this unit;
- Equations of a line, from lesson 5 of this unit;
- The cross product of vectors, from lesson4 of this unit;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Instruct the learners to form small groups under your supervision;
- Ask them to discuss the activity2.2.3 from student-teacher's book for a while; Check the contribution of each member of the group in the discussion; ensure that the discussion is running smoothly without confrontation;
- Choose at random a group to present his/her findings to the whole class while other learners are following attentively;
- Have them notice that cross product is used in the calculation of the distance from a point to a line;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Through well-chosen questions, bring learners to summarize the main points of the lesson, ensure that learners mentioned how to calculate the distance between a point and a line;
- Let learners practice individually, by solving application activity 2.2.3.As they are working go around different learners and provide assistance where necessary;
- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.

Answers to activity 2.2.3

a.



b. OA

c. \overline{AB}

$$d. \|\overline{AO}\| = \|\overline{BO}\| \sin \theta = \frac{\|\overline{BO}\| \|\overline{AB}\| \sin \theta}{\|\overline{AB}\|} = \frac{\|\overline{OB} \wedge \overline{AB}\|}{\|\overline{AB}\|}$$

Therefore, the distance from point P to the line through points A and B is

$$\frac{\|\overline{PB} \wedge \overline{AB}\|}{\|\overline{AB}\|}$$

Answers to application activity 2.2.3

a. $2\sqrt{30}$ b. 0 c. $\frac{9\sqrt{42}}{7}$ d. $\frac{3\sqrt{10}}{2}$ e. $\frac{\sqrt{2}}{3}$

Lesson 7: Distance between two lines

a. Learning objective:

Calculate accurately the distance between two lines

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other reference textbooks containing Space Geometry

c. Prerequisites/Revision/Introduction:

For better understanding and mastering of the concepts involved in the lesson, the following background points are needed:

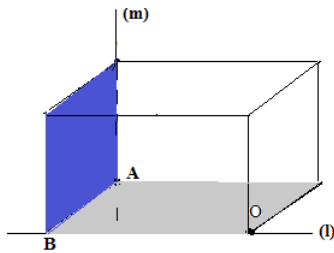
- Distance from a point to a line, from lesson 6 in this unit;
- Distance between two points in 3D, from lesson 2 of this unit;
- Equations of a line, from lesson 5 of this unit;
- The cross product of vectors, from lesson 4 of this unit;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Set criteria for formation of small groups (ensure that not the same members are in the same group all the time, and not the same members for the same task) ;
- Let them discuss the activity 2.2.4 from student-teacher's book for a while; as they are discussing, monitor the contribution of each member;
- Vary the technique of choosing the group for presentation (for example, taking into account the gender, where applicable) of the group work to the whole class
- Be vigilant on the sitting arrangement of the learners and on the effectiveness of the discipline during the presentation;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Ask clear and concise questions to learners to capture the main points of the lesson, ensure that learners mention the different equations of a straight line;
- Give learners time to practice individually, by solving application activity 2.2.4. As they are working check the effectiveness of the work and give further explanation for learners in need;
- Get a clear picture of the lesson you gave to notice the strong points to keep up, so as to improve your next lesson.

Answers to activity 2.2.4

From a to d



e. The distance is AB

Answers to application activity 2.2.4

a. $\frac{95\sqrt{1817}}{1817}$ b. 0 c. $\frac{68\sqrt{230}}{115}$

Lesson 8: Equations of planes in 3D

a. Learning objective:

Determine equations of a plane (vector equation, parametric equation, Cartesian equation).

b. Teaching resources:

Student-teacher's book and other Reference textbooks containing Space Geometry

c. Prerequisites/Revision/Introduction:

For better understanding and mastering of the concepts involved in the lesson, the following background points are needed:

- Points and straight lines, from year 1: unit9;
- Operations on vectors in 3D, from lesson4 of this unit;
- Calculation of determinant of order two, from lesson4 of this unit;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Instruct the learners to form small groups under your supervision;
- Ask them to discuss the activity 2.3.1 from student-teacher's book for a while; Check the contribution of each member of the group in the discussion; ensure that the discussion is running smoothly without confrontation;
- Choose at random a group to present his/her findings to the whole class while other learners are following attentively;
- Have them notice that a plane can be defined in six different ways and that all the ways can be brought to one;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Through well-chosen questions, bring learners to summarize the main points of the lesson, ensure that learners mentioned how to obtain vector equation, parametric equations and Cartesian equation of a plane;
- Let learners practice individually, by solving application activity 2.3.1. As they are working go around different learners and provide assistance where necessary;
- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.

Answers to activity 2.3.1

a. A plane is determined in one of the following six ways:

- A point and two vectors of different directions
- Three non collinear points
- A straight line and a point outside the line
- Two intersecting lines
- Two distinct parallel lines
- A point and a vector normal to the plane

b. $\vec{AM} = t\vec{u} + s\vec{v}$ for some scalars t and s

c. $\vec{r} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + t(a\vec{i} + b\vec{j} + c\vec{k}) + s(a'\vec{i} + b'\vec{j} + c'\vec{k})$

$$\begin{cases} x = x_0 + at + a's \\ y = y_0 + bt + b's \\ z = z_0 + ct + c's \end{cases}$$

$\overrightarrow{AM} \cdot (\vec{u} \wedge \vec{v}) = 0$, which can be expressed as $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$, where $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is a vector normal to the plane.

Answers to application activity 2.3.1

a. $\vec{r} = 2\vec{i} + 4\vec{j} + \vec{k} + t(-\vec{i} - \vec{j} - 2\vec{k}) + s(-3\vec{j} + 2\vec{k})$

$$\begin{cases} x = 2 - t \\ y = 4 - t - 3s \\ z = 1 - 2t + 2s \end{cases}$$

$$-8x + 2y + 3z + 5 = 0$$

b. $\vec{r} = \vec{i} + \vec{j} + \vec{k} + t(\vec{i} - \vec{j}) + s(-3\vec{i} + 3\vec{j} + 2\vec{k})$

$$\begin{cases} x = 1 + t - 3s \\ y = 1 - t + 3s \\ z = 1 + 2s \end{cases}$$

$$x + y - 2 = 0$$

c. $\vec{r} = 3\vec{i} + 6\vec{j} + t(-2\vec{i} - 6\vec{j} + \vec{k}) + s(2\vec{i} - 5\vec{j} + 7\vec{k})$

$$\begin{cases} x = 3 - 2t + 2s \\ y = 3 - 6t - 5s \\ z = t + 7s \end{cases}$$

$$-37x + 16y + 22z + 11 = 0$$

$$d. \vec{r} = 4\vec{i} + 3\vec{j} + 8\vec{k} + t(8\vec{i} + 2\vec{j} + 7\vec{k}) + s(6\vec{i} - 5\vec{j} + 2\vec{k})$$

$$\begin{cases} x = 4 + 8t + 6s \\ y = 3 + 2t - 5s \\ z = 8 + 7t + 2s \end{cases}$$

$$39x + 26y - 52z - 182 = 0$$

Lesson 9: Distance from a point to a plane

a. Learning objective:

Calculate accurately the distance between a point and a plane, line and plane and between two planes

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other Reference textbooks to facilitate research

c. Prerequisites/Revision/Introduction:

Student-teachers will learn better this lesson if they have mastered:

- The distance between two points, studied in lesson 2 of this unit;
- The distance from a point to a line, studied in lesson 6 of this unit;
- The use of a ruler to measure the distance between two points as studied in ordinary level senior one, unit 7;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

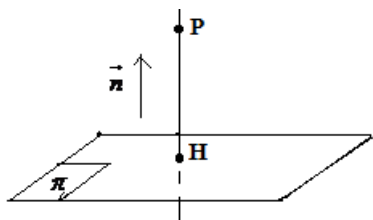
- Start by giving clear instructions for the learners to organize themselves into small groups;
- Ask them to discuss the activity 2.3.2 from student-teacher's book for a while;
- Let a group present his/her findings to the whole class, after a proper sitting arrangement of the learners, and after ensuring their readiness to follow;
- During the harmonization, help student-teachers to discover that the distance between a point and a plane is a measure of how far is a plane

from the point and how it is calculated;

- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book ;
- Have learners practice individually, by solving application activity 2.3.2 and provide assistance for learners in need;
- Assess whether learning has occurred or not to feed forwards.

Answers to activity 2.3.2

a.



b. PH

c. $\overrightarrow{HP} = t\vec{n}$ for some scalar t

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

d. $t = \frac{-(ax_0 + by_0 + cz_0 + d)}{a^2 + b^2 + c^2}$

e. $\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Answers to application activity 2.3.2

a. 3 b. $\frac{19}{5}$ c. $\frac{5}{3}$ d. $\frac{9\sqrt{41}}{41}$ e. $\frac{5\sqrt{6}}{18}$

Lesson 10: Angles

a. Learning objective:

Calculate the angle (between two lines, between a line and a plane or between two planes)

b. Teaching resources:

Charts, Student-teacher's book and other Reference textbooks containing the concept Space Geometry

c. Prerequisites/Revision/Introduction:

For student-teachers to master well the concepts in this lesson, the following prerequisites are required:

- Relations in a right angled triangle, from year 1, unit8;
- The dot product of two vectors, studied in lesson 4 of this unit;
- Equations of a straight line, studied in lesson of this unit;
- Equations of a plane, studied in lesson of this unit;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

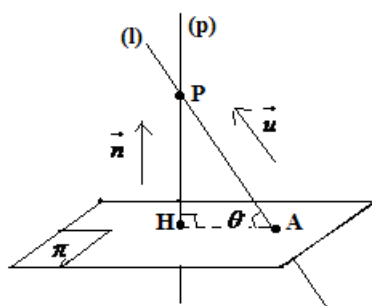
- Instruct the learners to form small groups;
- Ask them to discuss the activity2.3.3 from student-teacher's book for a while;
- Choose at random a group to present his/her findings to the whole class while other learners are following attentively;
- Have them notice that the angle between a line and a plane is the angle between the line and its projection on the plane;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book;
- Through well-chosen questions, bring learners to summarize the main points of the lesson, ensure that learners mentioned how to calculate the angle between a line and a plane, the angle between two planes and the angle between two lines.
- Let learners practice individually, by solving application activity 2.2.3.As they are working go around different learners and provide

assistance where necessary;

- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.

Answers to activity 2.3.3

a and b:



c. Right angled triangle

$$d. \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta = \frac{\vec{n} \cdot \vec{u}}{\|\vec{n}\| \|\vec{u}\|}$$

$$e. \sin \theta = \frac{an_1 + bn_2 + cn_3}{\sqrt{a^2 + b^2 + c^2} \sqrt{n_1^2 + n_2^2 + n_3^2}}$$

Answers to application activity 2.3.3

1. 77.3°
2. 35.6°
3. 30°
4. 100.9°
5. 47°
6. 79°
7. 45°

2.6 Summary of the unit

In this unit we studied three dimensional elements: points, straight lines and planes.

- We described a point by its three coordinates, three real numbers indicating how to reach the point, starting from the origin, and moving successively in the OX- direction, OY-direction and OZ-direction.
- We were then interested in how to obtain the components of vector \overline{AB} given the coordinates of points A and B , how to find the distance between two points given by their coordinates, and how to find the coordinates of the midpoint of line segment $[AB]$, when points A and B are given by their coordinates
- Straight lines were determined in two ways: either by a point through which the line is passing and the direction vector of the line, or by two distinct points
- We focused on how to determine the vector equation, the parametric equations and the Cartesian equations (symmetric equations) of a straight line
- Problems, such as distance from a point to a straight line, distance between two straight lines, and angle between two straight lines, were discussed
- Planes were determined in six ways:
 - a point and two vectors of different directions;
 - Three non collinear points
 - A straight line and a point outside the line;
 - Two intersecting lines;
 - Two distinct parallel lines;
 - A point and a vector normal (perpendicular) to the plane
- Problems, such as distance from a point to a plane, angle between two planes, angle between a straight line and a plane, were discussed.

2.7 Additional Information for teachers

For the educative action of the teacher to be effective (in order to respond to all aspects of the learners' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding

that is required from the learner. It is therefore, it is imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

2.8. Answers for end of unit assessment

1. $\vec{r} = \vec{i} + \vec{j} + \vec{k} + t(2\vec{i} + 3\vec{j} - \vec{k})$

2. $\vec{r} = -\vec{i} + 2\vec{j} + \vec{k} + t(\vec{i} + 2\vec{j} + 3\vec{k})$

3. $4\vec{i} - \vec{j} + 12\vec{k} = 2\vec{i} + 3\vec{j} + 4\vec{k} + 2(\vec{i} - 2\vec{j} + 4\vec{k})$

4. $a = 6$ and $b = 8$

5. a. $\overline{AB} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$

b. $\vec{r} = 2\vec{i} - \vec{j} + \vec{k} + t(\vec{i} + \vec{j} - \vec{k})$

6. 80°

7. 40.2°

8. a. 114°

b. 54.7°

9. $\frac{8\sqrt{14}}{7}$

10. a. $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z+1}{1}$

b. $\frac{x-3}{3} = \frac{y+1}{2} = \frac{z-2}{-4}$

c. $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-1}{-1}$

$$11. \vec{r} = 2\vec{i} + 5\vec{j} + 4\vec{k} + t(3\vec{i} - 2\vec{j} - \vec{k})$$

$$12.a. \vec{r} = 2\vec{i} + 2\vec{j} - \vec{k} + t(3\vec{i} + 2\vec{j} + 4\vec{k})$$

$$b. \vec{r} = 3\vec{i} - 2\vec{j} + 3\vec{k} + t(\vec{i} + 4\vec{j} - \vec{k})$$

2.9 Additional activities

2.9.1. Remedial activity

For students who learn at a pace a little behind others of their level, it is preferable to start with activities demanding less effort for them catch up their peers. The teacher is requested to give them, initially, simple (or factual) activities, and progress gradually to more elaborated (or conceptual) activities. For this unit, some non-exhaustive examples of simplified activities include:

1. Read carefully the following passage and answer the related questions about the passage:

“The coordinates of a point P describe how to reach the point, starting from the origin O and moving successively in the direction of the OX-axis, then in the direction of the OY-axis and finally in the direction of the OZ-axis”.

We agree to describe the positions of points as follow:

- Along the OX-axis: for any positive real number x,
 - x units **in front** of O is denoted x
 - x units **behind** O is denoted -x
- Along the OY-axis: for any positive real number y,
 - y units at the **right side** of O is denoted y
 - y units at the **left side** of O is denoted -y
- Along the OZ-axis: for any positive real number z,
 - z units **Above** O is denoted z
 - z units **Below** O is denoted -z

a. i. How many steps are required to move from the origin O to point P?

ii. Can the motion start from any point? Explain your answer.

b. According to the passage, can we change the order in which we are moving from O to P? Explain your answer.

c. The following statement “P moves x units in front of O, then y units at the left side of O, and finally, z units above O” is denoted mathematically as $P(x, -y, z)$. We say that $(x, -y, z)$ are **coordinates** of point P.

i. Write down the coordinates of point M described by following statement: “Point M moves a units behind O, then b units at the right side of O, and finally, c units below O”

ii. Explain in words the mathematical statements:

$$N(3, -1, -1);$$

$$T(0, -2, -1);$$

$$K(2, 3, 5)$$

2. a. Write down the formula for the distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

b. Use the formula to find the distance between points $C(2, 1, 0)$ and $D(0, -1, 1)$

3. a. Write down the formula for the coordinates of the midpoint of line segment $[AB]$ if $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

b. Use the formula to find the midpoint of line segment $[XY]$ if points

$$X(2, 1, -3) \text{ and } Y(4, -5, 3)$$

4. Given vectors $\vec{u} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{v} = \vec{i} - \vec{j} + \vec{k}$ find:

a. The dot product $\vec{u} \cdot \vec{v}$

b. The cross product $\vec{u} \wedge \vec{v}$

5. Find the parametric equations of the straight line through point $A(3, 1, 2)$ and the line has direction vector $\vec{u} = \vec{i} - 4\vec{j} + 3\vec{k}$

6. Given that $\vec{u} = (1, -1, 1)$, $\vec{v} = (2, 3, -1)$ and $\vec{w} = (1, -4, -3)$, find $2\vec{u} + \vec{v} - \vec{w}$.

7. Write down the coordinates of a point on the line $(l_1): \frac{x-5}{3} = \frac{y-7}{-16} = \frac{z-3}{7}$

8. Determine the components of a direction vector of the line

$$(l_2): \frac{x-9}{3} = \frac{y-13}{8} = \frac{z-15}{-5}$$

2.9.2. Consolidation activity

To enforce the information learned in the unit and making sure that none of the students are left uncertain, more practice for learners to test the skills they have gained through the unit is necessary. The following are some non-exhaustive examples of such activities:

1. Vectors $\vec{u} = 2\alpha\vec{i} + \vec{j} - \vec{k}$ and $\vec{v} = \vec{i} - \alpha\vec{j} + \vec{k}$ are perpendicular. Find the value of α
2. Find the area of the triangle with vertices at points A(6,4,-1); B(8,5,-3) and C(2,8,-5)
3. Given that vectors $\vec{u} = (a+2, 2a-b, b+3)$ and $\vec{v} = (b+3, 3, a+2)$ are equal, find the values of real numbers a and b.
4. Find the vector equation, the parametric equations and the symmetric equations of the line through points A(2,3,1) and B(3,-1,4).
5. Find the distance from:
 - a. Point P(5,6,2) to the line with symmetric equations: $x-2 = y-1 = \frac{4-z}{2}$
 - b. Point A(2,3,-1) to the plane π with equation: $2x+2y-z-20=0$
6. Find the symmetric equations of the line ^(P) perpendicular to both lines $(l_1): \frac{x-5}{3} = \frac{y-7}{-16} = \frac{z-3}{7}$ and $(l_2): \frac{x-9}{3} = \frac{y-13}{8} = \frac{z-15}{-5}$
7. Given the points A(2,1,-4) and B(-2,4,-2), find the vector equation, the parametric equations and the symmetric equations of the line (AB)
8. Consider the points A(3,3,-2) and B(5,-1,4). Find the vector equation of the line through the midpoint of line segment [AB] and parallel to vector $\vec{u} = \vec{i} - 2\vec{j} - 4\vec{k}$

2.9.3. Extended activity

Some students learn very quickly, while the rest of the class requires more time to synthesize. It is important to respond to this quicker pace of learning by providing advanced students with more challenging and engaging activities.

The following are non-exhaustive examples of such activities:

1. Given that point $M(3, -2, 1)$ is the midpoint of line segment $[AB]$, where $A(2x-1, 1, z-3)$ and $B(2, \frac{y-1}{2}, 1-\frac{z}{3})$, find the real numbers x , y and z
2. Points $A(1, 2, 4)$; $B(a, 4, 8)$; $C(-3, -6, b)$ are collinear. Find a and b
3. Given the points $A(1, 2, 2)$, $B(2k+1, 3, -k+3)$ and the vector $\vec{u} = 2\vec{e}_1 - \vec{e}_3$, find the value of k for which vectors \vec{AB} and \vec{u} are perpendicular
4. Points $A(0, 2, 5)$; $B(-1, 3, 1)$; $C(1, 4, 3)$; $D(3+2t, 2-2t, 2-t)$ are vertices of a tetrahedron whose volume is 9. Find the value of t
5. Find the symmetric equations of line (l) through point $(1, 0, -1)$ that is perpendicular to each of the lines $(l_1): \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z+3}{1}$ and $(l_2): \frac{x}{3} = \frac{y-1}{1} = \frac{z+2}{-2}$
6. Given the points $A(11, 0, -1)$ and $B(-9, 4, 5)$, find:
 - a. The vector equation, the parametric equations and the symmetric equations of line (AB) .
 - b. The coordinates of point P on (AB) such that OP is perpendicular to (AB) . O is the origin.
 - c. The distance from O to line (AB) .
7. Consider the tetrahedron with vertices at points $S(-1, 3, 4)$, $M(2, -4, 5)$, $N(-3, 5, -1)$ and $P(0, -1, 2)$.
 - a. Find the vector equation, the parametric equations and the Cartesian equation of the plane π through points M , N and P .
 - b. Calculate the area of triangle MNP
 - c. Find the symmetric equations of line (p) through vertex S and perpendicular to plane π .

d. Find the coordinates of point H, intersection of (p) and π . Hence, find the distance from point S to plane π . Verify your result by using the formula of the distance from a point to a plane

8. (P) and (Q) are two planes given by parametric equations below.

$$(P) \begin{cases} x = 2 + \alpha + 2\beta \\ y = 2 + 2\alpha + \beta \\ z = 1 - \alpha - \beta \end{cases} \quad (Q) \begin{cases} x = 1 + 3m - t \\ y = 3 + 2m + 2t \\ z = 1 - 2m \end{cases}$$

,where α , β , m and t are real numbers.

- a.** Determine a vector normal to plane (P) and a vector normal to plane (Q).
- b.** Show that the two planes (P) and (Q) are intersecting.
- c.** Find a Cartesian equation for each plane (P) and (Q)
- d.** Calculate the measure of the acute angle between the two planes
- e.** Determine the parametric equations of the line of intersection of the two planes.

UNIT 3

TRIGONOMETRIC EQUATIONS

3.1 Key unit competence

Solve trigonometric equations and related problems using trigonometric identities and transformation formulas.

3.2 Prerequisite

Student-teachers will perform better in this unit if:

- They have a good background on “trigonometry”, from unit 8 of year 1
- They can formulate easily verbally, mathematic expressions, and vice versa
- Perform numerical calculations correctly, mentally or using a calculator.

3.3 Cross-cutting issues to be addressed

- **Inclusive education:** promote the participation of all student-teachers while teaching
- **Peace and value Education:** During group activities, the teacher will encourage student teachers discuss without confrontation their points of view
- **Cooperation:** the teacher will encourage student teachers to help each other and to respect opinions of colleagues.
- **Gender:** Give equal opportunities to all learners (girls and boys) to participate actively in all learning activities from the beginning to the end of the lesson.

3.4 Guidance on introductory activity

- Guide the student-teachers to form small groups and to work on the introductory activity;
- Through class discussions, let student-teachers discover the trigonometric concepts involved in the activity and the relationship between them;
- Walk around to monitor the work of each group and to assist any group in need;
- After a given time, invite student teachers to present their findings and harmonize them.
- Ensure that the learners are eager to study the unit and sustain their curiosity

Answer for introductory activity

Rather than just answering the introductory activity, take the opportunity to go through the general approach of solving a problem:

- Reading the problem and knowing to state what is requested to find (if possible in your own words), to ensure that the problem is understood,
- Discovering the quantities involved in the problem and assigning a letter to represent each quantity;
- Expressing mathematically the relationships between the quantities;
- Solving mathematically the problem and expressing the answer in terms of the asked question.

The answer to the introductory activity is as follows:

a. $\sin 2\theta = \frac{1}{2}$
 $\theta = 15^\circ$

b. The height is $h = \frac{(9.8)^2 \sin^2 15^\circ}{2(9.8)} = 0.33 \text{ meters}$

3.5. List of lessons and sub-heading

No	Lesson title	Learning objectives	Periods
0	Introduction activity	To arouse the curiosity of student teachers on the content of unit 3	1
1	Transformation formulas: Addition and subtraction formulas.	Use addition and subtraction formulas to simplify trigonometric expressions	3
2	Double angle formulas.	Use double angle formulas to simplify trigonometric expressions	4
3	Half-angle formulas.	Use half angle formulas to simplify trigonometric expressions	2
4	Transformation of a product into a sum.	Convert a product into a sum or difference	2
5	Transformation of Sum or difference to a product	Transform a sum or difference to a product	2
6	Trigonometric equations	Solve trigonometric equations and discuss the solution	4
7	Application of trigonometry in real life	Solve real life word problems involving trigonometry.	2
8	End unit assessment		2
Total number of periods in this unit			22

Lesson 1: Addition and subtraction formulas

a. Learning objective:

Use addition and subtraction formulas to simplify trigonometric expressions

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other Reference textbooks containing addition and subtraction formulas to make research easy

c. Prerequisites/Revision/Introduction:

Student-teachers will learn better this lesson if they have a good back ground on:

- Angles (vertically opposite, corresponding, alternate), studied in Ordinary level, senior one, unit 6;
- Unit circle, studied in year one, unit 8;
- Right angled triangle, studied in year one, unit 8;

d. Learning activities

- Have the learners organized themselves into small groups under your management;
- Ask them to discuss the activity 3.1.1 from student-teacher's book;
- After a given time, choose randomly a group to present his/her findings to the whole class and note how the members are organizing the presentation; Ensure that each member participates actively;
- During the harmonization, help student-teachers discover that a trigonometric function of a sum or difference is not the same as the sum of the trigonometric functions of the terms;
- Guide students to explore examples and the content given in the student's book to enhance how to transfer the procedure to similar cases;
- Request learners to practice individually through application activity 3.1.1 and go round to provide assistance where needed;
- Assess the learners' competences and evaluate whether lesson objectives were achieved.

Answers to activity 3.1.1

a.i. $\frac{EA}{OA}; \frac{OE}{OA}$

ii. $\frac{CA}{OA}; \frac{OC}{OA}$

b.i. $\alpha; 90^\circ - \alpha; \alpha$

ii. $\cos \alpha$

c. True; True; True; True

d. $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$

e. $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

f. $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$;

$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

g. $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

h.i. False

ii. False

iii. False

Answers to application activity 3.1.1

1. $2 \cos 3\theta$

2.a. $\frac{\sqrt{6} + \sqrt{2}}{4}$

b. $\frac{\sqrt{3}}{2}$

c. $-\frac{\sqrt{3}}{3}$

3. $\tan 56^\circ = \tan(45^\circ + 11^\circ) = \frac{\tan 45^\circ + \tan 11^\circ}{1 - \tan 45^\circ \tan 11^\circ} = \frac{1 + \frac{\sin 11^\circ}{\cos 11^\circ}}{1 - \frac{\sin 11^\circ}{\cos 11^\circ}} = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$

4. a. $2 + \sqrt{3}$

b. $\frac{\sqrt{6} - \sqrt{2}}{4}$

c. $\frac{\sqrt{3}}{2}$

d. $\frac{1}{2}$
5. a. 1; 0.5

b. $\frac{\pi}{3}$; $\frac{\pi}{6}$

Lesson 2: Double angle formulas

a. Learning objective:

Use double angle formulas to simplify trigonometric expressions

b. Teaching resources:

Student-teacher's book and other Reference textbooks containing double angle formulas to make research easy.

c. Prerequisites/Revision/Introduction:

Student-teachers will learn better this lesson if they have mastered:

- Addition formulas , covered in lesson 1 of this unit;
- Multiplication by an integer as a repeated addition;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Start by setting criteria for forming groups, and then request learners to organize themselves into small groups from given criteria you have set ;
- Ask them to discuss the activity 3.1.2 from student-teacher's book for a while;
- Let a group present his/her findings to the whole class , after a proper sitting arrangement of the learners , and after ensuring their readiness to follow;
- During the harmonization, help student-teachers to discover that a trigonometric function of the double of an angle is not the same as the double of the trigonometric function of the angle; Let learners be aware of the three ways of expressing $\cos 2\theta$;
- Request the learners to try in groups the example and to check their answers against the one proposed in the students' book ;

- Have learners practice individually, by solving application activity 3.1.2 and provide assistance for learners in need;
- Assess whether learning has occurred on not to feed forwards.

Answers to activity 3.1.2

A and b: i. $\sin 2\theta = 2 \sin \theta \cos \theta$

ii. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

iii. $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

c. i. False

ii. False

iii. False

Answers to application activity 3.1.2

1. $4 \sin x \cos^3 x - 4 \cos x \sin^3 x$

2. $\cos^8 x + \sin^8 x - 28 \cos^2 x \sin^6 x + 70 \cos^4 x \sin^4 x - 28 \cos^6 x \sin^2 x$

3. $\frac{1}{2}$

4.a. $\frac{4}{5}; \frac{3}{5}; \frac{4}{3}$

b. $-\frac{4}{5}; \frac{3}{5}; -\frac{4}{3}$

Lesson 3: Half angle formulas

a. Learning objective:

Use half angle formulas to simplify trigonometric expressions

b. Teaching resources:

Student-teacher's book and other Reference textbooks containing half angle formulas

c. Prerequisites/Revision/Introduction:

Student-teachers will learn better this lesson if they have mastered:

- The expression of $\cos 2\theta$, in terms of $\cos \theta$ only, or in terms of $\sin \theta$ only ;
- The square roots, studied in Ordinary level, Senior 2, unit 1, senior2;
- The trigonometric ratios in a right angled triangle, studied in year one, unit8;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Ensure that the learners are organized into small groups ;
- Ask them to discuss the activity 3.1.3 from student-teacher's book for a while;
- Let a group present his/her findings to the whole class , after a proper sitting arrangement of the learners and their readiness to follow;
- During the harmonization, help student-teachers to discover that a trigonometric function of the half of an angle is not the same as the half of the trigonometric function of the angle; ensure that the learners are able to derive the *t – formulas* using right angled triangles and to use them in simplification of trigonometric expressions; Focus on the importance of knowing in which quadrant the half of the angle lies in order to choose the proper sign;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book ;
- Have learners practice individually, by solving application activity 3.1.3 and provide assistance for learners in need;
- Assess whether learning has occurred on not to feed forwards

Answers to activity 3.1.3

$$\text{a. } \cos \gamma = 1 - 2 \sin^2 \frac{\gamma}{2}$$

$$\cos \gamma = 2 \cos^2 \frac{\gamma}{2} - 1$$

$$\text{b and c: } \cos \frac{\gamma}{2} = \pm \sqrt{\frac{1 + \cos \gamma}{2}}$$

$$\sin \frac{\gamma}{2} = \pm \sqrt{\frac{1 - \cos \gamma}{2}}$$

$$\tan \frac{\gamma}{2} = \pm \sqrt{\frac{1 - \cos \gamma}{1 + \cos \gamma}}$$

d. i. False

ii. False

iii. False

Answers to application activity 3.1.3

$$1. \cot \frac{x}{2}$$

$$2. \cos x = \frac{4}{5}; \sin x = \frac{3}{5}$$

$$3. \sin \frac{A}{2} = \pm \frac{4}{5}; \cos \frac{A}{2} = \pm \frac{3}{5}; \tan \frac{A}{2} = \pm \frac{4}{3}$$

$$4. \tan A = \frac{1}{7}$$

$$5. \sqrt{\frac{2 - \sqrt{2}}{2}}; \sqrt{\frac{2 + \sqrt{2}}{2}}; \frac{2 - \sqrt{2}}{2}$$

Lesson 4: Transformation of a product to a sum (or difference)

a. Learning objective:

Convert a product into a sum or difference

b. Teaching resources:

Student-teacher's book and other Reference textbooks containing transformation of a product to a sum or difference

c. Prerequisites/Revision/Introduction:

For student-teachers to master well the concepts in this lesson, the following prerequisites are required:

- Addition formulas and subtraction formulas;
- Factorization of algebraic expressions, from unit 4, year one;
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Instruct the learners to form small groups ;
- Ask them to discuss the activity 3.1.4 from student-teacher's book for a while;
- Choose at random a group to present his/her findings to the whole class while other learners are following attentively;
- Have them notice that the product of trigonometric functions is not the same as the trigonometric function of a product and draw learners' attention on the sign "-" as they are transforming a product of "sines";
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book ;
- Through well chosen questions, bring learners to summarize the main points of the lesson, ensure that learners mentioned the different formulas and how to use them in the transformation of a product to a sum or a difference.
- Let learners practice individually, by solving application activity 3.1.4 .As they are working go around different learners and provide assistance where necessary;
- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.

Answers to activity 3.1.4

a. i. $2 \sin \alpha \cos \beta$

ii. $2 \sin \beta \cos \alpha$

iii. $2 \cos \alpha \cos \beta$

iv. $-2 \sin \alpha \sin \beta$

b. i. $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

ii. $\sin \alpha \sin \beta = -\frac{1}{2} [\cos(\alpha + \beta) - \cos(\alpha - \beta)]$

iii. $\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$

c. i. False

i i. False

Answers to application activity 3.1.4

1. i. $\frac{1}{4}$

ii. $\frac{2 - \sqrt{3}}{4}$

iii. $-\frac{1}{4}$

2. i. $\sin x \sin 3x = -\frac{1}{2} (\cos 4x - \cos 2x)$

ii. $\cos 6x \cos 4x = \frac{1}{2} (\cos 10x + \cos 2x)$

iii. $\cos 2x \sin 4x = \frac{1}{2} (\sin 6x - \sin 2x)$

iv. $\sin x \cos 3x = \frac{1}{2} (\sin 4x - \sin 2x)$

v. $\cos \frac{5x}{2} \cos \frac{3x}{2} = \frac{1}{2} (\cos 4x + \cos x)$

Lesson 5: transformation of a sum (or difference) to a product

a. Learning objective:

Transform a sum or difference to a product

b. Teaching resources:

Student-teacher's book and other Reference textbooks containing Trigonometry

c. Prerequisites/Revision/Introduction:

For better understanding and mastering of the concepts involved in the lesson, the following background points are needed:

- Product to sum or difference formulas, from lesson: unit9;
- Algebraic expressions(factorization), from unit4, year one.
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Set criteria for formation of small groups(ensure that not the same members are in the same group all the time, and not the same the same members for the same task) ;
- Let them discuss the activity 3.1.5 from student-teacher's book for a while; as they are discussing, monitor the contribution of each member;
- Vary the technique of choosing the group for presentation(for example, taking into account the gender, where applicable) of the group work to the whole class
- Be vigilant on the sitting arrangement of the learners and on the effectiveness of the discipline during the presentation;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book ;
- Ask clear and concise questions to learners to capture the main points of the lesson, ensure that learners do not confuse the "sum of trigonometric functions" and the "trigonometric functions of a sum";
- Give learners time to practice individually, by solving application activity 3.1.5 .As they are working ,check the effectiveness of the work and give further explanation for learners in need;

- Get a clear picture of the lesson you gave to notice the strong points to keep up and the area where you need to improve for your next lesson.

Answers to activity 3.1.5

a. i. $2 \sin \alpha \cos \beta$

ii. $2 \sin \beta \cos \alpha$

iii. $2 \cos \alpha \cos \beta$

iv. $-2 \sin \alpha \sin \beta$

b.i. $\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$

ii. $\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$

iii. $\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$

c. i. false

ii false

iii. False

iv. False

Answers to application activity 3.1.5

1. $\frac{\sin 3x + \sin 5x}{\cos 5x - \cos 3x} = \frac{2 \sin 4x \cos x}{-2 \sin 4x \sin x} = -\cot x$

2. $(-\sin 2x)(1 + 2 \sin x)$

3. $\frac{\sin 3x - \sin 2x + \sin x}{\cos 3x - \cos 2x + \cos x} = \frac{(\sin 3x + \sin x) - \sin 2x}{(\cos 3x + \cos x) - \cos 2x} = \tan 2x$

4. i. $2 \sin 2x \cos x$

ii. $2 \cos 4x \cos 3x$

iii. $2 \sin 3x \sin x$

iv. $-2 \sin \frac{5x}{2} \cos \frac{13x}{2}$

$$5.a. \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}} = \tan \frac{A+B}{2}$$

$$b. \frac{\cos 2B - \cos 2A}{\sin 2A + \sin 2B} = \frac{-2 \sin \frac{2A+2B}{2} \sin \frac{2B-2A}{2}}{2 \sin \frac{2A+2B}{2} \cos \frac{2B-2A}{2}} = \frac{\sin(A+B) \sin(A-B)}{\sin(A+B) \cos(A-B)} = \tan(A-B)$$

Lesson 6: Trigonometric equations

a. Learning objective:

Solve trigonometric equations and discuss the solution

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other Reference textbooks containing Trigonometry

c. Prerequisites/Revision/Introduction:

For better understanding and mastering of the concepts involved in the lesson, the following background points are needed:

- Trigonometric identities, from unit 8, year one.
- Trigonometric formulas, from lesson 1 in this unit;
- Algebraic equations, from unit 5, year one.
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Instruct the learners to form small groups under your supervision ;
- Ask them to discuss the activities 3.2.1 and 3.2.2 from student-teacher's book for a while; Check the contribution of each member of the group in the discussion; ensure that the discussion is running smoothly without confrontation;
- Choose at random a group to present his/her findings to the whole class while other learners are following attentively;
- Have them notice that the trigonometric or unit circle is a powerful tool used in solving trigonometric equations;

- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book ;
- Through well chosen questions, bring learners to summarize the main points of the lesson, ensure that learners mentioned how to solve simple trigonometric equations and how to use transformation formulas in solving trigonometric equations;
- Let learners practice individually, by solving application activities 3.2.1 and 3.2.2 .As they are working go around different learners and provide assistance where necessary; check the mathematical instruments such as the pair of compasses and the ruler and their proper use;
- Make introspection about the lesson, as it took place, to write down the area to improve for next lesson.

Answers to activity 3.2.1

1. a. $\cos \alpha = \cos \beta$

b. Opposite; $\beta = -\alpha$

c. $\alpha + 2k\pi; \beta + 2k\pi, k \in \mathbb{Z}$

d. If $|a| > 1$ then the equation $\cos x = a$ has no solution in the set of real numbers.

If $|a| \leq 1$ then the solutions of the equation $\cos x = a$ are of the type

$x = \alpha + 2k\pi, k \in \mathbb{Z}$,or $x = -\alpha + 2k\pi, k \in \mathbb{Z}$,where α is an angle such that $\cos \alpha = a$

2. a. $\sin \alpha = \sin \beta$

b. Supplementary; $\beta = \pi - \alpha$

c. $\alpha + 2k\pi; \beta + 2k\pi = \pi - \alpha + 2k\pi, k \in \mathbb{Z}$

d. If $|a| > 1$ then the equation $\sin x = a$ has no solution in the set of real numbers.

If $|a| \leq 1$ then the solutions of the equation $\sin x = a$ are of the type

$x = \alpha + 2k\pi, k \in \mathbb{Z}$,or $x = \pi - \alpha + 2k\pi, k \in \mathbb{Z}$,where α is an angle such that $\sin \alpha = a$

3. a. $\tan \alpha = \tan \beta$

b. $\beta = \pi + \alpha$

c. $\alpha + k\pi; k \in \mathbb{Z}$

d. The solutions of the equation $\tan x = a$ are of the type $\alpha + k\pi, k \in \mathbb{Z}$, where α is an angle such that $\tan \alpha = a$, and a is any real number

Answers to activity 3.2.2

1.a. $\frac{\pi}{2}$

b. $\beta = \frac{\pi}{10}$

c. $\cos 2\beta = \sin 3\beta = \cos\left(\frac{\pi}{2} - 3\beta\right)$

$\beta = \frac{\pi}{10} + \frac{2k\pi}{5}; k \in \mathbb{Z}$ or $\beta = \frac{\pi}{2} + 2k\pi; k \in \mathbb{Z}$

2. a. Using the change of variable $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$, the equation becomes $5\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) - 2 = 0$, which is equivalent to $-7t^2 - 4t + 3 = 0$

b. $t_1 = -1; t_2 = \frac{3}{7}$

c. For $\tan \frac{x}{2} = -1$, $x = -\frac{\pi}{2} + 2k\pi; k \in \mathbb{Z}$

For $\tan \frac{x}{2} = \frac{3}{7}$, $x = 46.4^\circ + 360^\circ k; k \in \mathbb{Z}$

d. The solution set is $S = \{-45^\circ + 360^\circ k; k \in \mathbb{Z}\} \cup \{46.4^\circ + 360^\circ k; k \in \mathbb{Z}\}$

3. a. $\cos 2\theta = 2\cos^2 \theta - 1$

b. The equation becomes $2\cos^2 \theta - 5\cos \theta + 2 = 0$

c. Solving for $\cos \theta$, we have $\cos \theta = 2$, no value of θ verifies this equation, or

$\cos \theta = \frac{1}{2}$, which is verified for $\theta = \pm \frac{\pi}{3} + 2k\pi; k \in \mathbb{Z}$

4.a. $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}; \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$

b. $\overline{OP} \cdot \overline{OQ} = a \cos x + b \sin x;$

$\|\overline{OP}\| = \sqrt{a^2 + b^2}; \|\overline{OQ}\| = 1$

$$\text{c. } \cos(x - \alpha) = \frac{\overline{OP} \cdot \overline{OQ}}{\|\overline{OP}\| \|\overline{OQ}\|} = \frac{a \cos x + b \sin x}{\sqrt{a^2 + b^2}}$$

$$\text{d. } a \cos x + b \sin x = \sqrt{a^2 + b^2} \cos(x - \alpha)$$

Answers to application activity 3.2.1

$$1. \text{ a. } \frac{\pi}{3}$$

$$\text{b. } -\frac{\pi}{3}$$

$$\text{c. } \frac{\pi}{3}$$

$$\text{d. } \frac{\pi}{6}$$

$$\text{e. } \frac{\pi}{3}$$

$$2. \text{ a. } x = \frac{\pi}{12} + \frac{2k\pi}{3}; k \in \mathbb{Z}, x = -\frac{\pi}{12} + \frac{2k\pi}{3}$$

$$\text{b. } x = \frac{\pi}{6} + \frac{k\pi}{2}; k \in \mathbb{Z}$$

$$\text{c. } x = \frac{\pi}{24} + \frac{k\pi}{2}; k \in \mathbb{Z}, x = \frac{5\pi}{12} + k\pi$$

$$\text{d. } x = \frac{11\pi}{120} + \frac{k\pi}{2}; k \in \mathbb{Z}, x = \frac{\pi}{120} + \frac{k\pi}{2}$$

$$\text{e. } x = \frac{7\pi}{36} + \frac{2k\pi}{3}; k \in \mathbb{Z}, x = -\frac{\pi}{84} + \frac{2k\pi}{7}$$

Answers to application activity 3.2.2

a. The equation is equivalent to $\cos\left(\frac{x}{2} - \alpha\right) = -\frac{\sqrt{3}}{2}$, where $\cos \alpha = \frac{1}{2}$; $\sin \alpha = \frac{\sqrt{3}}{2}$

$$\alpha = \frac{\pi}{3} \text{ so that } \cos\left(\frac{x}{2} - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} = \cos \frac{5\pi}{6}$$

The general solutions are $x = \frac{7\pi}{3} + 4k\pi; k \in \mathbb{Z}$ and $x = -\pi + 4k\pi; k \in \mathbb{Z}$

There is no integral value of k such that $0 < x < 2\pi$.

Therefore, the equation has no particular solution between 0 and 2π

b. The equation $\tan^2 x = 2 \cos^2 x$ is equivalent to $\frac{1}{\cos^2 x} - 1 = 2 \cos^2 x$ or $2 \cos^4 x + \cos^2 x - 1 = 0$

By letting $\cos^2 x = t$, we obtain the quadratic equation $2t^2 + t - 1 = 0$. Then $t = \frac{1}{2}$

$$\text{or } t = -1 \quad \cos^2 x = \frac{1}{2} \Leftrightarrow \cos x = \pm \frac{\sqrt{2}}{2};$$

This gives $x = \frac{\pi}{4} + 2k\pi; k \in \mathbb{Z}$ or $x = \frac{3\pi}{4} + 2k\pi; k \in \mathbb{Z}$. The only value of x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is

$$x = \frac{\pi}{4}$$

c. The equation $\sin \theta \cos \theta = -\frac{1}{2}$ is equivalent to $\sin 2\theta = -1 = \sin \frac{3\pi}{2}$;

The values of θ between 0 and 2π are $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$

d. The equation is equivalent to $\cos(\theta + 30^\circ) = \frac{1}{2} = \cos 60^\circ$

$$\theta + 30^\circ = 60^\circ + 360^\circ k; k \in \mathbb{Z} \text{ or } \theta + 30^\circ = -60^\circ + 360^\circ k; k \in \mathbb{Z}$$

$$\theta = 30^\circ + 360^\circ k; k \in \mathbb{Z} \text{ or } \theta = -90^\circ + 360^\circ k; k \in \mathbb{Z}$$

The values of θ between -180° and 180° are $\theta = 30^\circ$ and $\theta = -90^\circ$

e. a. The equation $\sin^2 x + \sin x \cos x = 0$ is equivalent to
 $\sin x(\sin x + \cos x) = 0 \Leftrightarrow \sin x = 0; \cos x = -\sin x = \cos(90^\circ + x)$

Solving the two equations, we obtain $x = -45^\circ + 180^\circ k; k \in \mathbb{Z}$ or $x = 180^\circ k; k \in \mathbb{Z}$

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The values of x between 0° and 360° are $0^\circ; 135^\circ; 180^\circ; 360^\circ$

2. a. The equation $\cos(\theta + 60^\circ) = \sin \theta$ is equivalent to

$$\cos(\theta + 60^\circ) = \sin \theta = \cos(90^\circ - \theta)$$

Solving the two equations, we obtain $\theta = 15^\circ + 180^\circ k; k \in \mathbb{Z}$

b. The equation $\sin(x - 30^\circ) \sin(x + 30^\circ) = \frac{1}{2}$ is equivalent to

$$\left(\frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x \right) \left(\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x \right) = \frac{1}{2} \Leftrightarrow 3 \sin^2 x - \cos^2 x - 2 = 0$$

$$\Leftrightarrow 1 - 4 \cos^2 x = 0$$

$$\Leftrightarrow (1 - 2 \cos x)(1 + 2 \cos x) = 0$$

$$\Leftrightarrow \cos x = \frac{1}{2} \text{ or } \cos x = -\frac{1}{2}$$

Therefore, $x = \pm \frac{\pi}{3} + 2k\pi; k \in \mathbb{Z}$ or $x = \pm \frac{2\pi}{3} + 2k\pi; k \in \mathbb{Z}$

Lesson 7: Applications of trigonometric equations

a. Learning objective:

Solve real life word problems involving trigonometry

b. Teaching resources:

Ruler, T-square, Student-teacher's book and other reference textbooks containing Trigonometry

c. Prerequisites/Revision/Introduction:

For better understanding and mastering of the concepts involved in the lesson, the following background points are needed:

- Trigonometric formulas, from lesson 1 in this unit;
- Trigonometric equations, from lesson 6 of this unit;
- Algebraic equations, from unit 5, year one.
- Trigonometric identities, from unit 8, year one.
- The calculation of numerical values of algebraic expressions.

d. Learning activities

- Set criteria for formation of small groups (ensure that not the same members are in the same group all the time, and not the same members for the same task);
- Let them discuss the activity 3.3. from student-teacher's book for a while; as they are discussing, monitor the contribution of each member;
- Vary the technique of choosing the group for presentation (for example, taking into account the gender, where applicable) of the group work to the whole class
- Be vigilant on the sitting arrangement of the learners and on the effectiveness of the discipline during the presentation;
- Request the learners to try in groups the examples and to check their answers against the one proposed in the students' book ;
- Ask clear and concise questions to learners to capture the main points of the lesson, ensure that learners mention the different steps in modelling a problem by an equation;
- Give learners time to practice individually, by solving application activity 3.3. As they are working check the effectiveness of the work and give further explanation for learners in need;
- Get a clear picture of the lesson you gave to notice the strong points to keep up, and the weak points to improve for your next lesson.

Answers to activity 3.3

a. $\sin 2\theta = \frac{1}{2}$
 $\theta = 15^\circ$

b. The height is $h = \frac{(9.8)^2 \sin^2 15^\circ}{2(9.8)} = 0.33 \text{ meters}$

Answers to application activity 3.3

$$1. a. OA = \frac{\sqrt{3}}{\sin x}; OB = \frac{1}{\cos x}; AB = \frac{\sqrt{3}}{\sin x} + \frac{1}{\cos x}$$

$$b. f(x) = \frac{\sqrt{3}}{\sin x} + \frac{1}{\cos x}$$

$$= \frac{\sqrt{3} \cos x + \sin x}{\sin x \cos x}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x\right)}{\sin x \cos x}$$

$$= \frac{4(\cos 30^\circ \cos x + \sin 30^\circ \sin x)}{\sin 2x}$$

$$= \frac{4 \cos(x - 30^\circ)}{\sin 2x}$$

$$c. i. \frac{4 \cos(x - 30^\circ)}{\sin 2x} = 4 \Leftrightarrow \cos(x - 30^\circ) = \sin 2x = \cos(90^\circ - 2x)$$

$$\Leftrightarrow x = 40^\circ \text{ or } x = 60^\circ.$$

$$ii. \frac{\sqrt{3}}{\sin x} = \frac{1}{\cos x} \Leftrightarrow \tan x = \sqrt{3}$$

$$\Leftrightarrow x = 60^\circ$$

$$2. a. \begin{cases} a+b=17 \\ a-b=1 \end{cases} \Leftrightarrow \begin{cases} a=9 \\ b=8 \end{cases}$$

$$b. 13 = 9 + 8 \cos 18t \Leftrightarrow \cos 18t = \frac{1}{2} = \cos 60^\circ$$

$$\Leftrightarrow t = \frac{10}{3} \text{ seconds}$$

3.6 Summary of the unit

The main aim of this unit was to acquire skills for solving trigonometric equations and modelling problems from daily life using trigonometric equations. For this purpose, we proceeded as follows:

- We described different trigonometric transformation formulas, such as:
 - Addition formulas;
 - Subtraction formulas;
 - Double angle formulas;
 - Half angle formulas;
 - Product to sum or difference formulas;
 - Sum to product formulas
- We then went through techniques for solving trigonometric equations, considering:
 - Simple cases;
 - Cases requiring trigonometric transformations
- Finally, we gave examples of how problems from daily life are modelled, and solved, using trigonometric equations

3.7 Additional Information for teachers

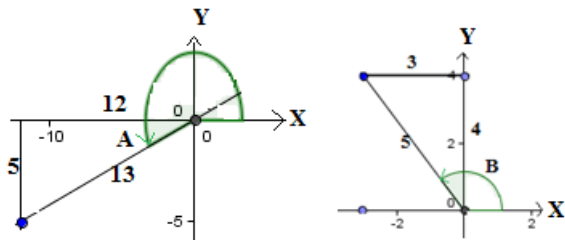
For the educative action of the teacher to be effective(in order to respond to all aspects of the learners' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding that is required from the learner. It is, therefore, imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

3.8. End of unit assessment

1.a. $\tan A + \tan B = -p$; $\tan A \tan B = q$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{-p}{1 - q}$$

b.



We have: $\tan A = \frac{5}{12}$; $\tan B = -\frac{3}{4}$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{5}{12} + \frac{3}{4}}{1 - \left(\frac{5}{12}\right)\left(\frac{3}{4}\right)} = \frac{56}{33}$$

2.a. $\cos 2\theta = -\frac{7}{25}$

b. $\tan 2\theta = -\frac{120}{199}$

3.a. $\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta} = \tan \theta$

b. $\tan \frac{\pi}{8} = \frac{\sin \frac{\pi}{4}}{1 + \cos \frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}}$; hence, $\cot \frac{\pi}{8} = \frac{2 + \sqrt{2}}{\sqrt{2}} = 1 + \sqrt{2}$

4.a. $\cos 2x = 1 - 2 \sin^2 x = 1 - 2 \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right)^2 = \frac{\sqrt{3}}{2}$

$$x = \frac{\pi}{12}$$

$$\begin{aligned}
 5. \sin^2 t - 2(1 - \cos t) &= \left(2 \sin \frac{t}{2} \cos \frac{t}{2}\right)^2 - 2\left(2 \sin^2 \frac{t}{2}\right) \\
 &= 4 \sin^2 \frac{t}{2} \cos^2 \frac{t}{2} - 4 \sin^2 \frac{t}{2} \\
 &= -4 \sin^2 \frac{t}{2} \left(1 - \cos^2 \frac{t}{2}\right) = -4 \sin^4 \frac{t}{2}
 \end{aligned}$$

$$6. \text{a. } \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{7}{24}$$

$$\Leftrightarrow 7 \tan^2 A + 48 \tan A - 7 = 0$$

$$\tan A = \frac{1}{7}; \tan A = -7$$

The negative value is not considered, since the angle is acute

$$\text{b. } \sin \frac{A}{2} = \frac{4}{5}; \cos \frac{A}{2} = \frac{3}{5}; \tan \frac{A}{2} = \frac{4}{3}$$

$$7. \text{a. } \sin 4x \cos 11x = \frac{1}{2} \sin 15x - \frac{1}{2} \sin 7x$$

$$\text{b. } \cos 7x \sin 9x = \frac{1}{2} \sin 16x + \frac{1}{2} \sin 2x$$

$$8. \text{a. } \cos 8x - \cos 9x = 2 \sin \frac{17x}{2} \sin \frac{x}{2}$$

$$\text{b. } \sin 3x + \sin 11x = 2 \sin 7x \cos 4x$$

$$\begin{aligned}
 9. \frac{\sin A + 2 \sin 3A + \sin 5A}{\sin 3A + 2 \sin 5A + \sin 7A} &= \frac{2 \sin 3A \cos 2A + 2 \sin 3A}{2 \sin 5A \cos 2A + 2 \sin 5A} \\
 &= \frac{2 \sin 3A(\cos 2A + 1)}{2 \sin 5A(\cos 2A + 1)} \\
 &= \frac{\sin 3A}{\sin 5A}
 \end{aligned}$$

$$10. \text{a. } x = \frac{\pi}{6}; x = \frac{\pi}{4}$$

$$\text{b. } x = \frac{\pi}{6}; x = \frac{5\pi}{6}; x = \frac{7\pi}{6}$$

$$c. x = \frac{11\pi}{120} + \frac{k\pi}{2}; k \in \mathbb{Z}, x = \frac{\pi}{120} + \frac{k\pi}{2}; k \in \mathbb{Z}$$

Give integral values to k as long as the value of x is in the required interval

$$d. x = \frac{9\pi}{5}; x = \frac{\pi}{15}; x = \frac{11\pi}{15}; x = \frac{7\pi}{5}$$

$$11. x = \frac{\pi}{6} + 2k\pi; k \in \mathbb{Z}, x = \frac{\pi}{2} + 2k\pi; k \in \mathbb{Z}$$

$$12.a. 16^{\circ}C$$

b. Minimum: $130^{\circ}C$

Maximum: $250^{\circ}C$

3.9. Additional activity

3.9.1. Remedial activity

For slow learners to catch up their peers, the teacher is requested to give them, initially, activities where skills are broken into smaller components. For this unit, some non exhaustive examples include:

1. The expression $E = \cos x \cos y + \sin x \sin y$ involves four trigonometric functions: $\cos x, \cos y, \sin x, \sin y$

Express E as a single trigonometric function

2. Complete $2 \sin 17^{\circ} \cos 29^{\circ} = \sin \dots$,

3. Complete:

If $\tan \frac{\alpha}{2}$ is replaced by t then, in an expression containing $\cos \alpha$ and $\sin \alpha$, $\cos \alpha$ is replaced by...and $\sin \alpha$ is replaced by...

4. Complete: $\frac{1}{2}[\cos(2x+x) - \cos(2x-x)] = \dots$ and use it to express $\cos 3x - \cos x$ as a product of two trigonometric functions

5. Write down an acute angle x such that $\sin 2x = \frac{1}{2}$

6. The motion of a particle is modeled by the equation $y = 10 \cos \frac{\pi t}{3}$, where y is the distance of the particle, in meters, from the origin after time t second. Find the initial distance from the origin and the time when the distance is $5m$.

3.9.2. Consolidation activity

Not all students have the same level of understanding. Give opportunity for learners to move at their individual pace through a variety of exercises. To reinforce learning, to clarify and address any doubts, the following are some non exhaustive examples:

1. Simplify the expression $\tan \frac{\theta}{2} + \cot \frac{\theta}{2}$, writing your final answer in terms of θ
2. Given that $\tan A = 8$ and $\tan B = 3$, find $\tan(A - B)$
3. Simplify, assume the denominator $\neq 0$: $\frac{\cos 2x - \sin 2x + 1}{\cos 2x + \sin 2x - 1}$
4. Given $\sin \alpha = \frac{4}{5}$, $\frac{\pi}{2} < \alpha < \pi$ and $\sin \beta = \frac{-2\sqrt{5}}{5}$, $\pi < \beta < \frac{3\pi}{2}$, find $\cos \alpha, \cos \beta, \cos(\alpha + \beta), \sin(\alpha + \beta)$
5. A particle is projected at an angle of 60° to the horizontal with a velocity of 20 m/s.

Calculate:

- a. the maximum height attained
 - b. the time of flight
 - c. the horizontal range
6. Solve for x the equations:

a. $\cos^2 x + \frac{1}{2} \sin x = 1$ (general solution)

b. $\sin(2x - 45^\circ) = \frac{\sqrt{2}}{2}$, where $0^\circ < x < 360^\circ$

3.9.3. Extended activity

Some students learn very quickly, while the rest of the class requires more time to synthesize. It is important to respond to this quicker pace of learning by providing advanced students with more challenging and engaging activities.

The following are non exhaustive examples of such activities:

- Two simple harmonic motions are given by $x_1 = 3 \cos 2\pi t$ meters and $x_2 = 3 \cos(2\pi t + \frac{\pi}{4})$ meters.

Express the resulting motion $x_1 + x_2$ in the form $x = r \cos(\omega t + \alpha)$

- Factorize $\cos(x + y) + \cos(x - y)$ and solve the simultaneous equations

$$\begin{cases} x - y = \frac{\pi}{4} \\ \cos x \cos y = \frac{\sqrt{2}}{2} \end{cases}, \text{ where } x \text{ and } y \text{ are angles in the interval } [0, 2\pi]$$

- Given that $\cos x \cos y = \frac{\sqrt{3} + 1}{4}$ and $\sin x \sin y = \frac{\sqrt{3} - 1}{4}$, where x and y are angles in the interval $[0, \pi]$, find $\cos(x + y)$ and $\cos(x - y)$. Hence, solve the

simultaneous equations $\begin{cases} \cos x \cos y = \frac{\sqrt{3} + 1}{4} \\ \sin x \sin y = \frac{\sqrt{3} - 1}{4} \end{cases}$

- In coastal areas, the depth of water depends on time of the day. The depth of water is greatest at high tide and least at low tide.

The depth d of water t hours after the low tide is given by the formula $d = a + b \cos kt$, where a, b and k are constants, kt is expressed in degrees.

- Given that the depth of water at low tide is 2 meters and low tide occurs every 12 hours, find the value of k
- Given that the level of water at high tide is 6 meters, find the values of a and b
- A sailor needs a depth of at least 3 meters of water to sail his boat. Given that the low tide on a particular day was at 9:00 am, find the earliest time that the sailor could sail his boat.

5. An object O of weight 100 N is prevented from sliding down a smooth inclined plane by a force F in a string acting up the plane. The angle of the plane to the horizontal is 30° .

a. Represent the inclined plane, showing the angle, the object and the weight W

b. Resolve the weight into the direction:

i. Parallel to the inclined plane

ii. Perpendicular to the inclined plane.

c. Hence, write down the force F, and the reaction of the plane.

UNIT 4

BIVARIATE STATISTICS

4.1 Key unit Competence:

Extend understanding, analysis and interpretation of bivariate data to correlation coefficients and regression lines.

4.2 Prerequisites

Student-teachers will easily learn this unit, if they have a good background on descriptive statistics (Unit 6, Year 1) and on Arithmetic (Unit 1, Year 1).

4.3 Cross-cutting issues to be addressed

Inclusive education (promote education for all while teaching)

Peace and value Education (respect others' view and thoughts during class discussions)

Gender (provide equal opportunity to boys and girls in the lesson)

4.4 Guidance on introductory activity

Invite student-teachers to work in groups and give them instructions on how they can do the introductory activity 4.0 found in unit 4 of student's book;

Guide students to read and analyse the questions insisting on the analysis of statistical data with two variables (x , y) and how they can interpret the bivariate data using correlation coefficients and regression lines.

Invite some group members to present groups' findings, then try to harmonize their answers;

Basing on student-teachers' experience, prior knowledge and abilities shown in answering the questions for this activity, use different questions to facilitate them to give their predictions and ensure that you arouse their curiosity on what is going to be learnt in this unit.

Answer for Introductory activity:

	x_i	y_i	x_i^2	$x_i y_i$
	1	4	1	4
	2	8	4	16
	3	2	9	6
	4	12	16	48
	5	10	25	50
	6	14	36	84
	7	16	49	112
	8	6	64	48
	9	18	81	162
Σ	45	90		

$$x = \frac{1}{n} \sum x_i = \frac{45}{9} = 5 \quad , \quad y = \frac{1}{n} \sum y_i = \frac{90}{9} = 10$$

$$\delta_{x,y} = \frac{\sum x_i \sum y_i - \frac{1}{n} \sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n} (\sum x_i)^2} = \frac{530 - \frac{1}{9} (45)(90)}{285 - \frac{1}{9} (45)^2} = \frac{4}{3} = 1.33$$

The equation of regression line of y on x is

$$y - y_i = \delta_{x,y} (x - x_i)$$

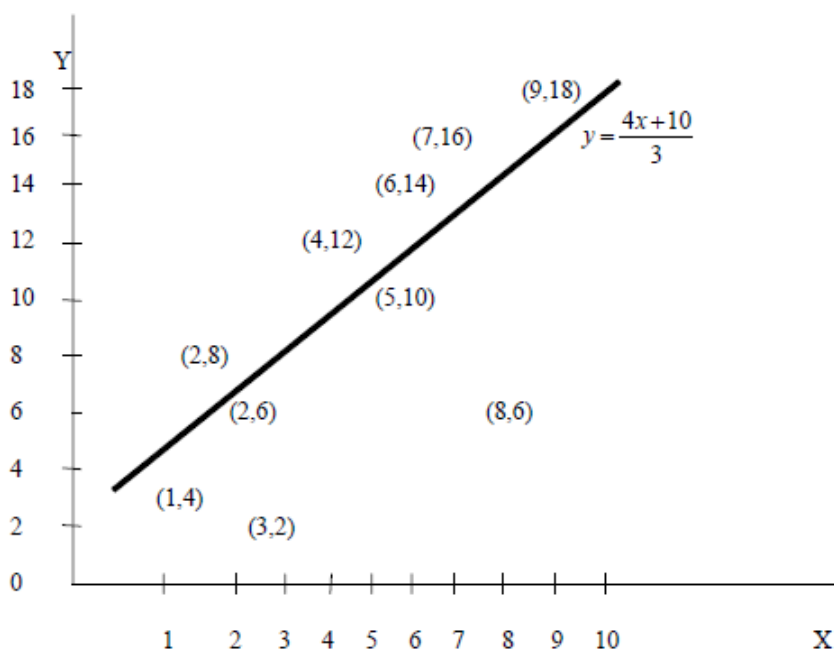
$$y - 10 = 1.33(x - 5)$$

$$y = 10 + 1.33x - 6.65$$

$$y = 1.33x + 3.35$$

Scatter diagram: plotting the 9 sample points.

The first point on the line is . Another point on the line is so the regression line of y on x passes through the two points and plot these points and join them the required line of regression of is obtained.



4.5. List of lessons

#	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student-teacher on the content of unit 4.	1
1	Bivariate data ,scatter diagram and types of correlation	To define bivariate statistics data and plot the scatter diagram.	2
2	Covariance	Explain and determine the covariance for bivariate statistics data	3
3	Coefficient of correlation	To define and calculate the Coefficient of correlation for Bivariate statistics data. Appreciate the importance of using Coefficient of correlation to interpret data to infer conclusion	4

4	Regression line	To define the regression line and establish its equation, Appreciate the importance of using regression line to interpret data.	3
5	Interpretation of statistical data	Analyze, interpretation and predict bi-variate statistical data from various areas (Business, Geography, Demography ...)	6
6	End unit assessment		1
	Total		21

Lesson 1: Bivariate data, scatter diagram and types of correlation

a) Learning objectives

Define bivariate statistics data , plot the scatter diagram and distinguish types of correlation

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on descriptive statistics (Unit 6, Year 1) and on Arithmetic (Unit 1, Year 1).

d) Learning activities:

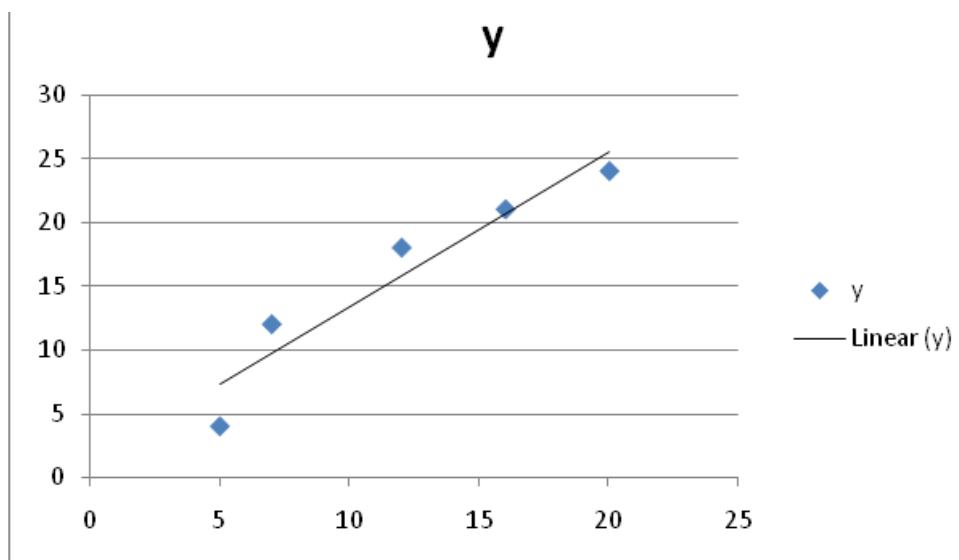
- Invite student-teachers to work in groups and do the activity 4.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover

how to define bivariate statistics data , plot the scatter diagram and to distinguish different types of data.

- After this step, guide students to do the application activity 4.1 and evaluate whether lesson objectives were achieved.

Answer to activity 4.1

a)



a) As time is increasing the chemical reaction also is increasing.

Answer to Application activity 4.1

t	P	t^2	P^2	tP
0.5	125	0.25	15625	62.5
1	113	1	12769	113
1.5	102	2.25	10404	153
2	94	4	8836	188
3	81	9	6561	243
4	83	16	6889	332
5	71	25	5041	355
$\sum_{i=1}^7 t_i = 17$	$\sum_{i=1}^7 P_i = 669$	$\sum_{i=1}^7 t_i^2 = 57.5$	$\sum_{i=1}^7 P_i^2 = 66125$	$\sum_{i=1}^7 t_i P_i = 1446.5$

We need the line $P = at + b$

Use the formula

$$\begin{cases} \sum_{i=1}^7 P_i = a \sum_{i=1}^7 t_i + bn \\ \sum_{i=1}^7 t_i P_i = a \sum_{i=1}^7 t_i^2 + b \sum_{i=1}^7 t_i \end{cases}$$

We have

$$\begin{cases} 669 = 17a + 7b \\ 1446.5 = 57.5a + 17b \end{cases}$$

Solving we have

$$\begin{cases} a = -11 \\ b = 122.3 \end{cases}$$

Then $P = -11t + 122.3$

So,

Norman's pulse rate 2.5 minutes after stopping the exercise program is estimated to be $P = -11(2.5) + 122.3$ or 94.8.

Lesson 2: Covariance

a) Learning objectives

Explain and determine the covariance for bivariate statistics data

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on descriptive statistics (Unit 6, Year 1) , on Arithmetic (Unit 1,Year 1) and on lesson 1 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine and explain the covariance for bivariate statistics data.
- After this step, guide students to do the application activity 4.2 and evaluate whether lesson objectives were achieved.

Answer to activity 4.2

Answers

x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})(y - \bar{y})$
3	6	-1.3	-2.6	-3.9
5	9	0.7	0.4	1.1
7	12	2.7	3.4	6.1
3	10	-1.3	1.4	0.1
2	7	-2.3	-1.6	-3.9
6	8	1.7	-0.6	1.1
$\sum_{i=1}^6 x_i = 26$	$\sum_{i=1}^6 y_i = 52$			$\sum_{i=1}^6 (x_i - \bar{x})(y_i - \bar{y}) = 0.6$
$\bar{x} = 4.3$	$\bar{y} = 8.6$			

If you divide by total frequency you get **variance**

1. If you divide by total frequency you get **covariance**

Answer to application activity 4.2

1. $\text{cov}(x, y) = \frac{71}{12}$

2. $\text{cov}(x, y) = 98.75$

Lesson 3: Coefficient of correlation

a) Learning objectives

Define and calculate the Coefficient of correlation for Bivariate statistics data.

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on lesson 1 & lesson 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and calculate the Coefficient of correlation for Bivariate statistics data.
- After this step, guide students to do the application activity 4.3 and evaluate whether lesson objectives were achieved.

Answers to activity 4.3

1) $\sigma_x = 1.8, \sigma_y = 1.97$

2) $\text{cov}(x, y) = \frac{41}{18}$

3) $\frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = 0.64$

Answer to application activity 4.3

- 1) $r = 0.94$. As the correlation coefficient is very close to 1, the correlation is very strong.
- 2) $r = -0.26$. As the correlation coefficient is very close to zero, the correlation is very weak.
- 3) $\sigma = 0.14$. There is a weak positive correlation between the English and Mathematics rankings.

Lesson 4: Regression line

a) Learning objectives

Define the regression line and establish its equation

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on lesson 1, lesson 2 and lesson 3 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.4 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define the regression line, establish its equation and appreciate the importance of using regression line to interpret data.
- After this step, guide students to do the application activity 4.4 and evaluate whether lesson objectives were achieved.

Answer to activity 4.4

$$1) D'_b = 2 \sum_{i=1}^k (y_i - ax_i - b)(-1) \text{ or } D'_b = -2 \sum_{i=1}^k (y_i - ax_i - b)$$

$$2) \sum_{i=1}^k (y_i - ax_i - b) = 0 \text{ or } \sum_{i=1}^k y_i - \sum_{i=1}^k ax_i - \sum_{i=1}^k b = 0 \text{ or } \sum_{i=1}^k b = \sum_{i=1}^k y_i - \sum_{i=1}^k ax_i$$

Dividing both sides by n gives

$$\frac{1}{n} \sum_{i=1}^k b = \frac{1}{n} \sum_{i=1}^k y_i - \frac{1}{n} \sum_{i=1}^k ax_i \text{ or } \frac{b}{n} \sum_{i=1}^k 1 = \frac{1}{n} \sum_{i=1}^k y_i - \frac{a}{n} \sum_{i=1}^k x_i \text{ or } b = \bar{y} - a\bar{x}$$

$$3) \sum_{i=1}^k (y_i - ax_i - b)^2 = \sum_{i=1}^k (y_i - ax_i - \bar{y} + a\bar{x})^2$$

Or

$$\sum_{i=1}^k (y_i - ax_i - b)^2 = \sum_{i=1}^k [(y_i - \bar{y}) - a(x_i - \bar{x})]^2$$

Differentiation with respect to a and equating to zero:

$$\sum_{i=1}^k 2[(y_i - \bar{y}) - a(x_i - \bar{x})][-(x_i - \bar{x})] = 0$$

$$-2 \sum_{i=1}^k [(y_i - \bar{y}) - a(x_i - \bar{x})](x_i - \bar{x}) = 0$$

$$\Leftrightarrow \sum_{i=1}^k [(y_i - \bar{y}) - a(x_i - \bar{x})](x_i - \bar{x}) = 0$$

$$\Leftrightarrow \sum_{i=1}^k \left[(x_i - \bar{x})(y_i - \bar{y}) - a(x_i - \bar{x})^2 \right] = 0$$

$$\Leftrightarrow \sum_{i=1}^k (x_i - \bar{x})(y_i - \bar{y}) - \sum_{i=1}^k a(x_i - \bar{x})^2 = 0$$

$$\Leftrightarrow \sum_{i=1}^k a(x_i - \bar{x})^2 = \sum_{i=1}^k (x_i - \bar{x})(y_i - \bar{y})$$

$$\Leftrightarrow a \sum_{i=1}^k (x_i - \bar{x})^2 = \sum_{i=1}^k (x_i - \bar{x})(y_i - \bar{y})$$

Dividing both sides by n gives

$$\Leftrightarrow \frac{a}{n} \sum_{i=1}^k (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^k (x_i - \bar{x})(y_i - \bar{y})$$

$$\Rightarrow a = \frac{\frac{1}{n} \sum_{i=1}^k (x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n} \sum_{i=1}^k (x_i - \bar{x})^2}$$

4) The variance for variable x is $\sigma_x^2 = \frac{1}{n} \sum_{i=1}^k (x_i - \bar{x})^2$ and the variance for variable y is $\sigma_y^2 = \frac{1}{n} \sum_{i=1}^k (y_i - \bar{y})^2$ and the covariance of these two variables is

$$\text{cov}(x, y) = \frac{1}{n} \sum_{i=1}^k (x_i - \bar{x})(y_i - \bar{y})$$

$$\text{Then } a = \frac{\text{cov}(x, y)}{\sigma_x^2}$$

5) Now, we have that the regression line y on x is $y = ax + b$, where

$$\begin{cases} a = \frac{\text{cov}(x, y)}{\sigma_x^2} \\ b = \bar{y} - a\bar{x} \end{cases}$$

Or

$$y = \frac{\text{cov}(x, y)}{\sigma_x^2} x + \left(\bar{y} - \frac{\text{cov}(x, y)}{\sigma_x^2} \bar{x} \right)$$

Answer to application activity 4.4

1. a) $y = 0.19x - 8.098$ b) $y = 4.06$

2. $x = -5.6y + 163.3$, $y = -0.06x + 21.8$

Lesson 5: Interpretation of statistical data

a) Learning objectives

Analyze, interpret and predict bi-variate statistical data from various areas (Business, Geography, Demography ...)

b) Teaching resources

Learner's book and other Reference books to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn better in this lesson if they have a good background on lesson 1 , lesson 2 , lesson 3 and lesson 4 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 4.5 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to

present their work where they must explain the working steps;

- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define the regression line, establish its equation and appreciate the importance of using regression line to interpret data.
- After this step, guide students to do the application activity 4.5 and evaluate whether lesson objectives were achieved.

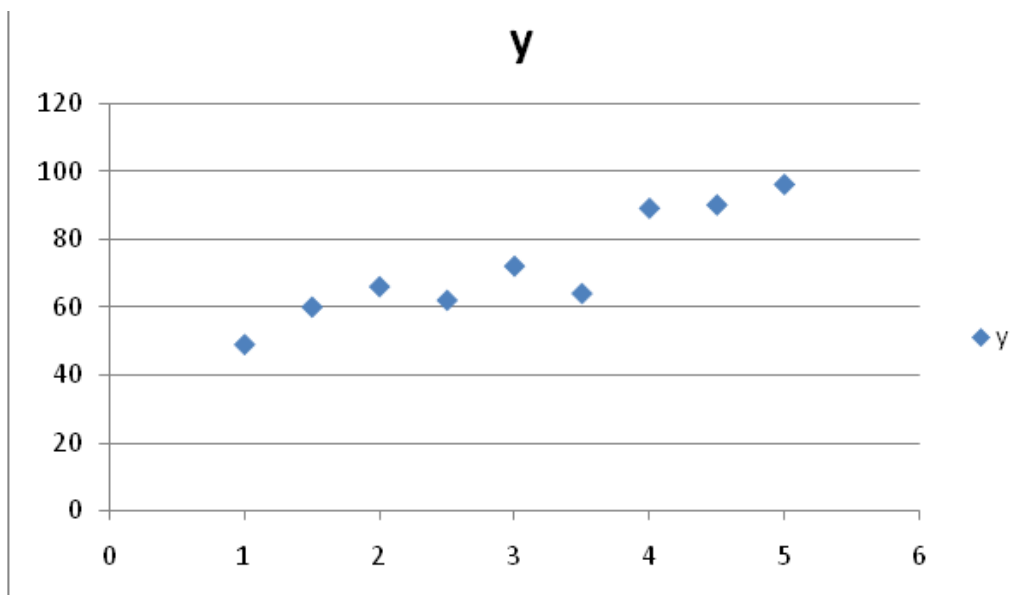
Answer to activity 4.5

(see the content summary in the student book)

Bivariate analysis is a statistical method that helps people study relationships (correlation) between data sets. Many business, marketing, and social science questions and problems could be solved using bivariate data sets.

Answer to application activity 4.5

(a) Scatter diagram



(b) Sample F was damaged.

$$(c) \bar{x} = \frac{\sum x}{n} = \frac{23.5}{8} = 2.9375 \text{ and } \bar{y} = \frac{\sum y}{n} = \frac{584}{8} = 73$$

$$\text{To calculate } r: s_{xy} = \frac{1}{n} \sum xy - \bar{x}\bar{y} = \frac{1}{8} \times 1883 - 2.9375 \times 73 = 20.9375$$

$$s_{xx} = \frac{1}{n} \sum x^2 - \left(\bar{x}\right)^2 = \frac{1}{8} \times 83.75 - (2.9375)^2 = 1.839\dots$$

$$s_{yy} = \frac{1}{n} \sum y^2 - \left(\bar{y}\right)^2 = \frac{1}{8} \times 44622 - (73)^2 = 248.75$$

$$r = \frac{s_{xy}}{s_x s_y} = \frac{20.9375}{\sqrt{1.839\dots} \sqrt{248.75}} = 0.9787\dots, \text{ Then, } r = 0.98 \text{ (2 s.f.)}$$

d) Yes it is sensible to conclude that x and y are related. Since $r = 0.98$ (2 s.f.) is very closed to 1, it would appear to indicate a very strong positive linear correlation.

$$\text{e) For the regression line } y = a + bx, a = \bar{y} - b\bar{x} \text{ and } b = \frac{s_{xy}}{s_{xx}} = \frac{20.9375}{1.839\dots} = 11.38\dots;$$

$$b = \frac{s_{xy}}{s_{xx}} = \frac{20.9375}{1.839\dots} = 11.38\dots \text{ Then, } a = \bar{y} - b\bar{x} = 73 - 11.38\dots \times 2.9375 = 39.57\dots$$

$$y = 39.6 + 11.4x \text{ (3 s.f.)}$$

f) When $x = 3.5$, $y = 38.57\dots + 11.38\dots \times 3.5 = 79$ (2 s.f.) The constant index would have been 79.

g) No, it would not be sensible to use the regression equation when $x = 0$, since this is outside the range of data. Extrapolating outside the data is unreliable.

4.6 Unit summary

1. **Bivariate** or **double series** includes technique of analysing data in two variables.
2. If each point (x, y) of the data is plotted in an x, y coordinate plane, we say that we have **the scatter plot or Scatter diagram**
3. If x -coordinates increases as y -coordinates increases also; We say that x and y have a **positive correlation**. When y tends to decrease as x increases,

then x and y have a **negative correlation**.

4. The **covariance of variables x and y** is a measure of how these two variables change together.

If covariance is zero the variables are said to be **uncorrelated**, means that there is no linear relationship between them. Then, $\text{cov}(x, y) = \frac{1}{n} \sum_{i=1}^k f_i (x_i - \bar{x})(y_i - \bar{y})$

5. The **Pearson's coefficient of correlation** (or **Product moment coefficient of correlation** or simply **coefficient of correlation**), denoted by r , is a measure of the strength of linear relationship between two variables.

The coefficient of correlation between two variables x and y is given by

$$r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$$

Where, $\text{cov}(x, y)$ is covariance of x and y

σ_x is the standard deviation for x

σ_y is the standard deviation for y

6. We use the regression line to **predict** a value of y for any given value of x and vice versa. The "best" line would make the best predictions: the observed y -values should stray as little as possible from the line. This straight line is the regression line from which we can adjust its algebraic expressions and it is written as $y = ax + b$, where a is the gradient and c is the y -intercept.

The regression line y on x is written as $y = \frac{\text{cov}(x, y)}{\sigma_x^2} x + \left(\bar{y} - \frac{\text{cov}(x, y)}{\sigma_x^2} \bar{x} \right)$

We may write $L_{y/x} \equiv y - \bar{y} = \frac{\text{cov}(x, y)}{\sigma_x^2} (x - \bar{x})$

4.7 Additional information for the tutor

Here the tutor has to emphasize on how to analyze, interpret and predict bivariate statistical data using regression line and coefficient of correlation from various areas (Business, Geography, Demography ...)

4.8 End Unit assessment

1) $y = 0.611x + 10.5, x = 1.478y - 1.143, y = 28.83$

2) $y = -8 + 1.2x$

3) $\bar{x} = 13, \bar{y} = 17, y = 0.8x + 6.6, x = 0.45y - 5.35, r = 0.6, \sigma_y = 4$

4.9 Additional activities

4.9.1 Remedial activities

1. For a given set of data it is known that $\bar{x} = 10$ and $\bar{y} = 4$. The gradient of the regression line y on x is 0.6. Find the equation of this regression line and estimate y when $x = 12$

Solution:

The equation of the regression line is $y = a + bx$, where $b = 0.6$; Then $y = a + 0.6x$

The regression line goes through (\bar{x}, \bar{y}) , so $\bar{y} = a + 0.6\bar{x}$

$4 = a + 0.6 \times 10 \Rightarrow a = -2$; Thus, the equation of the regression line is

$$y = -2 + 0.6x; \text{ When } x = 12, y = -2 + 0.6 \times 12 = 5.2$$

4.9.2 Consolidation activities

1. Find the regression line of x on y if the line goes through (1,4) and has gradient 2.

Solution:

Equation of regression line x on y is $x = c + dy$; rearranging

$$dy = x - c \Leftrightarrow y = \frac{1}{d}x - \frac{c}{d}$$

Gradient = $\frac{1}{d}$, then $2 = \frac{1}{d} \Leftrightarrow d = 0.5$; So $x = c + 0.5y$

You are given that $(1, 4)$ lies on the line $1 = c + 0.5 \times 4 \Leftrightarrow c = -1$.

The equation of regression line x on y is $x = -1 + 0.5y$

4.9.3 Extended activities

1. A student found the following data for the female life expectancy, x years, and the Gross Domestic Production (GDP) per head, \$ y , in six countries in South Asia in 1988.

Country	x	y
Afghanistan	42	143
Bangladesh	50	179
Bhutan	47	197
India	58	335
Pakistan	57	384
Sri Lanka	73	423

$$[n = 6, \sum x = 327, \sum y = 1661, \sum x^2 = 18415, \sum y^2 = 529909, \sum xy = 96412]$$

- (a) It is required to estimate the value of x for Nepal, where the value of y was 160. (i) Find the equation of a suitable line of regression. Simplify your answer as far as possible, giving the constants correct to three significant figures. (ii) Use your equation to obtain the required estimate.
- (b) Use your equation to estimate the value of x for North Korea, where the value of y was 858. Comment on Your answer.

Solution:

- (a) Neither variable has been controlled in the given data and since you are required to estimate the life expectancy, x years, When the Gross Domestic Product per head, \$ y is \$160, it is sensible to use the regression line of x on y .

The least squares regression line of x on y . Has the equation $x = c + dy$

Where $c = \bar{x} - d\bar{y}$ and $d = \frac{s_{xy}}{s_{yy}}$. $\bar{x} = \frac{\sum x}{n} = \frac{327}{6}$ and $\bar{y} = \frac{\sum y}{n} = \frac{1661}{6}$

' Then $s_{xy} = \frac{1}{n} \sum xy - \bar{x}\bar{y} = \frac{1}{6} \times 96412 - \frac{327}{6} \times \frac{1661}{6} = 981.25$

$$s_{xy} = \frac{1}{n} \sum y^2 - \left(\bar{y}\right)^2 = \frac{1}{6} \times 529909 - \left(\frac{1661}{6}\right)^2 = 11681.47\dots$$

$$d = \frac{s_{xy}}{s_{yy}} = \frac{981.25}{11681.47} = 0.084000\dots$$

$$c = \bar{x} - d \bar{y} \Leftrightarrow c = \frac{327}{6} - 0.08400\dots \times \frac{1661}{6} = 31.24\dots$$

The equation of regression line of x on y is $x = 31.2 + 0.084y$ (3s.f.)

(ii) When $y = 160$, $x = 31.2 + 0.0840 \times 160 = 45$ (2s.f.)

The estimated value of the life expectancy in Nepal is 45 years.

(b) From the equation, when $y = 858$, $y = 160$, $x = 31.2 + 0.0840 \times 858 = 103$ (3s.f.)

This would give the life expectancy in North Korea as 103 years, which is clearly not sensible. The value of $y = 858$ is a long way outside the range of the data, and should not be used to estimate a value of x .

UNIT 5

POLYNOMIAL, RATIONAL AND IRRATIONAL FUNCTIONS

5.1. Key Unit competence

Use concepts and definitions of functions to determine the domain of rational functions and represent them graphically in simple cases and solve related problems.

5.2 Prerequisite

Student-teachers will perform better in this unit if:

- They have a good background on “linear functions”, from unit6 of ordinary level, senior 3
- They have mastered “linear functions” studied in unit2 of Year 1
- They can convert easily a verbal a problem into a mathematical statement

5.3 Cross-cutting issues to be addressed

- **Financial education:** Use examples of the application of functions in economics
- **Inclusive education** :Promote the participation of all student-teachers while teaching numerical functions by a proper distribution of activities;
- **Peace and value Education** :Ensure that the members of a group respect others’ view and thoughts during group discussions, in the activities about polynomial, rational and irrational functions;
- **Gender** :When applicable, give equal opportunity for boys and girls to participate in a lesson about polynomial, rational and irrational functions ,ensure that the examples are chosen in such a way that neither boys nor girls are inconvenienced;

5.4 Guidance on introductory activity

Help student-teachers to grasp the general idea of the whole unit: definition, classification (polynomial, rational and irrational functions), operations (composite, inverse, etc), qualities (such as being odd, even, etc.), some specific sets (such as domain, range, etc.).

You can proceed as follows:

- Give clear instructions for students to form small groups and to work on the introductory activity;
- As they are discussing, circulate around to note the relevancy of the discussion and to provide guidance where necessary;
- Ensure that the learners have understood what the unit will be about and they are eager to learn; you can observe this through a clear and concise presentation of a group chosen randomly and the degree of attention other students are paying to the presentation;
- Sustain the curiosity of the learners by a proper management of your class;
- A prior knowledge of software such as geogebra and malmath would be very useful for you and for your students;

Answers to “Introductory activity5.0”

a.i. The use (what something is made for);

ii. Social event (a ceremony);

iii. A quantity whose value depends on the value of another quantity.

b. i; ii; iii

c. i.- Independent: x ;

-Dependent: y ;

ii.- Independent: r ;

-Dependent: A

iii. -Independent: A ;

-dependent: s

d. Similarity: all of them are equalities

Difference: i. the denominator contains the independent variable: **rational function**

ii. the independent variable is neither in the denominator, nor in the radicand: **polynomial function**

iii. the independent variable is in the radicand: **irrational function**

e. Irrational; rational; irrational; polynomial; rational

f. i. Domain: $\text{Dom } f =]-\infty; 1[\cup]1; +\infty[$

Range: $\text{Im } f =]-\infty; 0[\cup]0; +\infty[$: The function is equivalent to $y = \frac{4}{x-1}$; the only value that y cannot assume is 0 ;

ii. Domain: $\text{Dom } f =]-\infty; +\infty[$: as any expression, r can assume any value;

Range: $\text{Im } f = [0; +\infty[$: the expression $A = \pi r^2 \geq 0$, for any value of r

Note: If we consider $A = \pi r^2$ as the formula for the area of a circle with radius r , then the domain is: $\text{Dom } f =]0; +\infty[$, since, practically, the radius of a circle cannot be negative or zero. In this case, the range is $\text{Im } f =]0; +\infty[$

g.i. For $f: \{0\}$: only 0 is not in the domain; for $g:]-\infty; 0[$: all strictly negative real numbers are not in the domain; for $h: \Phi$: no real number is not in the domain.

ii. For $f: \{0\}$: only 0 is not in the range; for $g:]-\infty; 0[$: all strictly negative real numbers are not in the range; for $h:]-\infty; 0[$: all strictly negative numbers are not in the range.

5.5. List of lessons/sub-heading

	Lesson title	Learning objectives	Number of periods
0	Introductory activity	Develop curiosity about unity 5	1
1	Types of functions	Classify numerical functions into polynomial, rational and irrational functions	1
2	Injective, surjective and bijective functions	To determine whether a given function is injective, surjective bijective or neither To interpret graphically injective, surjective and bijective functions	1

3	Existence condition for a given function	Determine the intervals for which a given function exists.	1
4	Domain and range of polynomials	Determine the Domain and range of polynomial functions	1
5	Domain and range of rational function	Determine the domain and range of simple rational functions Interpret graphs to find the domain and range of a function	3
6	Domain and range of irrational function	Determine domain and range of simple irrational functions.	2
7	Composite functions	To find the composite of functions and provide some real life examples of composition of functions. Define the composition of functions. Demonstrate understanding of composition of functions.	1
8	Inverse of a function	To determine whether a given function is invertible To find the inverse of a function.	1
9	Even function and symmetry	Analyze whether a function is even algebraically and from the graph	2
10	Odd function and symmetry	Analyze whether a function is odd algebraically and from the graph	1
End unit assessment			2
Total number of periods			17

Lesson 1: Types of numerical functions

a) Learning objective

Differentiate the types of functions

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Student-teachers will learn better this lesson if they have mastered:

- Numerical functions studied in ordinary level and in year 1;
- Operations on algebraic expressions studied in ordinary level and in year 1;
- Calculation of numerical values with or without a calculator.

d) Learning activities

- Invite student-teachers to work in group and do the activity 5.1 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from a group to present his/her findings; then the teacher and the students will discuss the questions and answers. During the discussion, the teacher will monitor the participation of each member of the class to further comprehension in this lesson.
- As a tutor, harmonize the findings from presentation and guide them to explain why they take such type of function.
- Use different probing questions , guide them to explore the content and examples given in the student's book ,lead them to be able to differentiate different types of functions: polynomial, rational and irrational functions.
- Throughout this time, the teacher will circulate in the room to ensure students understanding and to answer any questions they may have.
- The teacher will evaluate and determine whether more instruction is needed.

- Invite student-teachers to work individually the application activities 5.1. for improving their skills

Answer for activity 5.1.

Polynomial	Rational	Irrational
$f(x) = (x+1)^2$	$h(x) = \frac{x^3 + 2x + 1}{x - 4}$	$f(x) = \sqrt{x^2 + x - 2}$

Answer of application activity 5.1.

Polynomial	Rational	Irrational
$f(x) = x^3 + 2x^2 - 2$	$g(x) = \frac{x^3 + 2x^2 - 2}{x - 5}$	$h(x) = \sqrt{x^3 + 2x^2 - 2}$

Lesson 2: Injective, surjective and bijective functions

a) Learning objectives:

- Determine whether a given function is Injective function.
- Find out any function which is injective.

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites

In this lesson, Student-teachers will perform better if they revise the content on relations and functions learnt in S2 and S3 including linear and quadratic functions learnt in TTC year1.

d) Learning activities

- Invite student-teachers to discuss in small groups the activity 5.2
- Walk around each group and ask probing questions leading them to come up with idea about existence of a function.
- In the plenary class, invite representatives from some groups to present their findings;

- As a tutor, harmonize their answers and guide students to discover that the function may or may not exist at a given value (x_0).
- After this step, invite students to do the application activity 5.2 and evaluate whether lesson objectives are achieved.

Answers of learning activity 5.2

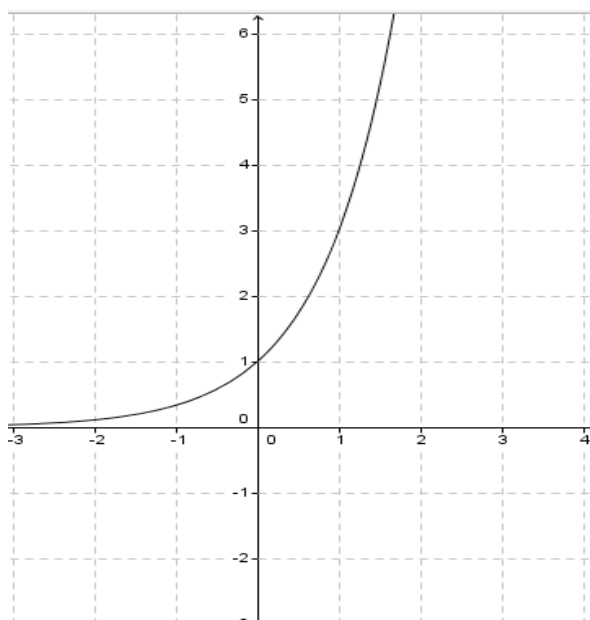
- a. i.** No, there is no real number missing image;
- ii.** Yes f is a mapping
- iii.** Yes, for example, $f(2) = f(-2) = 4$; 4 is image of more than one real number.
- iv.** f is not one-to-one, since some elements are sharing images
- v.** Yes, for example -3 is not image; there is no real number x such that $x^2 = -3$
- vi.** f is not onto.
- b.iii.** Yes, for example, $f(2) = f(-2) = 4$; 4 is image of more than one real number.
- iv.** f is not one-to-one, since some elements are sharing images
- v.** No, all elements of \mathbb{R}^+ are images under f .
- vi.** f is onto.
- c.iii.** No, all elements of \mathbb{R}^+ are images of either one or zero element of \mathbb{R}^+ under function f .
- iv.** f is one-to-one, since no elements are sharing images
- v.** No, all elements of \mathbb{R}^+ are images under f .
- vi.** f is onto.
- d.** $A = [2; +\infty[$ and $A =]-\infty; 4]$; or $A =]-\infty; 2]$ and $A =]-\infty; 4]$

Answers to the application activity 5.2.

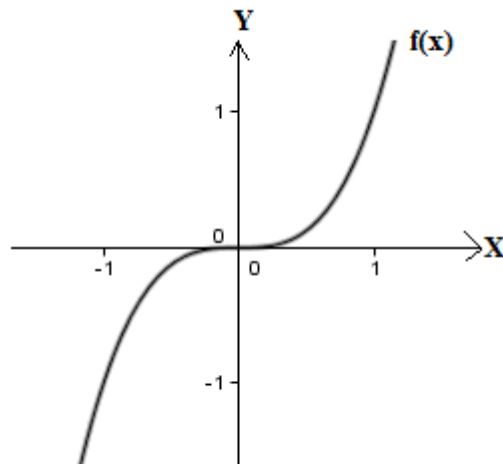
1. T is not a function since there is an element (2) which has two images. For a function one input cannot have many outputs.
 U is a function and V is also a function.

2. i) V is not injective since two elements from the domain are sharing image.
- ii) V is surjective (onto) since there no elements from B which is is not image.
- iii) No, A relation which is not one to one cannot have inverse. Only injective functions have inverses.
3. a) The relation from $\mathbb{N} \rightarrow \mathbb{N}$ defined by $x \rightarrow x + 3$ is injective but not surjective. There is no element whose image is 1.
- b) The relation from $\mathbb{N} \rightarrow \mathbb{N}$ defined by $x \rightarrow |x - 5|$ is not injective but it is surjective since there are elements from the domain which are sharing the same image. Example is the images of 3 and 7 which are given by $=|3 - 5| = 2$ and $|7 - 5| = 2$. It is surjective since any natural number is image under the function
4. The function defined from $x \rightarrow 3^x$ is injective but not surjective.

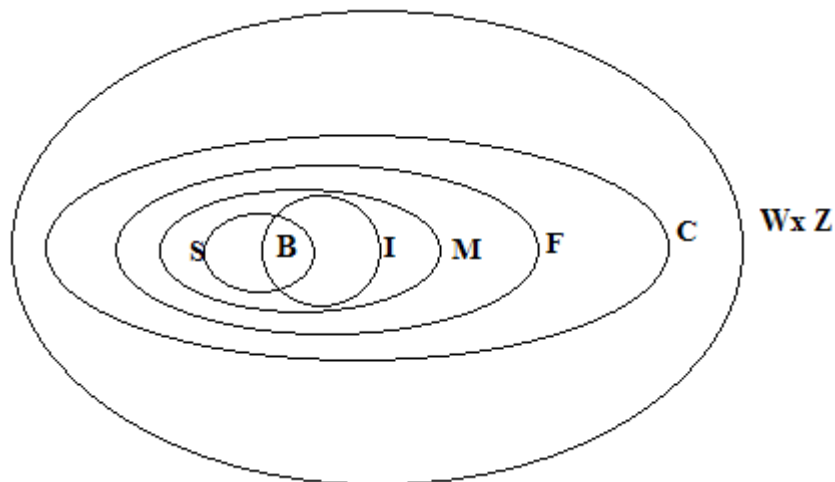
Reasons: all images are positive which means that there are negative elements from \mathbb{R} which are not images



a) $g(x) = x^3$ is both injective and surjective. Hence Bijective, see the diagram below:



5. Relationship between mappings, functions and correspondences



The following inclusion is true: $S \subset M \subset F \subset C \subset (W \times Z)$.

A bijection is a type of mapping which is at the same time surjective (S) and injective (I) mapping.

All mappings are functions $f(x)$ for which every point x has an image.

All functions are types of correspondences where you cannot observe a point which has more than one image.

Lesson 3: Existence condition for a given function

a) Learning objective: Determine the intervals for which a given function exists.

b) Teaching resources:

Manila papers, calculators.

c) Prerequisites/Revision/Introduction:

Students will perform well in this lesson if they have background knowledge on:

- Sets of numbers
- Solving equations and inequalities (unit 7) learnt in ordinary level,
- Propositional and predicate logic learnt in TTC year 1
- Calculation of numerical values with ,or without calculators

d) Learning activities

- Invite students to discuss in small groups the activity 5.3.1
- Walk around each group and ask probing questions leading them to come up with idea about existence of a function.
- In the plenary class invite representatives from groups with different findings to present for a whole class , their results;
- As a tutor, harmonize their answers and guide students discover that the function may or may not exist at a given x-value x_0 .
- After this step, guide students to do the application activity 5.2 and evaluate whether lesson objectives were achieved.

Answers to the activity 5.3.1

a) $f(x) = \frac{1}{x}$ at $x = 0$,you would be dividing by 0. So $x \neq 0$

The numerical value of $f(x)$ does not exist when $x = 0$.

b) $f(x) = \frac{2+x}{x-3}$ at $x = 3$, you would be dividing by 0. So $x \neq 3$

$f(1) = \frac{-3}{2}$, $f(2) = \frac{4}{-1} = -4$, $f(3)$, the numerical value of $f(x)$ does not exist.

c) $f(x) = \frac{2(x-1)}{x-1}$ at $x=1$, you would be dividing by 0. Therefore, $x \neq 1$

$f(0) = 2$, $f(1)$ does not exist.

d) $f(x) = \frac{x+1}{x^2-1}$ at $x=-1, x=1$, you would be dividing by 0. Therefore, $x \neq 1$ and $x \neq -1$ the function can be simplified to $f(x) = \frac{1}{x-1}$, $f(-1)$ and $f(1)$ do not exist, $f(2) = 1$

e) For the function $f(x) = \frac{2(x-1)}{x^2+1}$ there is no restriction on the independent variable, even though there is a variable in the denominator. Since $x^2 \geq 0$, $x^2 + 1$ can never be 0. The least it can be is 1.

$$f(-1) = \frac{2(-1-1)}{(-1)^2+1} = \frac{-4}{2} = -2$$

$$f(-1) = -2, f(1) = 0 \text{ and } f(2) = \frac{2}{5}$$

Answers for the application activity 5.3.1

i) $x - 4x^3 \neq 0 \Leftrightarrow x(1 - 2x)(1 + 2x) \neq 0$

$$\Leftrightarrow x \neq 0; x \neq \frac{1}{2}; x \neq -\frac{1}{2}$$

ii) $4 - x^2 > 0 \Leftrightarrow (2 - x)(2 + x) > 0$

$$\Leftrightarrow -2 < x < 2$$

iii) $x \neq 0$

iv) No restrictions

Lesson 4: Domain and range of polynomials

a) Learning objective

Determine the domain and range of polynomial functions

b) Teaching resources

Student -teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software such as Geogebra should be used etc.

c) Prerequisites/Revision/Introduction

For learners to feel comfortable in this lesson, the following prerequisites are required:

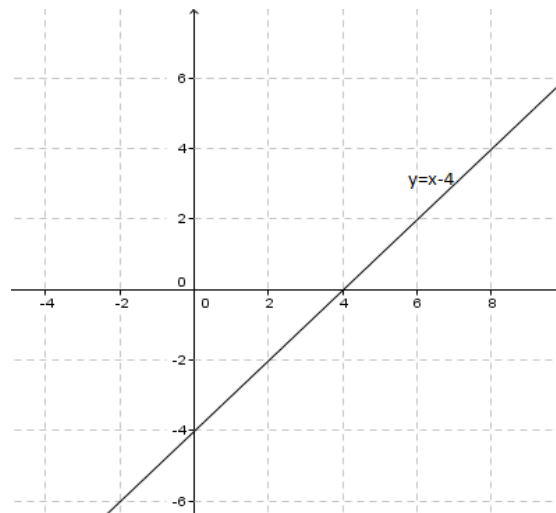
- Polynomial functions, studied in lesson1 of this unit;
- Restrictions on an independent variable, and on the depended variable, studied in lesson3 of this unit;
- Calculation of numerical values, using a calculator or not

d) Learning activities

- Invite student-teachers to work in pairs and do the activity 5.3.2 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Request two pairs to exchange their works and to discuss the set of values obtained;
- Verify and identify groups with different working steps;
- Invite one member to present the work of his/her group.
- As a tutor, harmonize the findings from presentation and guide them to explain why they took such values.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to determine the domain and range for specified functions: Constant, linear, quadratic, Polynomial functions.
- After this step, guide students to do the application activity 5.3.2 and evaluate whether lesson objectives are achieved or not for eventual improvement for the following lessons.

Answer for Activity 5.3.2

1. For any question you may graph the functions using geogebra (if possible) and projector.

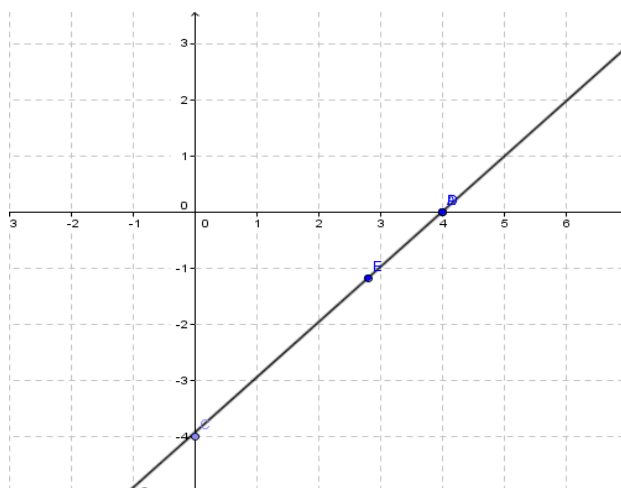


One end of the graph goes nonstop to negative infinity and another goes nonstop to plus infinity.

Alternative way of plotting

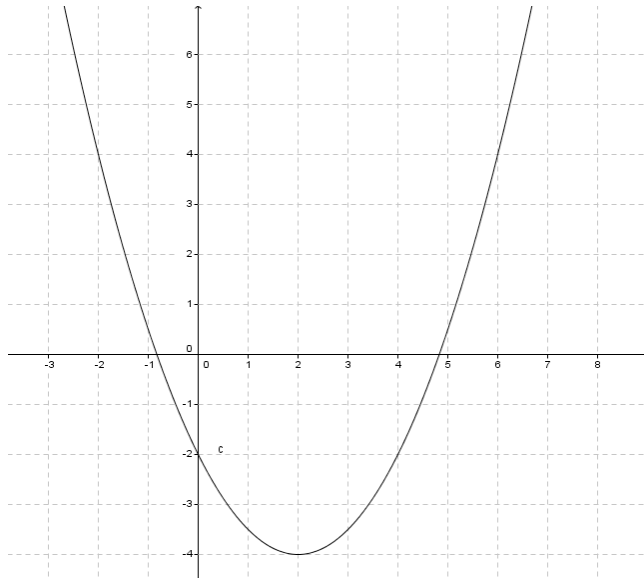
Two points are enough to plot a straight line.

x	0	-4
y	5	0



For each value of x we can get the value of y . Observing the curve, it goes from negative infinity to plus infinity on both axes.

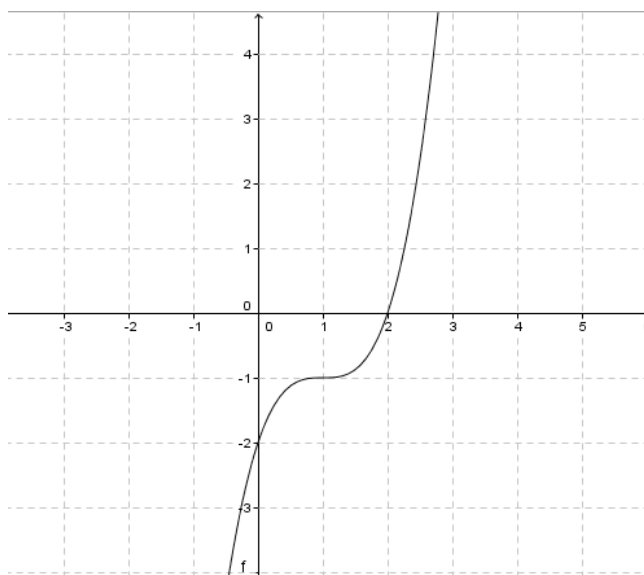
2. For any question of graphing geogebra or malmath may be used .



Using this graph, you can easily fill the table for values of x such as $x=-2; x=-1; x=0; x=1; x=2; x=3; x=4; x=5; \dots$

All x -values can get the outputs. But observing on the graph, the y -values lie in from -4 up to infinity.

3. Again using geogebra



Analysis of the graph:

Using this graph, you can easily fill the table for values of x such as $x=0; x=1; x=2; \dots$

All x -values are taken into consideration. The same as for y -values.

4. Any polynomial of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, where $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real coefficients and $n = 0, 1, 2, 3, 4, \dots$ is defined in the set of all real numbers.

Answers for application activity 5.3.2

a. Domain: $Dom f =]-\infty; +\infty[$

Range: $Im f =]-\infty; +\infty[$

b. Domain: $Dom f =]-\infty; +\infty[$

Range: $Im f =]-\infty; +\infty[$

c. Domain: $Dom f =]-\infty; +\infty[$

Range: $Im f =]-\infty; +\infty[$

Lesson 5: Domain and range of rational function

a) Learning objectives

- Determine the Domain and range rational functions
- Interpret graphs to find the domain and range of rational functions

b) Teaching resources

Student -teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software and math applications such as Geogebra, malmath, photomath should be used etc.

c) Prerequisites/Revision/Introduction

For learners to feel comfortable in this lesson, the following prerequisites are necessary:

- Solving equations studied in ordinary level and in year one;
- Polynomial functions, studied in lesson 1 of this unit;

- Restrictions on an independent variable, and on the depended variable, studied in lesson3 of this unit;
- Calculation of numerical values, using a calculator or not

d) Learning activities

- Invite student-teachers to work in small groups the activity 5.3.3 found in their Mathematics books;
- Give them time to explore and workout the activity and move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work.
- As a tutor, harmonize the findings from presentation and guide them to explain why they took such values.
- Use different probing questions and guide student-teachers to explore the content and examples given in the student's book and lead them to determine the domain of rational functions

Note: that the range will be determined only for elementary functions.

- After this step, guide students to do the application activity 5.3.3 and evaluate whether lesson objectives were achieved.

Answers to activity 5.3.3.

a) For $f(x) = \frac{1}{x}$ and if $x = 0$, we would be dividing by zero; therefore, $x=0$ is not in the domain

b) The function $f(x) = \frac{x}{(x-1)(x+3)}$ is not defined for $x = 1$ and $x = -3$

Answers for application activity 5.3.3.

1.

a) Condition of existence is that $x + 2 \geq 0 \Leftrightarrow x \geq -2$, and

$$x^2 - 9 \neq 0 \Leftrightarrow x \neq -3; x \neq 3 \text{ then the domain is } \text{dom}f = [-2, +\infty[- \{3\}$$

b) Condition of existence is that $6x^2 - x - 2 \neq 0 \Rightarrow \text{dom}f = \mathbb{R} \setminus \left\{ \frac{2}{3}, -\frac{1}{2} \right\}$

c) Condition of existence is that $25x^2 - 4 \neq 0 \Rightarrow \text{dom}f = \mathbb{R} \setminus \left\{ -\frac{2}{5}, \frac{2}{5} \right\}$.

d) Condition of existence is that $x^3 + 2x^2 - 8x \neq 0 \Rightarrow x \neq 0, x \neq -2$ or $x \neq 4$ therefore $\text{dom}f = \mathbb{R} \setminus \{-2, 0, 4\}$.

e) Condition of existence is that $x - 2 \neq 0 \Rightarrow x \neq 2$ therefore $\text{dom}f = \mathbb{R} \setminus \{2\}$.

2. Observing the graph we can see that there is a jump on $x = -1$ and $x = 1$, therefore this function is not defined on these values. Therefore,

$$\text{dom}f = \{x \in \mathbb{R} \setminus \{-1, 1\}\} \text{ and range is } \{y \in]-\infty, -1] \cup [1, +\infty[\}$$

Note: To find the range, just read the interval in which the curve is lying on y-axis. If the graph is not given use geogebra or Photomath and graph, hence read the intervals in which it is lying.

Lesson6: Domain and range of irrational function

a) Learning objectives

- Determine the Domain and range irrational functions
- Interpret graphs to find the domain and range of irrational functions

b) Teaching resources

Student -teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, graphing software and applications such as Geogebra, photomath should be used etc.

c) Prerequisites/Revision/Introduction

For learners to feel comfortable in this lesson, the following prerequisites are required:

- Solving equations studied in ordinary level and in year one;
- Solving inequalities, studied in ordinary level and in year one;
- Polynomial functions, studied in lesson1 of this unit;
- Restrictions on an independent variable, and on the depended variable, studied in lesson3 of this unit;
- Calculation of numerical values, using a calculator or not

d) Learning activities

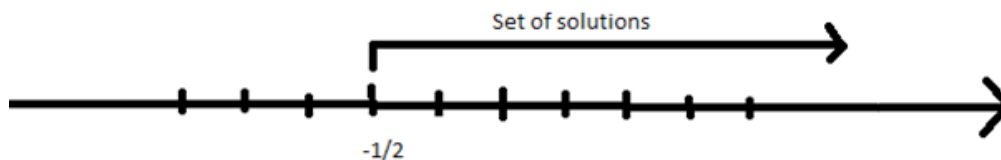
- Invite student-teachers to work in small groups the activity 5.3.4. found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from a group to present the work.
- As a tutor, harmonize the findings from presentation and guide them to explain why they took such values for which they think may cause the function undefined.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to determine the domain and range of irrational functions.

Note: that the range will be determined only for elementary functions basically using drawn graphs from geogebra or photomath.

- After this step, guide student-teachers to do the application activity 5.3.3 and evaluate whether lesson objectives were achieved.

Answers to activity 5.3.4

For this activity, by simple guided questions, the tutor will remind students that the square root of a negative number does not exist in \mathbb{R} . From this information, the condition of existence will be set up. For the function $f(x) = \sqrt{2x+1}$, the condition of existence is that $2x+1 \geq 0 \Rightarrow x \geq -\frac{1}{2}$. This is shown on a number line as follow :



The set of solutions is written as interval as follow: $\left[-\frac{1}{2}, +\infty\right[$

Therefore, the function is not defined for $x \in]-\infty; -\frac{1}{2}[$

For this activity, by simple guided questions, the tutor will remind students that a radical with an odd index is the same as a polynomial.

Therefore the function $f(x) = \sqrt[3]{x^2 + x - 2}$ is defined for every value in the set of real numbers; the function is not defined for the empty set.

The function is $g(x) = \sqrt{\frac{x-2}{x+1}}$ is not defined for the values of x such that

$$\frac{x-2}{x+1} < 0 \text{ or } x = -1$$

To identify this interval, one can solve $\frac{x-2}{x+1} < 0$ and $x = -1$ or use the table of signs.

x	$-\infty$		-1		2		$+\infty$		
$x-2$	-	-	-	-	-	0	+	+	+
$x+1$	-	-	-	0	+	+	+	+	+
$\frac{x-2}{x+1}$	+	+	+	undefined	-	-	0	+	+

If $x \in]-1; 2[$ the function is undefined.

Note: For $\frac{x-2}{x+1} \leq 0$, ensure that the students avoid the errors or wrong calculations consisting of cross-multiplying when solving an inequality. The use of a table is better to provide the all desired interval.

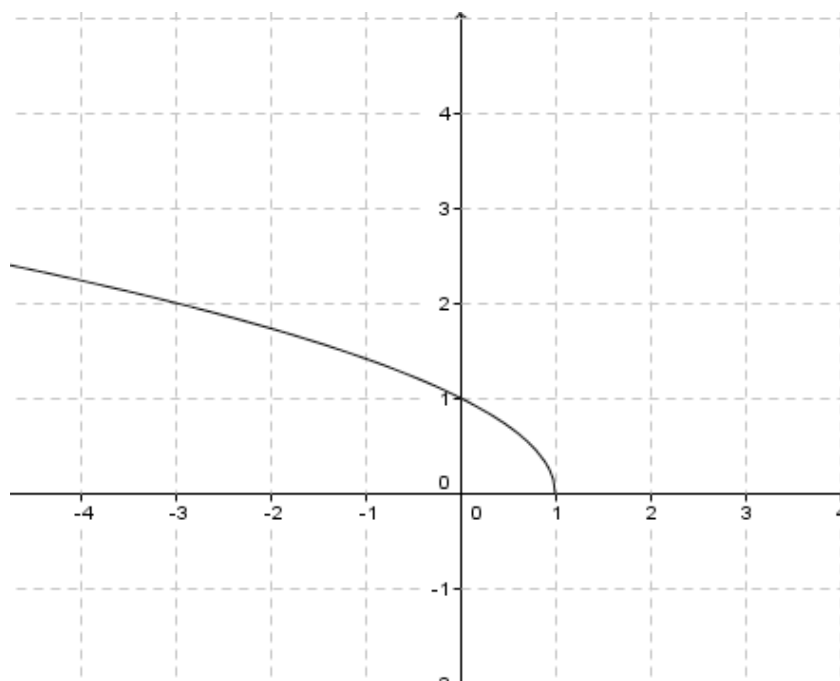
Answers for application activity 5.3.4.

1.

Function	Domain of definition.
$f(x) = \sqrt{4x-8}$	$[2, +\infty[$
$g(x) = \sqrt{x^2 + 5x - 6}$	$] -\infty, -6] \cup [1, +\infty[$
$h(x) = \frac{x^3 + 2x^2 - 2}{\sqrt[3]{x+4}}$	$] -\infty; -4[\cup] -4; +\infty[$
$f(x) = \frac{x-2}{\sqrt[4]{x^2-25}}$	$] -\infty, -5[\cup] 5, +\infty[$

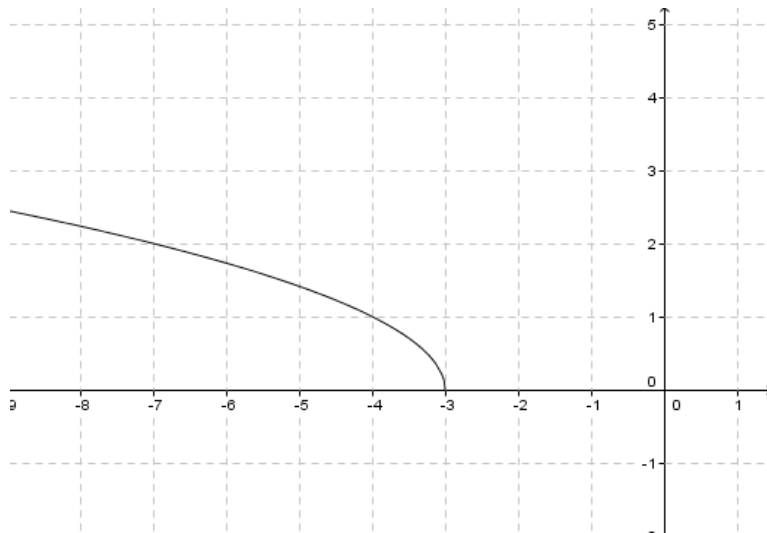
$f(x) = \sqrt{\frac{(x-1)^2}{x+4}}$	$] -4, +\infty[$
$h(x) = \sqrt{\frac{(x-1)(x+3)}{8-2x}}$	$] -\infty; -3] \cup [1; 4[$
$f(x) = \frac{x-1}{\sqrt{2-x}}$	$] -\infty, 2[$
$f(x) = \sqrt{4-x^2}$	$[-2, 2]$

a) Graph of the function $f(x) = \sqrt{1-x}$



The domain of $f(x) = \sqrt{1-x}$ is $Domf =]-\infty; 1]$ and the range is $Imf = [0; +\infty[$

b) Graph of the function $f(x) = \sqrt{-x-3}$



The domain is $Domf =]-\infty; -3]$ and the range is $Imf = [0; +\infty[$ range

Lesson 7: Composition of Functions

a) Learning objectives:

- Define the composition of functions.
- Find the composite of functions
- Demonstrate understanding of composition of functions.

b) Teaching resources:

Online resources, books, manila papers,

c) Prerequisites

Student-teachers will understand better this lesson if they have mastered:

- Composition of functions studied in ordinary level;
- Operations on polynomials studied in year one, unit 4;
- Polynomial functions studied in year one, unit 11.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 5.3.5. found in their Mathematics books;

- Move around for facilitating students where necessary and ask some challenging questions to lead them to find correct answer for the activity;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance concept of composition of functions.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the content developed
- Let students ask any remaining question to help them for better understanding.
- Invite students to work on the application activity 5.3.5. and evaluate whether lesson objectives were achieved.

Answer for activity 5.3.5.

- a) $18x^2 + 30x + 7$, yes the function is in terms of x .
- b) $6x^2 + 6x - 13$, the functions obtained is new function.
- c) The functions $f[g(x)]$ and $g[f(x)]$ are different.

Answers to application activity 5.3.5

- a. $f[g(x)] = x^2 + 2x + 2$ b. $g[f(x)] = x^2 + 2$ c. $fog(1) = 5$
- d. $gof(1) = 3$ e. $fog(x^2) = x^4 + 2x^2 + 2$ f. $gof(\sqrt{x}) = |x| + 2$

Lesson 8: inverse function

a) Learning objectives:

- Determine whether a given function is invertible
- Find the inverse of a function.

b) Teaching resources:

Digital materials including calculator, sticks, manila papers, markers, Mathematics book, etc

c) Prerequisites:

The student-teachers will feel comfortable in this lesson if they have mastered:

- Inverse function, studied in ordinary level;
- Bijective functions studied in lesson 2 of this unit;
- Making a letter the subject of a formula, studied in ordinary level;

d) Learning activities.

- Invite student-teachers to work in groups and do the activity **5.3.6** found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance concept of inverse of functions.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the content developed
- Let students any remaining question to help them for better understanding.
- Invite students to work on the application activity **5.3.6**. And evaluate whether lesson objectives were achieved.

Answers to the Activity 5.3.6

a) $y = 4x + 6 \Rightarrow x = \frac{y-6}{4}$

b) $f^{-1}(x) = \frac{x-6}{4}$

c) $f[f^{-1}(x)] = x$ and $f^{-1}[f(x)] = x$. Therefore, $f^{-1}[f(x)] = f^{-1}[f^{-1}(x)]$

Answers for the application activity 5.3.6

1. a) $x = \frac{14.75 - p}{0.01p}$

b) $x = \frac{14.75 - p}{0.01p} \Rightarrow x = \frac{14.75 - 10}{0.01(10)} = 47.5$ units.

2. a) let us find $f[g(x)] = 5\left(\frac{x-1}{5}\right) + 1 = x$ and $g[f(x)] = \frac{(5x+1)-1}{5} = x$

Hence, the two functions are inverse to each other since the condition is verified.

b) We need to show that $f[g(x)] = g[f(x)] = x$.

$$f[g(x)] = 9 - (\sqrt{9-x})^2 = x \text{ And } g[f(x)] = \sqrt{9 - (9-x^2)} = x, \text{ provided } 9-x \geq 0$$

Hence the two functions are inverse to each other.

c) We are supposed to show that $f[g(x)] = g[f(x)] = x$

$f[g(x)] = 1 - (\sqrt[3]{1-x})^3 = x$ and $g[f(x)] = \sqrt[3]{1 - (1-x^3)} = x$, then the two functions are inverse to each other.

Lesson 9: Even function and symmetry

a) Learning objective

Analyze whether a function is even, algebraically and from the graph

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, etc.

c) Prerequisites/Revision/Introduction

Student-teachers will participate fully in this lesson, if:

- they have mastered reflection and symmetry, studied in ordinary level;
- powers of negative numbers , studied in ordinary level;
- calculation of numerical values;
- Plotting points and drawing figures in the Cartesian plane;

d) Learning activities

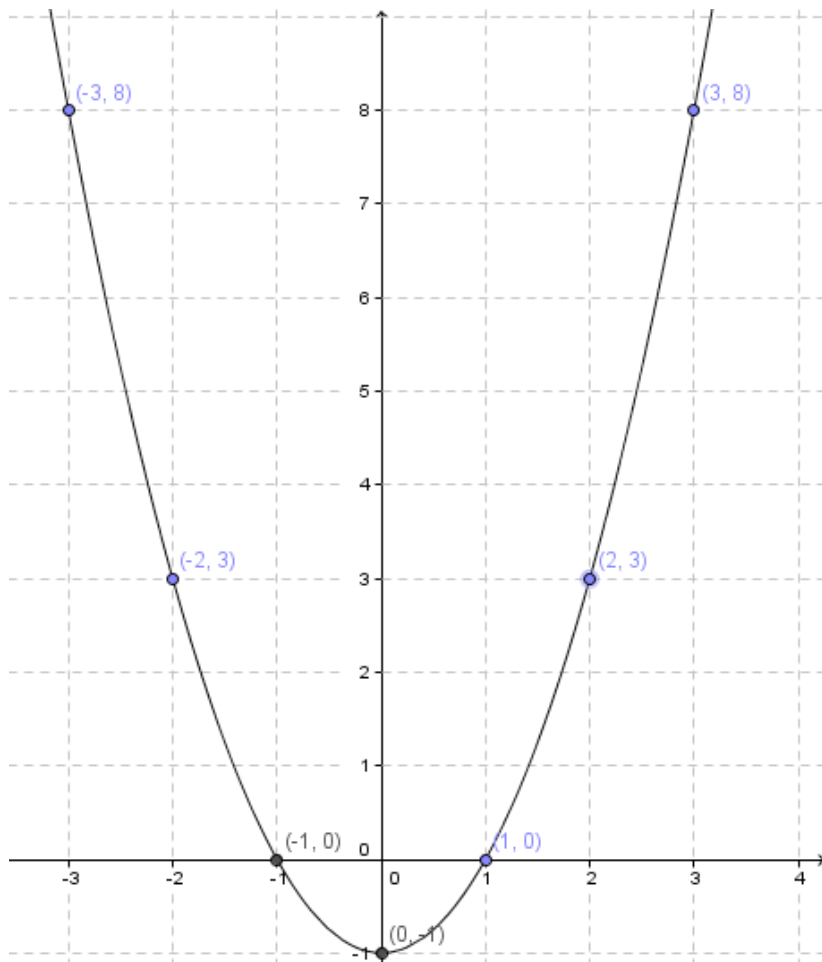
- Invite student-teachers to work in groups and do the activity 5.4.1. found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work.
- As a tutor, harmonize the findings from presentation and guide them to enhance the characteristics of even functions and odd functions.
- Use graphs for simple functions to illustrate the characteristics of even functions: The graph of even function is symmetric about the vertical axis (the line $x = 0$ is the axis of symmetry).
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to verify the parity of different functions.

- After this step, guide students to do the application activity 5.4.1. And evaluate whether lesson objectives were achieved.

Answers for activity 5.4.1

1.

x	-3	-2	-1	0	1	2	3	4	5
$y = x^2 - 1$	8	3	0	-1	0	3	8	15	24



2. The negative and positive values have the same images examples: image of -1 and 1 is 0. The line $x = 0$ divides the graph into two equal parts which are symmetric to each other.

Answers to application activity 5.4.1.

1. The function $f(x) = \frac{x^2 + 1}{x^4 + 3}$ will be even if $f(x) = f(-x)$

But $f(-x) = \frac{(-x)^2 + 1}{(-x)^4 + 3} = \frac{x^2 + 1}{x^4 + 3}$, Hence, the function is even, since $f(x) = f(-x)$

2. Not even because $f(x) \neq f(-x)$, this means $f(-x) = \sqrt[3]{x^2}(-x - 4) = -\sqrt[3]{x^2}(x + 4)$

3. Not even because $f(x) \neq f(-x)$, this means $f(-x) = -x\sqrt{9 + x}$

Lesson 10: Odd function and symmetry

a) Learning objective

Analyze whether a function is odd algebraically and from the graph

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, gridded paper, etc.

c) Prerequisites/Revision/Introduction

Student-teachers will participate fully in this lesson, if:

- they have mastered symmetry, studied in ordinary level;
- powers of negative numbers, studied in ordinary level;
- calculation of numerical values;
- Plotting points and drawing figures in the Cartesian plane;

d) Learning activities

- Invite student-teachers to work in groups and do the activity 5.4.2. found in their Mathematics books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work;
- Invite groups to present their work and as a tutor, harmonize the findings

from presentation and guide students to enhance the characteristics of even functions and odd functions.

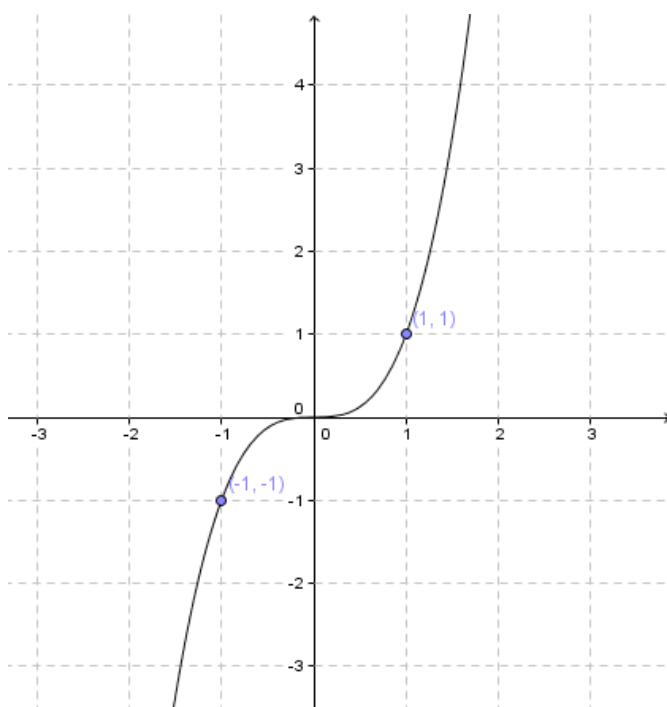
- Use graphs for simple functions to illustrate the characteristics of odd functions: the graph of odd function looks the same when rotated through half a revolution about the origin point $(0,0)$ is the centre of symmetry for its part.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead students to verify the parity of different functions.
- After this step, guide students to do the application activity 5.4.2. and evaluate whether lesson objectives were achieved.

Answers to activity 5.4.2.

For this activity geogebra may be used to plot graph

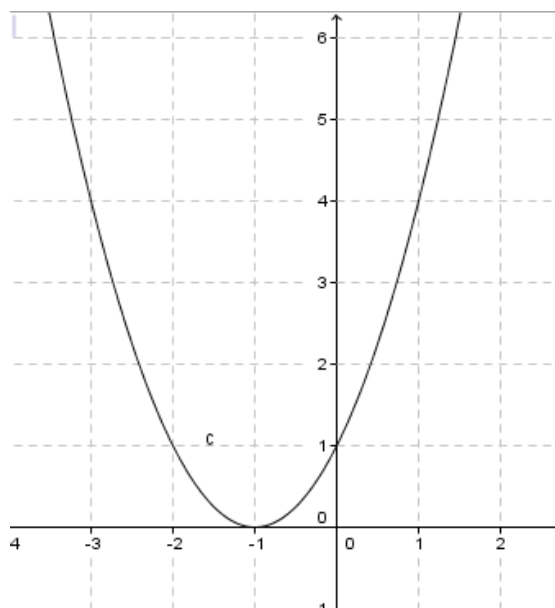
x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = x^3$	-125	-64	-27	-8	-1	0	1	8	27	64	125

Graph of the function $f(x) = x^3$



The images of negative and positive x-values are symmetric to each other compare to the point (0,0).The point (0,0) is the center of symmetry of the graph. Therefore, the function is odd.

Graph of the function $f(x) = x^2 + 2x + 1$



The graph is not symmetric about the origin (0,0) nor symmetric about the y-axis. Therefore, this function is neither even nor odd.

Answers to application activity 5.4.2.

1.

- Condition of existence for $f(x) = \frac{x^3}{9-x^2}$ is $9-x^2 \neq 0$ therefore, $dom f =]-\infty, -3[\cup]-3, 3[\cup]3, +\infty[$
- For this question the students-teacher will verify whether, $f(x) = f(-x)$ or whether $f(-x) = -f(x)$, the conclusion is that the function is **odd** because $f(-x) = -f(x)$.

2. The function $f(x) = \sqrt{x^2 + 5x + 6}$ is valid if $x^2 + 5x + 6 \geq 0$. the student-teacher can determine the exact interval in which this function is positive. can determine 'The students will verify whether $f(x) = f(-x)$ or whether $f(-x) = -f(x)$ the conclusion is that the function is neither even nor odd.

5.6. Summary of the unit

In this unit we dealt with numerical functions: **polynomials, rational and irrational** functions, how to determine whether a given function is a polynomial, rational or irrational function.

We focused on:

- Some particular properties that a numerical function can possess or not, such as: **“being one-to-one”, “being onto”, “being bijective”, “being even”, “being odd”**. In each case, we were able to determine the property algebraically and using graph.
- Some particular subsets of the sets between which the function is established: the **domain and the range**; in each case, we saw how to use equations and inequalities to determine the domain, and how to use graphs to determine the domain and the range;
- Some operations that can be performed with numerical functions: **composition** and **inverse** of functions;

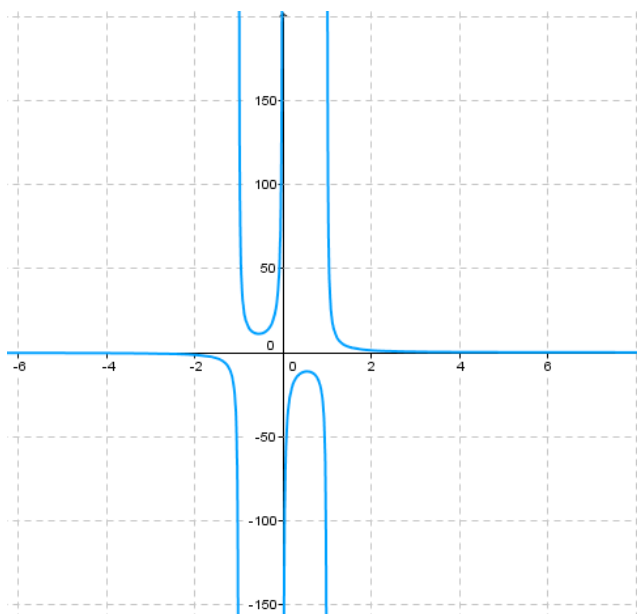
5.7. Additional Information for Teachers

- The teacher must be equipped with skills about using geogebra if possible install it the smart classroom computers to help learners for participation.
- It could be better to let learners graph and explore by themselves the graphs of functions in that case in smart classroom.
- Some concepts are tricky and you need to be aware so that your lesson does not turn to a mess.
- Emphasize the use of graph paper/gridded paper while student-teachers draw the graphs.
- Emphasize and facilitate students to use geometric materials to ameliorate the quality of graphs.
- Remind students to name axes (x-axis and y-axis).
- Recall them to mention/highlight the origin/intersection point of axes by (0.0).

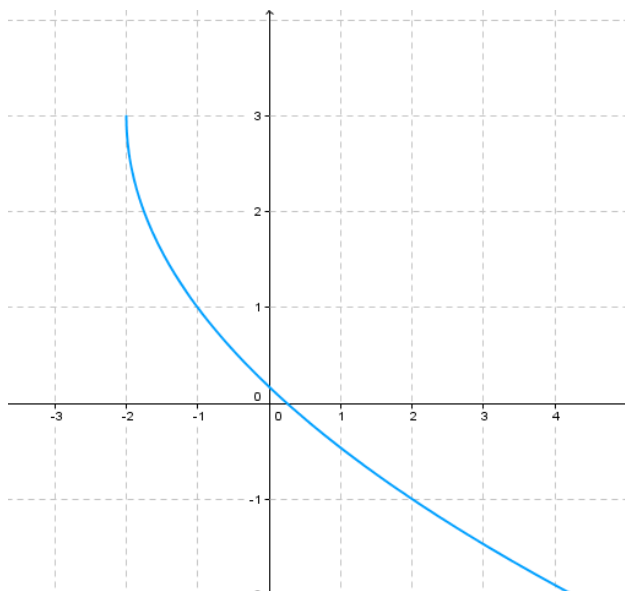
5.8 Answers for end unit assessment.

1. The students will verify whether $f(x) = f(-x)$ or whether $f(-x) = -f(x)$

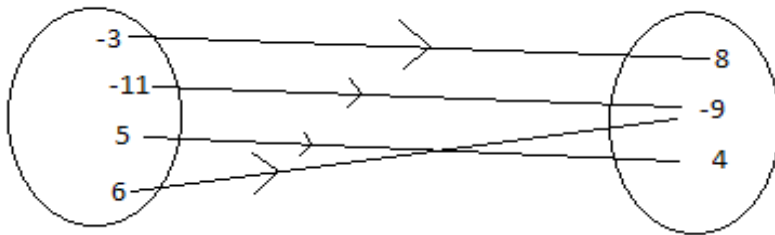
a) The function is odd. There, the graph is symmetric to the origin (see the graph)



b) The function is neither even nor odd. There, the graph is neither symmetric to the origin nor to the axis.



2.Hint: if drawn, using sagittal diagrams, the learners will easier get the point.



This function is subjective but not injective since elements are sharing the same image.

3. If the graph is needed use geogebra.

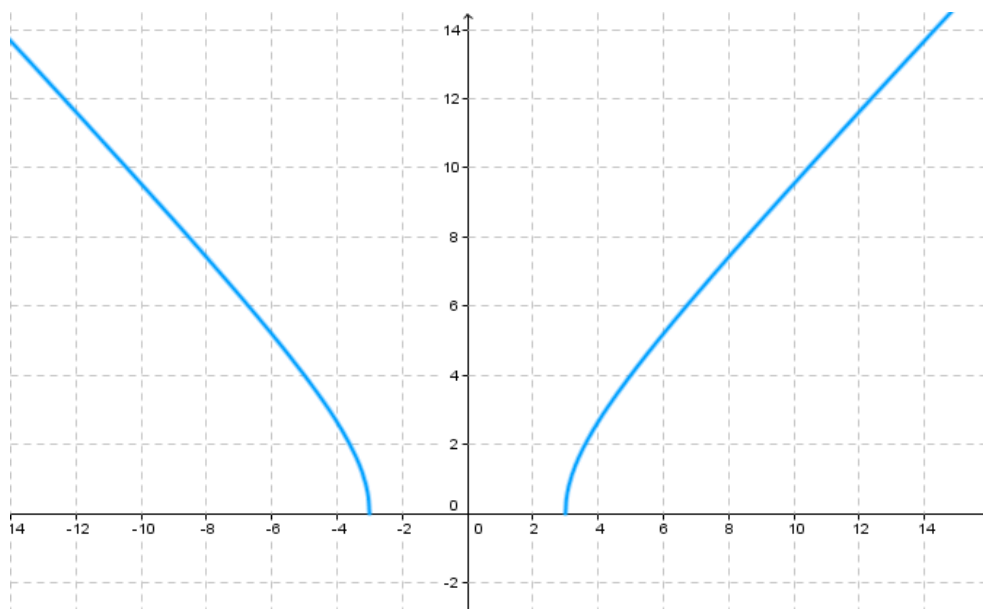
i. $h(5) = \sqrt{5^2 - 9} = 4$

ii. $h(x) = 2 \Rightarrow \sqrt{x^2 - 9} = 2$

$$x^2 = 11 \Rightarrow \begin{cases} x = \sqrt{11} \\ x = -\sqrt{11} \end{cases} \text{ Remember to verify the restrictions.}$$

iii. The domain is $]-\infty, -3] \cup [3, +\infty[$

Can be obtained from the graph of the function $h(x) = \sqrt{x^2 - 9}$



The range can be seen from the graph $[0; +\infty[$

4.

i. The idea is just to replace as usual. $h(-4) - f(-1) = 0$

ii. If $g(x) = 0$ that is $\frac{x^2}{x-3} = 0 \Rightarrow x=0$

iii. Existence condition is required for this sub-question.

- For $g(x)$ to be defined, $x-3 \neq 0 \Rightarrow \text{dom}g =]-\infty, 3[\cup]3, +\infty[$
- For $h(x)$ to be defined, $5-x \geq 0 \Rightarrow \text{dom}h =]-\infty, 5]$

5.

i. Let $y = 3x - 5$ if we undo each operation and solve for x , we get $x = \frac{y+5}{3}$

Therefore $f^{-1}(x) = \frac{x+5}{3}$

ii. Let $y = \frac{2}{x-4} \Leftrightarrow y(x-4) = 2$

$xy - 4y = 2$ Remember we are solving for x

$xy = 4x + 2 \Rightarrow x = \frac{4x+2}{y}$ or $f^{-1}(x) = \frac{4x+2}{x}$

6. $f^{-1}(x) = x - 2 = g(x)$

7. The functions are inverse to each other if $f[g(x)] = g[f(x)] = x$

$f[g(x)] = (x-2) + 2 = x$ And $g[f(x)] = (x+2) - 2 = x$

Hence the two functions are inverse to each other.

Rules for Excluding Numbers from the Domain of $f[g(x)]$	Applying the Rules to $f(x) = \frac{2}{x-1}$ and $g(x) = \frac{3}{x}$
<p>If x is not in the domain of $g(x)$ it must not be in the domain of $f[g(x)]$</p> <p>Any x for which $g(x)$ is not in the domain of $f(x)$ must not be in the domain of $f[g(x)]$</p>	<p>Because $g(x) = \frac{3}{x}$, 0 is not in the domain of $g(x)$. Thus, 0 must be excluded from the domain of $f[g(x)]$.</p> <p>Because $f[g(x)] = \frac{2}{g(x)-1}$, we must exclude from the domain of $f[g(x)]$ any x for which $g(x)$ is equal to 1.</p> $\frac{3}{x} = 1 \Rightarrow x = 3$ <p>3 must be excluded from the domain of $f[g(x)]$</p> <p>Hence, the domain of $f[g(x)]$ is $]-\infty, 0[\cup]0, 3[\cup]3, +\infty[$</p>

8.

- a. The model $f(x) = x - 400$ for the price of the computer, means that the reserved price of the computer is 400\$. A buyer to buy a computer must at least have more than 400\$.

The model $g(x) = 0.75x$ means that the price has been discounted by 25% of its original price.

- b. $f[g(x)] = 0.75x - 400$ \$. There is a discount of 25% but whatever the case, the cost price cannot go below 400\$

5.9 Additional activities

5.9.1 Remedial activities

1. Let the function $f(x) = -2x^2 + 6x - 3$ find $f(-2)$

Solution: $f(-2) = -23$

2. Consider the function $f(x) = -7x - 5$ and $g(x) = 10x - 12$ find $g[f(x)]$ and $f[g(x)]$

Solution

$$g[f(x)] = g(-7x - 5) = 10(-7x - 5) - 12 = -70x - 62$$

$$f[g(x)] = f(10x - 12) = -7(10x - 12) - 5 = -70x + 79$$

3. Use a dictionary or research on internet to define the following terms in mathematics context: Relations, functions, mappings, injection, surjection and bijection.

4. Find the domain of the real valued function h defined by $h(x) = \sqrt{x - 2}$

5.9.2 Consolidation activities

1. Find the domain of the function defined by $f(x) = \sqrt{-x^2 + 9} + \frac{1}{x - 1}$

Solution:

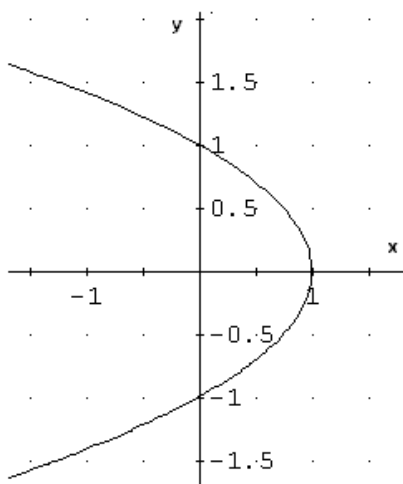
$$\text{dom} f = [-3, 1[\cup]1, +3]$$

2. Discuss whether the function defined from $\mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x^2$ is injective, surjective or neither.

Solution:

- Using counter example $f(-2) = 4$ and $f(2) = 4$, Hence the function is not injective since two elements from the domain are sharing the same image.
- The function $f(x) = x^2$ is always positive for any value of x that we can take. For that, the function is not surjective since negative y -values are missing their antecedents.

3. Does the graph below represents a function? Explain.



Solution:

No, the graph is not a function since for one x-value (0), there are two y-values (1 and -1).

5.9.3 Extended activities

1. A small ball is projected vertically upward from the top of a building with the initial velocity $v_0 = 144 \text{ m/sec}$. Its distance $s(t)$ in meter above the ground after t seconds is given by the equation $s(t) = -16t^2 + 144t + 100$.

- What is the distance $s(t)$ at the initial time when $t = 0$?
- Make a table of values for $s(t)$ to show the distance from the initial time $t=0$ to $t=10$ seconds.
- Use the table to draw the graph of $s(t)$ and show the position of the ball at $t = 5$ seconds
- Discuss the parity of the function $s(t)$.

Solution:

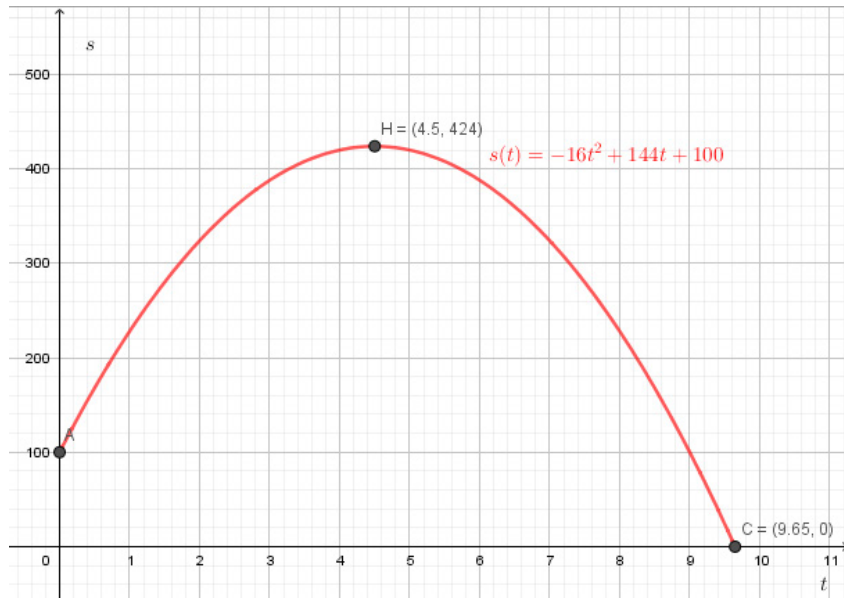
$s(t) = -16t^2 + 144t + 100$ represents the height or the vertical distance above the ground.

a) At the initial time $t = 0$, then the distance is $s(0) = -16(0) + 144(0) + 100 = 100$

This distance is 100m.

b)

T	0	1	2	3	4	5	6	7	8	9	$\frac{483}{50}$
$s(t)$	100	228	324	388	420	420	388	324	328	100	0



The graph shows that the ball falls on the ground between $t = 9$ and $t = 10$.

To find this time, you must solve the equation $s(t) = -16t^2 + 144t + 100 = 0$.

$$t = \frac{483}{50}$$

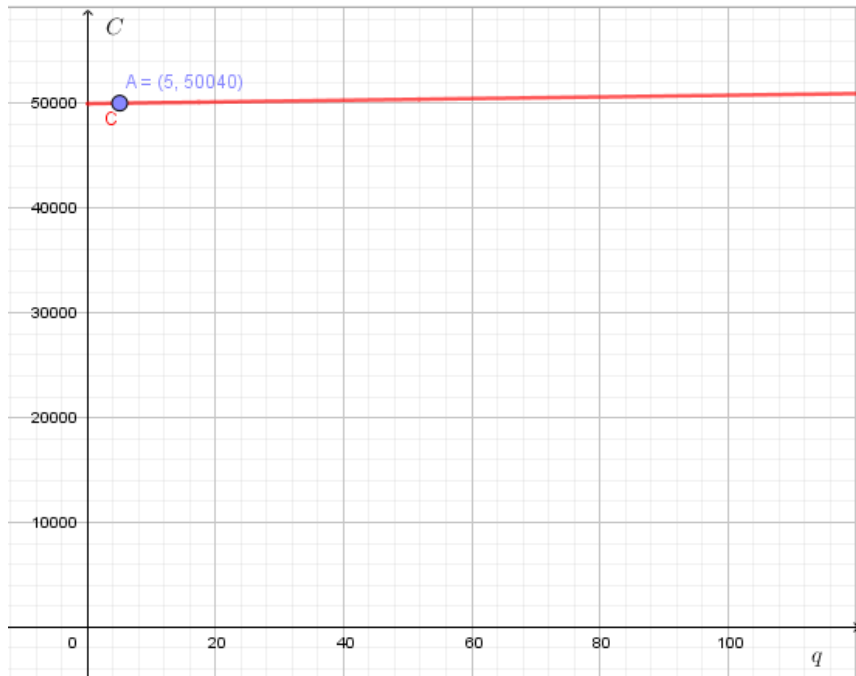
2. The total cost C for units produced by a company is given by $C(q) = 50000 + 7q$ where q is the number of units produced.

Solution:

a) The amount 50000 represents the fixed cost;

b) the number 8 represents the the marginal cost(cost of a unit of product);

c) The graph for $C(q) = 50000 + 7q$ is the following:



d) The real domain of C that corresponds to q which is positive is $[0; \infty[$. The range is $[50000; \infty[$

e) $C(q)$ is not an odd function because $C(-q) \neq -C(q)$.

UNIT 6

LIMITS OF POLYNOMIAL, RATIONAL AND IRRATIONAL FUNCTIONS

6.1. Key unit competence

Evaluate correctly limit of function and deduce asymptotes of a real function.

6.2 Prerequisite

Student-teachers will perform better in this unit if they:

- have a good background on numerical functions, from unit5 preceding this unit;
- have mastered sequences, studied in unit1 of this year;
- are familiar with graphs and their interpretation;
- are skilled enough on intervals in the set of real numbers;
- Are able to find the Cartesian equation of lines, and their positions, as studied in ordinary level.

6.3 Cross-cutting issues to be addressed

- Financial education (by providing examples reflecting economic issues)
- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)

6.4 Guidance on introductory activity

Help student-teachers to grasp the general idea of the whole unit: “limit of a function $f(x)$ as the independent variable x approaches x_0 as the behavior of the function as x assumes values closer and closer to x_0 ”

You can proceed as follows:

- Give clear instructions for students to form small groups and to work on the introductory activity;
- As they are discussing, circulate around to note the relevancy of the discussion and to provide guidance where necessary;
- Ensure that the learners have understood what the unit will be about and they are eager to learn; you can observe this through a clear and concise presentation of a group chosen randomly and the degree of attention other students are paying to the presentation;
- Sustain the curiosity of the learners by a proper management of your class;
- A prior knowledge of software and math applications such as geogebra and malmath would be very useful for you and for your students;

Answers for introductory activity 6.0

1) As $P = 200 - 0.24Q$, then

Q	1	0.5	0.1	0.01	0.001	0.0001	...	0
P	200	200	200	200	200	200		200

When Q approaches 0, the price gets closer and closer to 200.

2) The values of P when Q approaches 20 are given in the table below:

Q	19.5	19.9	19.9999	20	20.1	20.2	20.5	21
P	200	200	195.20024	195.2	195.176	195.152	195.08	194.96

When Q approaches 20, the price gets “closer and closer” to 195.2. This can be

written as $\lim_{Q \rightarrow 20} P = \lim_{Q \rightarrow 20} 200 - 0.24Q = 195.2$

6.5 List of lessons/sub-heading

No	Lesson title	Learning objectives	Number of periods
0	Introductory unit 6	To arouse the curiosity of student teachers on the content of unit 6	1
1	Introduction to limits of a function	Define the neighbourhood of a real number, the concept of limit for a real-valued function and the value of a function at a given point of one real variable	2
2	Graphical interpretation of limits of a function	Illustrate and interpret graphically the limit of a function To recognize geometrically whether a limit exists or not.	2
	Limit of a variable (Conditions, One-sided limits)	Evaluate the condition of existence of a limit for a function at a given point.	2
3	Properties of limits	Establish the properties of limit of functions and calculate limits for elementary functions.	2
4	Finite limits and Infinite limits	Calculate the finite and infinite limits of certain functions	1
	Limits of functions at infinity and involving infinity	Perform operations on limits involving infinity and discuss the Squeeze theorem	3
5	Infinity limits and rational functions.	<ul style="list-style-type: none"> To distinguish limits at infinity from infinity limits. To calculate finite and infinity limits of rational functions by simplification. 	1

6	Indeterminate cases $\left(\frac{\infty}{\infty}, \frac{0}{0}, \infty - \infty, 0 \cdot \infty\right)$	Calculate the true values of limits by removing the indeterminate forms $\left(\frac{\infty}{\infty}, \frac{0}{0}, \infty - \infty, 0 \cdot \infty\right)$	4
7	Graphs and limits of function	Guess the limit of a function on its graphical representation	1
8	Limit and Continuity of a function (Continuity of a function at a point or on interval, Properties on continuity, Point of discontinuity)	Solve problems involving continuity	2
	Asymptote		2
	Application of limits to determine asymptotes to curves of functions (Horizontal, vertical, oblique)	Extend the concept of limit to determine the asymptotes of a given function	2
9	Applications of limits in real life: Solving Problems involving limits in real life	solve problems involving limits in real life	2
10	End unit assessment		1
Total number of periods in this unit			28

Lesson 1: Introduction to limits and neighborhood of a function

Learning objective:

Define the concept of limit for a real-valued function, the neighbourhood of a real number and the value of a function at a given point of one real variable.

a) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, graph paper, ruler, markers, pens, pencils, tablets or computers for graphing etc.

b) Prerequisites/Revision/Introduction:

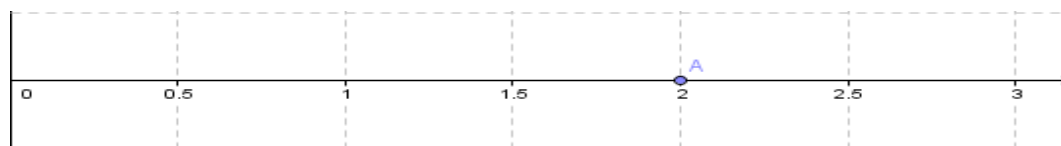
Student-teachers will perform well in this unit if they are skilled enough in the content of the second unit (Equations and Inequalities) of mathematics in student-teacher's book for year one social studies education.

c) Learning activities

- Invite student-teachers to work in group and do the activity 6.1.1 and activity 6.1.1 found in their Mathematics books;
- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite one member from each group to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to explore the content and examples given in the student's book where they will be able to differentiate the neighbourhood of a real number and the value of a function at a given point.
- After the lesson, guide students to do the application activity 6.1.1 and evaluate whether lesson objectives were achieved.

Answer for activity 6.1.1

a. Yes, it is possible



b. Open intervals: $]1.9; 2.1[$ and $]1.99; 2.01[$

c. $2.5 \approx 3$

$3.85714 \approx 3.86 \approx 3.9 \approx 4$

$5.0 \approx 5$

$4.157894737 \approx 4.157 \approx 4.2 \approx 4$

Answer of Application activity 6.1

1) Students may give different answers, verify if they are correct. The answers will depend to the location of the city given by the students. However, the teacher need to examine the with the students, to acquire well the concept of neighborhood.

Examples:

San Marino, a state surrounded by Italy

Vatican City, a state forming part of Rome, thereby surrounded by Italy

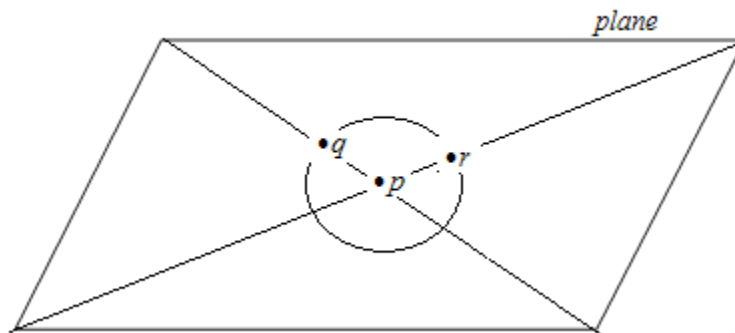
Nyarugenge surrounded by Kigali City

Nyamabuye surrounded by Muhanga.

2) There are many possible answers verify that -5 is the center of the interval.
For example: $] -6; -4[$, $] -5.1; -4.9[$, $] -5.01; -4.99[$.

3) No, if the point is on the circumference, then it is impossible to center a disc at the point such that the disc is contained in the circle.

4) The following plane is a neighborhood of points p , q and r .



5) From a-c let students use their calculator and approximate the limits by filling the table

They will get:

a. $\lim_{x \rightarrow 2} (3x - 2) = 4$

b. $\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1} - 1} = 0$

c. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$

d. $\lim_{x \rightarrow 1} (2x^2 + x - 4) = -1$

e. $\lim_{x \rightarrow -1} \frac{x + 1}{x^2 - x - 2} = -\frac{1}{3}$

Lesson 2: Graphical interpretation of limits of a function

a) Learning objective:

- Illustrate and interpret graphically the limit of a function.
- Recognize geometrically whether a limit exists or not.
- Evaluate the condition of existence of a limit for a function at a given point.

b) Teaching resources

Student-teacher's book and other textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, online resources etc.

c) Prerequisites/Revision/Introduction

In this lesson, Student-teachers will perform better if they are enough skilled in:

- Interpreting graphs and functions in terms of behavior in the neighborhood of a real number ;
- Cartesian plane: plotting points, drawing lines;
- Performing operations on intervals;

d) Learning activities

Invite student-teachers to work in groups and do the activity 6.2 in their Mathematics books;

- Move around in the class for facilitating where necessary and give more clarification on eventual challenges they may face during their work;
- Invite a student from each group to present their findings;
- As a tutor, harmonize the findings from presentation and guide them to guess the true definition of limit of a function at a given point basing on the value of the left hand limit and the right hand limit.
- Use different probing questions and guide them to explore the content and examples given in the student's book lead them to be able to determine the limit of a function at a point when applying related properties.
- After this step, assign students to do the application activity 6.2. and evaluate whether the lesson objectives were achieved.

Answers for activity 6.2

$$\text{a. } f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases} \quad g(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

b. If we stay to the left side, as x approaches 0, $f(x)$ gets closer to 0

c. If we stay to the right side, as x approaches 0, $f(x)$ gets closer to 1

d. If we stay to the left side, as x approaches 1, $g(x)$ gets closer to 0

e. If we stay to the right side, as x approaches 1, $g(x)$ gets closer to 1

Answers for application activity 6.2.

a. $\lim_{x \rightarrow -1^+} f(x) = 1$ and $\lim_{x \rightarrow -1^-} f(x) = 2$ therefore $\lim_{x \rightarrow -1} f(x)$ Does not exist

Since $\lim_{x \rightarrow -1^+} f(x) \neq \lim_{x \rightarrow -1^-} f(x)$.

b. $\lim_{x \rightarrow 2^+} f(x) = 1$ and $\lim_{x \rightarrow 2^-} f(x) = +\infty$ therefore the $\lim_{x \rightarrow 2} f(x)$ does not exist

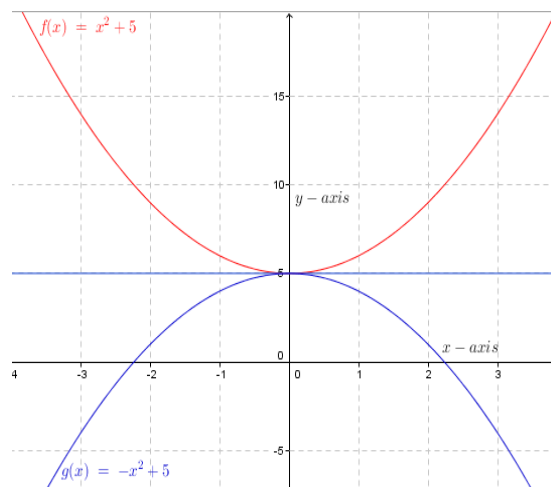
$$1. \lim_{x \rightarrow -1} \frac{|x+1|}{x^2-1} = \begin{cases} \lim_{x \rightarrow -1^-} \frac{-(x+1)}{x^2-1} = \frac{1}{2}, \text{ if } x < -1 \\ \lim_{x \rightarrow -1^+} \frac{(x+1)}{x^2-1} = -\frac{1}{2}, \text{ if } x \geq -1 \end{cases} \quad \text{Therefore } \lim_{x \rightarrow -1} \frac{|x+1|}{x^2-1} \text{ does not exist}$$

since the left limit is different from the right limit.

2. $\lim_{x \rightarrow -1^-} h(x) = -\infty$ and $\lim_{x \rightarrow -1^+} h(x) = +\infty$ this simply means that the $\lim_{x \rightarrow -1} h(x)$ does not exist.

$$\lim_{x \rightarrow +\infty} h(x) = 1 \text{ and } \lim_{x \rightarrow -\infty} h(x) = 1$$

3.



The curve of $h(x) = 5$ lies between other two curves and the three curves meet at the same point $(0, 5)$

$$\text{Therefore, } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} h(x) = 5$$

Lesson 3: properties of limits

a) Learning objective

Establish the properties of limit of functions and calculate limits for elementary functions.

b) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in:

- Properties of operations;
- Representation in the Cartesian plane of points and lines;
- Properties of inequalities in the set of real numbers.

d) Learning activities

- Invite student-teachers to work in group and do the activity 6.2 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
- After the lesson, guide students-Teacher to do the application activity 6.2 and evaluate whether lesson objectives were achieved.

Answers to activity 6.2

- a) $\lim_{x \rightarrow 0} [3(3x - 1)] = 3 \left[\lim_{x \rightarrow 0} (3x - 1) \right] = -3$
- b) $\lim_{x \rightarrow 0} (x^2) = 0, \lim_{x \rightarrow 0} (3x - 1) = -1$ and $\lim_{x \rightarrow 0} (x^2 + 3x - 1) = -1$
 $\Rightarrow \lim_{x \rightarrow 0} (x^2 + 3x - 1) = \lim_{x \rightarrow 0} (x^2) + \lim_{x \rightarrow 0} (3x - 1)$
- c) $\lim_{x \rightarrow 1} (x^2 + 3x - 6) = -2, \lim_{x \rightarrow 1} (x + 4) = 5, \lim_{x \rightarrow 1} \frac{x^2 + 3x - 6}{x + 4} = \frac{-2}{5}$
 $\Rightarrow \lim_{x \rightarrow 1} \frac{x^2 + 3x - 6}{x + 4} = \frac{\lim_{x \rightarrow 1} (x^2 + 3x - 6)}{\lim_{x \rightarrow 1} (x + 4)}$
- d) $\lim_{x \rightarrow 2} (x - 1) = 1, \lim_{x \rightarrow 2} (x + 4) = 6, \lim_{x \rightarrow 2} (x^2 + 3x - 4) = 6$
but $(x^2 + 3x - 4) = (x - 1)(x + 4)$ Therefore
 $\lim_{x \rightarrow 2} (x^2 + 3x - 4) = \lim_{x \rightarrow 2} (x - 1) \lim_{x \rightarrow 2} (x + 4) = 6$
- e) $\lim_{x \rightarrow -4} [(x^2 + 1)^2] = \left[\lim_{x \rightarrow -4} (x^2 + 1) \right]^2 = 289$

Answers to application activity 6.2

If $\lim_{x \rightarrow 3} f(x) = 3$ and $\lim_{x \rightarrow 3} g(x) = -3$, then

- a) $\lim_{x \rightarrow 3} [f(x) + g(x)] = \lim_{x \rightarrow 3} f(x) + \lim_{x \rightarrow 3} g(x) = 3 - 3 = 0$
- b) $\lim_{x \rightarrow 3} [f(x)g(x)]^3 = \left[\lim_{x \rightarrow 3} f(x) \lim_{x \rightarrow 3} g(x) \right]^3 = [3(-3)]^3 = -729$
- c) $\lim_{x \rightarrow 3} 5f(x) = 5 \lim_{x \rightarrow 3} f(x) = 5(3) = 15$

Lesson 4: Finite and infinite limits

a) Learning objective

- Calculate the finite and infinite limits of certain functions
- Perform operations on limits involving infinity and discuss the Squeeze theorem

b) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in:

- Operations in the set of real numbers
- Sequences

d) Learning activities

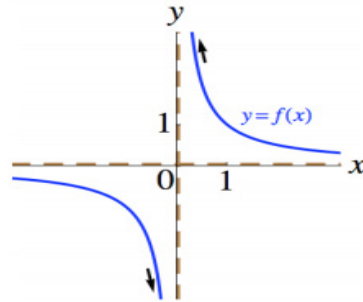
- Invite student-teachers to work in group and do the activity 6.3 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
- After the lesson, guide students-Teacher to do the application activity 6.3 and evaluate whether lesson objectives were achieved.

Answers to activity 6.3

$$1. f(0.97) = -65.\overline{66} \quad f(0.98) = -99 \quad f(0.99) = -199 \quad f(1.01) = 201$$

$$f(1.02) = 101 \quad f(1.03) = 67.\overline{66}$$

Let the learners use their calculators to fill the table



As x approaches 0 from the right, the function values **increase without bound** and as x approaches 0 from the left, the function values **decrease without bound**.

$$2. a) \lim_{x \rightarrow -\infty} \frac{1}{x} = 0 \quad b) \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \quad d) \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

$$3. a) -2 + \infty = +\infty \quad b) 3 + \infty = +\infty \quad c) \frac{3}{\infty} = 0$$

$$d) -\infty \times (+\infty) = -\infty \quad e) \frac{\infty}{-2} = -\infty$$

Note: the teacher may find an intuitive way to help learners to get the idea of the limits.

Answers to application activity 6.3

1. a)

x	$-\infty$	0	$+\infty$
$x-2$	-	undefined	+

$$\lim_{x \rightarrow 2^+} \frac{3}{x-2} = \frac{3}{0^+} = +\infty$$

$$\text{b) } \lim_{x \rightarrow 2^+} \frac{3}{x-2} = \frac{3}{0^-} \lim_{x \rightarrow -2^-} \frac{5}{x+2} = \frac{5}{0^-} = -\infty$$

$$\text{c) } \lim_{x \rightarrow 3} (-3x^2 + 7x) = -3(3)^2 + 7(3) = -6$$

$$\text{d) } \lim_{x \rightarrow -1} \frac{4x-5}{3-x} = \frac{-1}{3}$$

$$\text{e) } \lim_{x \rightarrow 5} \frac{\sqrt{x+4}-2}{x} = \frac{1}{5}$$

$$\text{f) } \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x} = \frac{1}{3}$$

$$\text{g) } \lim_{x \rightarrow 4} \sqrt[3]{x+4} = 2$$

$$\text{h) } \lim_{x \rightarrow +\infty} \frac{x}{x^2} = \lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

2.

$$\text{a) } \bar{C} = \frac{60x+1650}{x} = 60 + \frac{1650}{x}$$

$$\text{b) } \bar{C}(100) = \frac{60(100)+1650}{100} = 76.5\$$$

If 100 tons are sold, the average cost will be 76.5\$

$$\text{c) } \bar{C}(1000) = \frac{60(1000)+1650}{1000} = 61.65\$$$

$$\text{d) } \lim_{x \rightarrow +\infty} \frac{60x+1650}{x} = 60, \text{ as observed from the different average costs, the more quantity material is recycled, the more the average cost price reduces until the final average cost will be } 60\$$$

Lesson5: infinite limits and rational functions

a) Objectives

- To distinguish limits at infinity from infinity limits.
- To calculate finite and infinity limits of rational functions by simplification.

b) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in:

- Operations in the set of real numbers
- Sequences

d) Learning activities

- Invite student-teachers to work in group and do the activity 6.3.2 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
- After the lesson, guide students-Teacher to do the application activity 6.3.2 and evaluate whether lesson objectives were achieved.

Answers to activity 6.3.2

$$\text{a) } \lim_{x \rightarrow \infty} \frac{5}{x+2} = 0 \quad \text{b) } \lim_{x \rightarrow 10} \frac{x}{2} = 5 \quad \text{c) } \lim_{x \rightarrow \infty} \frac{x}{x^2} = 0$$

Answers to application activity 6.3.2.

I.

$$1. \lim_{x \rightarrow -\infty} \frac{4x^3 + 5x - 3}{x^2 + 3x + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 \left(4 + \frac{5}{x^2} + \frac{3}{x^3} \right)}{x^2 \left(1 + \frac{3}{x} + \frac{1}{x^2} \right)} = -\infty$$

$$2. \lim_{x \rightarrow \infty} \frac{5x + 2}{3x^2 + 1} = \frac{x \left(5 + \frac{2}{x} \right)}{x^2 \left(3 + \frac{1}{x^2} \right)} = 0$$

$$3. \lim_{x \rightarrow -\infty} (-6) = -6$$

II.

$$a. \lim_{x \rightarrow 7} \frac{49 - x^2}{x - 7} = \frac{(7 - x)(7 + x)}{-(7 - x)} = -14$$

$$\begin{aligned} b. \lim_{x \rightarrow -2} \frac{x^3 + 5x^2 + 6x}{x + 2} &= \lim_{x \rightarrow -2} \frac{x(x + 2)(x + 3)}{x + 2} \\ &= -2(-2 + 3) \\ &= -2 \end{aligned}$$

$$\begin{aligned} c. \lim_{x \rightarrow -2} \frac{x^3 + 5x^2 + 6x}{x + 2} &= \lim_{x \rightarrow -2} \frac{(x - 3)(x^2 - x + 2)}{(x - 3)} \\ &= \lim_{x \rightarrow -2} (x^2 - x + 2) \\ &= 8 \end{aligned}$$

III.

$$\begin{aligned} a. \lim_{x \rightarrow 2} \left(\frac{1}{x - 2} - \frac{2(2x - 3)}{x^3 - 3x^2 + 2x} \right) &= \lim_{x \rightarrow 2} \frac{(x - 2)(x - 3)}{x(x - 2)(x - 1)} \\ &= \lim_{x \rightarrow 2} \frac{x - 3}{x(x - 1)} = \frac{-1}{2} \end{aligned}$$

$$\begin{aligned} \text{d. } \lim_{x \rightarrow 1} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} &= \lim_{x \rightarrow 1} \frac{-x+3}{4x(x+4)} \\ &= \frac{1}{10} \end{aligned}$$

Lesson 6: Indeterminate cases

a) Learning objective:

Identify and remove indefinite cases.

b) Teaching resources

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

- Student-teachers will perform well in this unit if they have good background in : Lesson two of this unit,
- Radicals(surds)learnt in senior two;
- Polynomials factorization;
- Simplifying algebraic expressions.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 6.5.1 and then, 6.5.2 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify groups with different working steps;
- Invite a group , randomly chosen, with to present the work of his/her.
- As a tutor, harmonize the findings from presentation and guide them to enhance the limits involving infinity and the operations on limits in general.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the addition, multiplication and division on limits.

- Move to every group and verify their working steps.
- Invite all students for a whole class discussion and guide them to guess how to move out the indeterminate cases;
- Use different probing questions and guide them to explore the content and examples related to limits of functions at infinity and involving infinity, indeterminate cases as it is given in the student's book;
- After this step, guide students to do the application activity 6.5 and evaluate whether lesson objectives were achieved.

Answers for activity 6.5.1

For this activity, refer to the skills that learners have since primary the “dividing by zero is not possible”. They may also use calculators to recognize that this form is a math-error. The tutor may start from this already known term, to introduce the new term indefinite form (IF).

$$\text{a. } \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0} \text{ (IF)}$$

Removing the IF by factorization method.

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} = 2$$

$$\text{b. } \lim_{x \rightarrow 2} \frac{x^3 + x^2 - 5x - 2}{x^2 - 1} = \frac{0}{0} \text{ (IF)}$$

Removing the IF by factorization method

$$\lim_{x \rightarrow 2} \frac{x^3 + x^2 - 5x - 2}{x^2 - 1} = \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 3x - 1)}{(x - 2)(x + 2)}$$

$$= \frac{9}{4} = 2\frac{1}{4}$$

Answers for activity 6.5.2

- Learners already know conjugate from senior two. The tutor will use the skills from this unit of S2 radicals and indices to facilitate student-teacher to do this activity.

- The perfect product $(a-b)(a+b) = a^2 - b^2$ will help for faster calculation.
- a. The conjugate of $\sqrt{x^2 - 2} + 3$ is $\sqrt{x^2 - 2} - 3$, just changing the sign between terms.
- b. For this question $f(x) = \frac{\sqrt{x-2}-1}{x-3}$ we just find the conjugate of the numerator only.

That is the conjugate of $\sqrt{x-2} - 1$ which is $\sqrt{x-2} + 1$

If this conjugate is multiplied on both numerator and denominator, we will get

$$f(x) = \frac{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}{(x-3)(\sqrt{x-2}+1)}$$

Recall to learners that if the same term is

multiplied for both numerator and denominator, the expression does not change.

Answers for application activity 6.5

1.

a. $\lim_{x \rightarrow \infty} \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - x + 2} = \infty - \infty$ (IF)

Recall to learners that for the above expression, the denominator is 1.

$$\lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 2x - 1} - \sqrt{x^2 - x + 2})(\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2})}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2}}$$

But $(a-b)(a+b) = a^2 - b^2$

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 2x - 1) - (x^2 - x + 2)}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2}}$$

$$= \lim_{x \rightarrow \infty} \frac{3x - 3}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2}}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(3 - \frac{3}{x} \right)}{\sqrt{x^2 \left(1 + \frac{2}{x} - \frac{1}{x^2} \right)} + \sqrt{x^2 \left(1 - \frac{1}{x} + \frac{2}{x^2} \right)}}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(3 - \frac{3}{x} \right)}{2\sqrt{x^2}} \text{ but } \sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Therefore, $\lim_{x \rightarrow \infty} \frac{x \left(3 - \frac{3}{x} \right)}{2\sqrt{x^2}} = \begin{cases} \lim_{x \rightarrow +\infty} \frac{x \left(3 - \frac{3}{x} \right)}{2x} = \frac{3}{2} = 1\frac{1}{2} \text{ and} \\ \lim_{x \rightarrow -\infty} \frac{x \left(3 - \frac{3}{x} \right)}{-2x} = \frac{-3}{2} = -1\frac{1}{2} \end{cases}$

$$\left\{ \begin{array}{l} \lim_{x \rightarrow -\infty} \frac{x \left(3 - \frac{3}{x} \right)}{-2x} = \frac{-3}{2} = -1\frac{1}{2} \end{array} \right.$$

b. $\lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})}{x - 1} = \frac{0}{0}$ (IF)

$$\lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})}{x - 1} = \lim_{x \rightarrow 1} \frac{(1 - \sqrt{x})(1 + \sqrt{x})}{(x - 1)(1 + \sqrt{x})} = \frac{-1}{2}$$

c. $\lim_{u \rightarrow 0} \frac{\sqrt{4+u} - 2}{u} = \frac{0}{0}$ (IF)

By rationalization we get

$$\lim_{u \rightarrow 0} \frac{(\sqrt{4+u} - 2)(\sqrt{4+u} + 2)}{u(\sqrt{4+u} + 2)} = \frac{1}{4}$$

d. $\lim_{v \rightarrow 1} \frac{\sqrt{2v+1} - \sqrt{3}}{v - 1} = \frac{0}{0}$ (IF)

By rationalization we get

$$\lim_{v \rightarrow 1} \frac{\sqrt{2v+1} - \sqrt{3}}{v - 1} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\begin{aligned}
 2. \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right) &= \lim_{x \rightarrow 2} \frac{(x-2)(x-3)}{x(x-2)(x-1)} \\
 &= \lim_{x \rightarrow 2} \frac{x-3}{x(x-1)} = \frac{-1}{2}
 \end{aligned}$$

Lesson 7: Graphs and limits of function

a) Learning objective:

Predict the limit of a function from its graphical representation

b) Teaching resources

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction

Student-teachers will perform well in this unit if they have good background in:

- Lesson two of this unit;
- Radicals(surds) learnt in senior two, they should also be equipped with
- Factorization of polynomials;
- Simplifying algebraic expressions.

d) Learning activities

- Invite student-teachers to work in groups and do the activity 6.6 found in their Mathematics books;
- Move around in the class for facilitating students where necessary and ask some challenging questions to lead them to work correctly;
- Verify and identify different working steps;
- Invite the representative of one group chosen at random to present the work of his/her group
- As a tutor, harmonize the findings from presentation and guide them to enhance the limits involving infinity and the operations on limits in general.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to explore the

addition, multiplication and division on limits.

- Invite students to work on the activity application 6.6
- Move to every group and verify their working steps.
- Invite all students for a whole class discussion and guide them to discover how to remove indeterminate cases;
- Use different probing questions and guide them to explore the content and examples related to limits of functions at infinity and involving infinity, indeterminate cases as it is given in the student's book;
- After this step, guide students to do the application activity 6.6 and evaluate whether lesson objectives were achieved or not ,for later improvement

Answers for activity 6.6

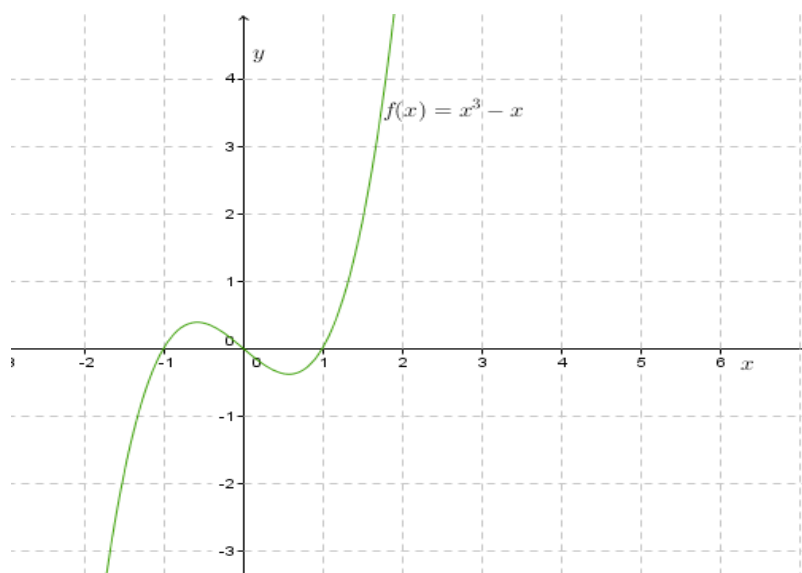
a. There are three parts for the graph.

b. $y = 0$ c. $y = 2$ d. $y = -2$ e. $y = 0$ f. $y = 4$

Answers for application activity 6.6

1. From the graph, $\lim_{x \rightarrow 2} f(x) = 4$

2. a.

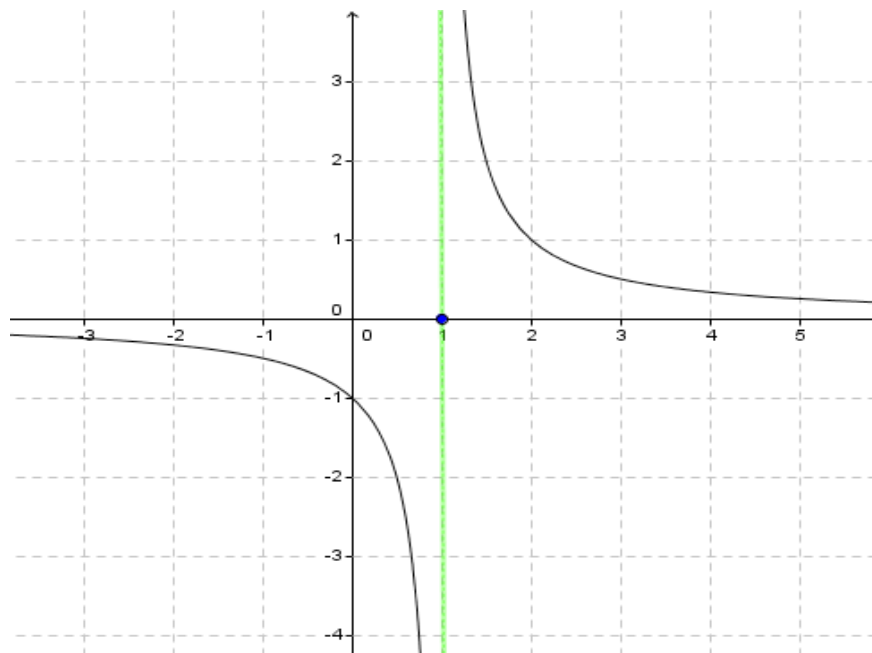


From this graph we can see that

i. $\lim_{x \rightarrow -\infty} x^3 - x = -\infty$

ii. $\lim_{x \rightarrow +\infty} x^3 - x = +\infty$

b.



i. $\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$

ii. $\lim_{x \rightarrow 1^+} \frac{1}{x-1} = +\infty$

iii. $\lim_{x \rightarrow +\infty} \frac{1}{x-1} = 0$

iv. $\lim_{x \rightarrow -\infty} \frac{1}{x-1} = 0$

Lesson 8: Applications of limits in mathematics, Continuity of a function, Asymptotes

Note: This section will be taught in 3 lessons of 2 periods each. One lesson will consider Continuity of a function; the second will be about asymptotes and the third lesson will deal with the applications of limits to calculate Horizontal, vertical and oblique asymptotes.

a) Learning objective:

- Solve problems involving continuity.
- Extend the concept of limit to determine the asymptotes of a given function

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, graph papers, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if:

- they have good background in previous lessons of this unit;
- They are skilled in analyzing functions on the graph.

d) Learning activities

- Invite student-teachers to work in group and do the activity 6.7.1 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite few groups to present their work;
- Guide them to explore the content and examples given in the student's book where they will be able to evaluate the continuity of limits toward a point.
- End the lesson by guiding students on the application activity 6.7.1 and evaluate whether lesson objectives were achieved.

Answer for activity 6.7.1

1. a) 4

b) 4

c) $f(2)$ and $\lim_{x \rightarrow 2} f(x)$ exist and are equal, $f(2) = \lim_{x \rightarrow 2} f(x)$

Answers of application activity 6.7.1

$$1. \begin{cases} \lim_{x \rightarrow 1^-} f(x) = 1 \\ \lim_{x \rightarrow 1^+} f(x) = 1 \end{cases}, \begin{cases} \lim_{x \rightarrow 3^-} f(x) = 3 \\ \lim_{x \rightarrow 3^+} f(x) = 3 \end{cases} \quad \text{and} \quad \begin{cases} \lim_{x \rightarrow 6^-} f(x) = 0 \\ \lim_{x \rightarrow 6^+} f(x) = 0 \end{cases}$$

Hence the function is continuous on the interval $1 \leq x \leq 6$.

2. The function is continuous on its domain

CE: $x^2 - 2x - 15 \neq 0$ therefore the function is not continuous at $x = 5$ and $x = -3$

3. Let us calculate the limit first.

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{x - 3} = 3 \text{ but the function is continuous if}$$

$$\lim_{x \rightarrow 3} f(x) = f(3)$$

$\Rightarrow k = 6$ Therefore for the function to be continuous $k = 6$.

Answers to activity 6.8

- Line A is vertical.
- Line B is an oblique line
- There exist horizontal line.
- The lines do not cross the curve.

Answers to the learning activity 6.9

$$f(x) = \frac{x}{x^2 - 1}$$

a) The domain is the set all real numbers such that $x^2 - 1 \neq 0$

$$(x - 1)(x + 1) \neq 0$$

$$x \neq 1 \text{ and } x \neq -1$$

Therefore, the domain is $]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[$

b)

$$\begin{cases} \lim_{x \rightarrow +\infty} \frac{x}{x^2 - 1} = 0 \\ \lim_{x \rightarrow -\infty} \frac{x}{x^2 - 1} = 0 \end{cases} \Rightarrow \text{H.A} \equiv y = 0$$

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{x}{x^2 - 1} = -\infty \\ \lim_{x \rightarrow 1^+} \frac{x}{x^2 - 1} = +\infty \end{cases} \Rightarrow \text{V.A} \equiv x = 1$$

$$\begin{cases} \lim_{x \rightarrow -1^-} \frac{x}{x^2 - 1} = -\infty \\ \lim_{x \rightarrow -1^+} \frac{x}{x^2 - 1} = +\infty \end{cases} \Rightarrow \text{V.A} \equiv x = -1$$

There no oblique asymptotes (Slant asymptotes does not exist).

b) The student-teachers do not know the meaning of an asymptote. But with simple guided questions, the teacher will facilitate the discussion to come up with the meaning of asymptotes.

Answers to application activity 6.9

i) The function $f(x) = \frac{x^2 + 1}{x^2 - x}$ exists if $x^2 - x \neq 0 \Rightarrow x \neq 0$ and $x \neq 1$

$\text{Dom}f = \mathbb{R} \setminus \{0, 1\}$ or $]-\infty, 0[\cup]0, 1[\cup]1, +\infty[$

$$\begin{cases} \lim_{x \rightarrow 0^-} \frac{x^2 + 1}{x^2 - x} = +\infty \\ \lim_{x \rightarrow 0^+} \frac{x^2 + 1}{x^2 - x} = -\infty \end{cases} \Rightarrow \text{V.A} \equiv x = 0$$

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{x^2 + 1}{x^2 - x} = -\infty \\ \lim_{x \rightarrow 1^+} \frac{x^2 + 1}{x^2 - x} = +\infty \end{cases} \Rightarrow \text{V.A} \equiv x = 1$$

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{x^2 + 1}{x^2 - x} = -\infty \\ \lim_{x \rightarrow 1^+} \frac{x^2 + 1}{x^2 - x} = +\infty \end{cases} \Rightarrow \text{V.A} \equiv x = 1$$

There no oblique asymptote.

The graph below describes this information.

ii) The function $f(x) = \frac{x^3}{1-x^2}$ is defined if $1-x^2 \neq 0$

Domain = $]-\infty, -1[\cup]-1, 1[\cup]1, +\infty[$.

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{x^3}{1-x^2} = +\infty \\ \lim_{x \rightarrow 1^+} \frac{x^2 + 1}{x^2 - x} = -\infty \end{cases} \Rightarrow \text{V.A} \equiv x = 1$$

$$\begin{cases} \lim_{x \rightarrow -1^-} \frac{x^3}{1-x^2} = +\infty \\ \lim_{x \rightarrow -1^+} \frac{x^2 + 1}{x^2 - x} = -\infty \end{cases} \Rightarrow \text{V.A} \equiv x = -1$$

$$\lim_{x \rightarrow \pm\infty} \frac{x^3}{1-x^2} = \pm\infty$$

Since the limit as x approaches to infinity is not a real number, the horizontal asymptote does not exist.

Let us verify whether the oblique asymptote exists or not:

O.A $\equiv y = ax + b$, where a and b are real numbers.

$$a = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x^3}{x - x^3} = -1 \in \mathbb{R}$$

$$b = \lim_{x \rightarrow \infty} f(x) - ax$$

$$b = \lim_{x \rightarrow \infty} \frac{x^3}{1-x^2} - (-x) = 0$$

$$\text{O.A} \equiv y = -x$$

iii. The function $f(x) = \frac{1}{x}$ is defined if $x \neq 0$, $\text{dom}f =]-\infty, 0[\cup]0, +\infty[$

$$\begin{cases} \lim_{x \rightarrow 0^+} \frac{1}{x} = -\infty \\ \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty \end{cases} \Rightarrow \text{V.A} \equiv x=0$$

$$\begin{cases} \lim_{x \rightarrow -\infty} \frac{1}{x} = 0 \\ \lim_{x \rightarrow +\infty} \frac{1}{x} = 0 \end{cases} \Rightarrow \text{H.A} \equiv y=0$$

There is no Slant or oblique asymptote .

Lesson 9: Real life problems about limits

a) Learning objective

Use the properties of limits to solve real life problems involving limits.

b) Teaching resources:

Student-teacher's book and other reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have mastered:

- Lessons from 1 to 8 of this unit
- To convert a word problem into a mathematical statement and vice versa

d) Learning activities

- Invite student-teachers to work in group and do the activity 6.10 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite some representatives to present the work of their when others are following attentively;

- As a tutor, harmonize the findings from presentation of student-teachers and guide them to evaluate the rate of change, instantaneous velocity and instantaneous acceleration of a moving body.
- Guide them to explore the content and examples given in the student's book where they will be able to solve real problems involving limits.
- After the lesson, guide students-Teacher to do the application activity 6.10 and evaluate whether lesson objectives were achieved.

Answers to activity 6.10

a) **The model is** $C = 0.5x + 5000$

b) **The average cost is given by** $\bar{C} = 0.5 + \frac{5000}{x}$

c) $C(100) = 0.5 + \frac{5000}{100} = 50.5$ \$

$C(1000) = 0.5 + \frac{5000}{1000} = 5.5$ \$

$C(10000) = 0.5 + \frac{5000}{10000} = 1$ \$

d) $\lim_{x \rightarrow \infty} 0.5 + \frac{5000}{x} = 0.5$ \$

e) The more quantities are sold, the average cost price decreases. But there is reserved price for the seller such that if he/she goes below that he/she can lose his/her money. So the reserved price is 0.5\$.

In the real life situation if you buy more quantities they reduce the price for you.

Answers to application activity 6.10

$\lim_{x \rightarrow 100^-} \left(\frac{528p}{100-p} \right) = +\infty$, this means that there are no more expenses of money when 100% of illegal drug is stopped. The country is safe at that time!

The tutor goes from this to inform to student-teacher that drugs destroy their brain.

2.

a. $\bar{C} = 0.5 + \frac{500}{x}$

b. $\bar{C}(250) = 0.5 + \frac{500}{250} = 2.5 \$$

c. $\bar{C}(1250) = 0.5 + \frac{500}{1250} = 0.9 \$$

d. $\lim_{x \rightarrow \infty} \left(0.5 + \frac{500}{x} \right) = 0.5 \$$

The more quantities are sold, the average cost price decreases gradually. But there is a reserved price for the seller such that if he/she goes below that he/she can lose his/her money. So the reserved price is 0.5\$.

6.6 Summary of the unit.

In this unit we dealt with limits and their applications.

We started with the concept of the “neighborhood”, in general, and the “neighborhood of a real number”, in particular. At this level, we were able to determine a δ -neighborhood of x_0 , deleted or not. We were also able to find the center and the radius of an open interval.

We, then, focused on the meaning of the “limit of a variable” and the “limit of a function as the independent variable approaches x_0 (finite or infinite). We found that:

“ $x \rightarrow x_0$ ” means that “ x is assuming values closer and closer to x_0 ”, from the left side and from the right side, for $x_0 \in \mathbb{R}$;

“ $x \rightarrow +\infty$ ” means that “the values of x are increasing without bound”, and

“ $x \rightarrow -\infty$ ” means that “the values of x are decreasing without bound”

The statement “If $x \rightarrow x_0$, then $f(x) \rightarrow L$ ” is denoted by $\lim_{x \rightarrow x_0} f(x) = L$

We set the goal of being able to determine the limit of a function algebraically, from numerical approach (table) and graphically. The calculation of the limit of a function $\lim_{x \rightarrow x_0} f(x)$ brought us to three cases:

1) $\lim_{x \rightarrow x_0} f(x) = f(x_0)$, provided $f(x_0)$ can be found by properties of elementary operations in the set of real numbers;

2) The calculation of $\lim_{x \rightarrow x_0} f(x)$ involves operations in the set $\mathbb{R} \cup \{-\infty; +\infty\}$. In this set, operations, such as the following held:

$$\frac{a}{\infty} = 0; \frac{a}{0} = \infty; \frac{\infty}{b} = \infty; \infty + a = \infty; (\infty)(\infty) = \infty; \text{ Etc}$$

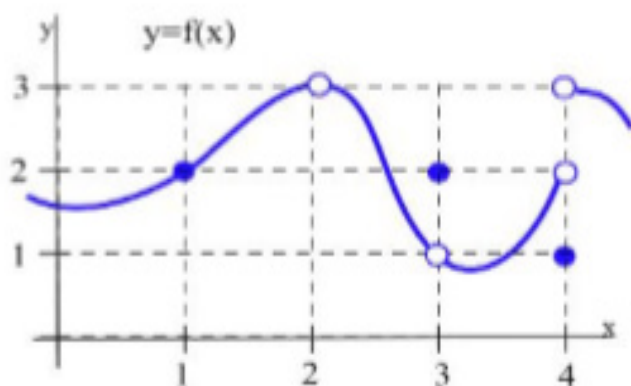
3) The calculation of $\lim_{x \rightarrow x_0} f(x)$ leads to one of the following:

$$\frac{0}{0}; \frac{\infty}{\infty}; \infty - \infty; 0(\infty); 1^\infty; 0^0; \infty^0 : \text{ These were termed "indeterminate cases", they}$$

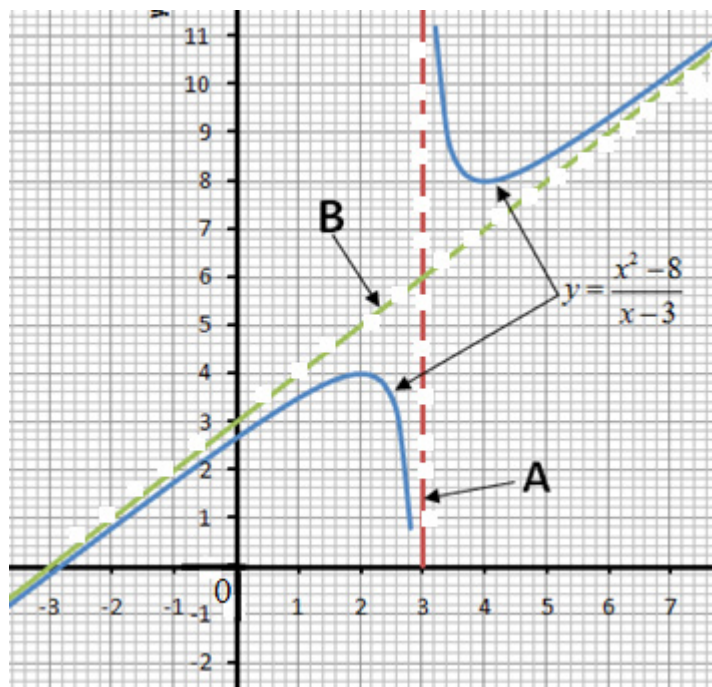
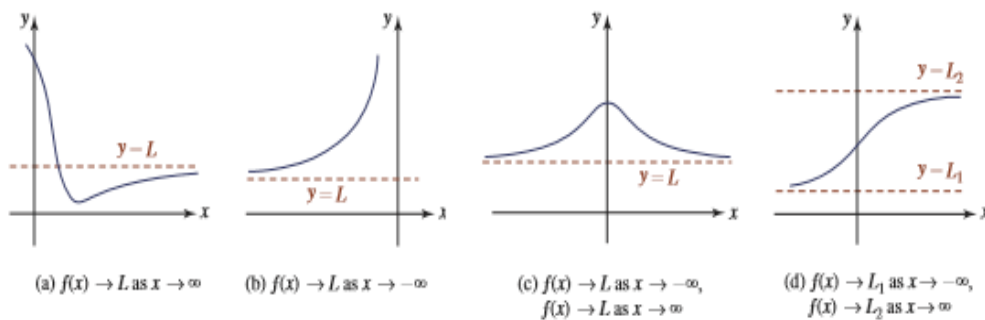
were hiding the true value towards which $f(x)$ gets closer and closer. To discover the true value, it was necessary to remove the indetermination. We focused on different ways of removing the indetermination, including factorization and multiplication by the conjugate.

Finally, we ended with the situations involving the use of limits: continuity and asymptotes.

We were able to interpret situations such as:



By saying that the function is discontinuous at 2, 3, and 4



In each of the graphs above, the lines in dots are known as asymptotes

6.7 Additional information to the teacher.

- Ensure that your students do not confuse superscripts with signs of numbers. They need to be careful about associating the “-” superscript with negative numbers. And the “+” superscript with positive numbers. The superscripts mean either the left or right.
- The symbol “ \rightarrow ” means approaches to. Your students should avoid using it as abbreviation sign
- Use parentheses when taking the limit of an expression consisting of more than one term. For example,

$$\lim_{x \rightarrow -1} (3x^3 + 4x^2 - 2x + 2) = 3(-1)^3 + 4(-1)^2 - 2(-1) + 2$$

This is not correct

Instead,

$$\lim_{x \rightarrow -1} (3x^3 + 4x^2 - 2x + 2) = 3(-1)^3 + 4(-1)^2 - 2(-1) + 2$$

- Your students should be careful while evaluating limits of piecewise function and functions involving absolute value signs (sometimes left and right limits confuse).
- In the calculation of limits involving infinity, they should precise the sign of infinity.

$\infty \neq +\infty$; For ∞ , consider both cases separately: $+\infty$ and $-\infty$

- Do not omit the limit operator until the substitution phase. For example, in the calculation of

$$\lim_{x \rightarrow \infty} \sqrt{x^2 + 2x - 1} - \sqrt{x^2 - x + 2} = \infty - \infty \text{ (IF)}$$

We have:

$$\lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 2x - 1} - \sqrt{x^2 - x + 2})(\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2})}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2}}$$

$$\text{But } (a - b)(a + b) = a^2 - b^2$$

$$= \frac{(x^2 + 2x - 1) - (x^2 - x + 2)}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2}} \text{ Wrong! (the limit sign is forgotten)}$$

$$\text{But } = \lim_{x \rightarrow \infty} \frac{(x^2 + 2x - 1) - (x^2 - x + 2)}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2}} \text{ correct!}$$

$$= \lim_{x \rightarrow \infty} \frac{3x - 3}{\sqrt{x^2 + 2x - 1} + \sqrt{x^2 - x + 2}}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(3 - \frac{3}{x} \right)}{\sqrt{x^2 \left(1 + \frac{2}{x} - \frac{1}{x^2} \right)} + \sqrt{x^2 \left(1 - \frac{1}{x} + \frac{2}{x^2} \right)}}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left(3 - \frac{3}{x} \right)}{2\sqrt{x^2}} \text{ but } \sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Therefore, $\lim_{x \rightarrow \infty} \frac{x \left(3 - \frac{3}{x} \right)}{2\sqrt{x^2}} = \frac{x \left(3 - \frac{3}{x} \right)}{2x}$ wrong! the limit sign is forgotten, and at

this level it is necessary to split the limit:

$$\lim_{x \rightarrow +\infty} \frac{x \left(3 - \frac{3}{x} \right)}{2\sqrt{x^2}} = \lim_{x \rightarrow +\infty} \frac{x \left(3 - \frac{3}{x} \right)}{2x} = \frac{3}{2} \text{ Correct!}$$

$$\text{And } \lim_{x \rightarrow -\infty} \frac{x \left(3 - \frac{3}{x} \right)}{2\sqrt{x^2}} = \lim_{x \rightarrow -\infty} \frac{x \left(3 - \frac{3}{x} \right)}{-2x} = -\frac{3}{2}$$

6.8 End unit assessment

1. The gradient of tangent is given by $\lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$

$$\lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = \frac{t^2 + 2t\Delta t + (\Delta t)^2 - t^2}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{2t\Delta t + (\Delta t)^2}{\Delta t} = \lim_{\Delta t \rightarrow 0} (2t + \Delta t) = 2t$$

At the point (1,1), the gradient becomes 2.

2.

a) For the function $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$

Domain is given by $\{x \in \mathbb{R} / x^2 - 4 \neq 0\}$

Therefore the domain is $]-\infty, -2[\cup]-2, 2[\cup]2, +\infty[$

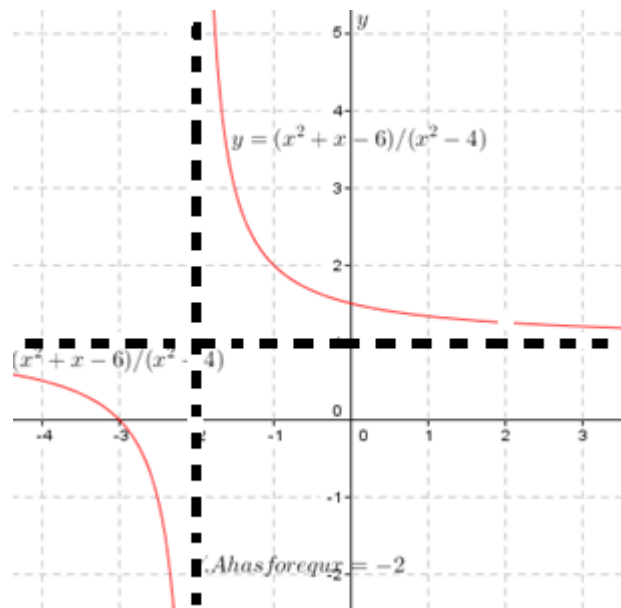
$$b) \begin{cases} \lim_{x \rightarrow +\infty} \frac{x^2 + x - 6}{x^2 - 4} = 1 \\ \lim_{x \rightarrow -\infty} \frac{x^2 + x - 6}{x^2 - 4} = 1 \end{cases} \Rightarrow \text{H.A} \equiv y=1 \text{ There is no slant asymptote.}$$

$$\begin{cases} \lim_{x \rightarrow -2^-} \frac{x^2 + x - 6}{x^2 - 4} = -\infty \\ \lim_{x \rightarrow -2^+} \frac{x^2 + x - 6}{x^2 - 4} = +\infty \end{cases} \Rightarrow \text{V.A} \equiv x=-2$$

$$\begin{cases} \lim_{x \rightarrow 2^-} \frac{x^2 + x - 6}{x^2 - 4} = \frac{5}{4} \\ \lim_{x \rightarrow 2^+} \frac{x^2 + x - 6}{x^2 - 4} = \frac{5}{4} \end{cases} \text{ At } x = 2, \text{ we have a removable discontinuity, Hence } x = 2 \text{ is}$$

not a vertical asymptote to the curve of our function.

Beware! Not all points at which the function is undefined are vertical asymptotes. The removable discontinuities are not vertical asymptotes.



You can see from the graph, that the curve is deleted at $x = 2$.

3.

$$a. \bar{P} = \frac{75.5x - 25.5x - 1000}{x} \Rightarrow \bar{P} = 50 - \frac{1000}{x}$$

b. $\bar{P}(100) = 50 - \frac{1000}{100} = 40$, $\bar{P}(500) = 50 - \frac{1000}{500} = 48$ and

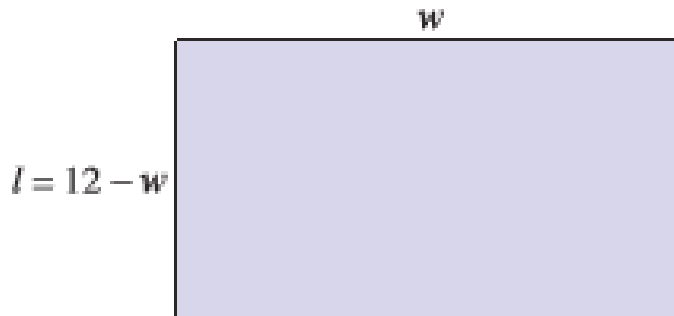
$$\bar{P}(1000) = 50 - \frac{1000}{1000} = 49$$

c. $\lim_{x \rightarrow \infty} \left(\frac{75.5x - 25.5x - 1000}{x} \right) = 50$, this is the maximum possible profit that the seller can make. This is the boundary of the average profit.

4. Let w be the width of the rectangle and l represent the length of the rectangle.

Because $2w + 2l = 24$ perimeter is 24

It follows that $l = 12 - w$ as shown below.



So the area of the rectangle is $A = lw$ formula for the area.

$$= (12 - w)w \text{ Substitute } (12 - w) \text{ for } l.$$

$$= 12w - w^2 \text{ After removing brackets.}$$

Using this model for area, you can experiment with different values of w to see how to obtain the maximum area (greatest area).

Lets try more values

width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	37.75	35.99	36.00	35.99	37.75	35.00

After trying several values, it appears that the maximum area occurs when $w = 6$.

In limit terminology, you can say that “the limit of A as w approaches 6 is 36.” This is written as

$$\begin{aligned}\lim_{x \rightarrow 6} A &= \lim_{x \rightarrow 6} (12w - w^2) \text{ Substituting the variable } w \text{ by } 6 \\ &= 12(6) - (6)^2 \\ &= 72 - 36 \\ &= 36\end{aligned}$$

5.

i. $\lim_{x \rightarrow 2^+} f(x) = (2)^2 - 4(2) + 1 = -3$

ii. $\lim_{x \rightarrow 2^-} f(x) = -2(2) = -4$

iii. $\lim_{x \rightarrow 2} f(x)$ does not exist, since left and right limits are different.

iv. The function is discontinuous at $x = 2$ since the limit does not exist, but the function is continuous at the left.

6.9 Additional activities

6.9.1 Remedial activities

1. For the given function
$$\begin{cases} x + 2, & x < 1 \\ x^2, & 1 \leq x \leq 2 \\ 5x - 6, & x \geq 2 \end{cases}$$

i) Identify the points where the function is discontinuity,

ii) where $f(x)$ is discontinuous or continuous

Answers

$$\lim_{x \rightarrow 1^-} f(x) = 3$$

$$\lim_{x \rightarrow 1^+} f(x) = 1$$

$$\lim_{x \rightarrow 2^-} f(x) = 4$$

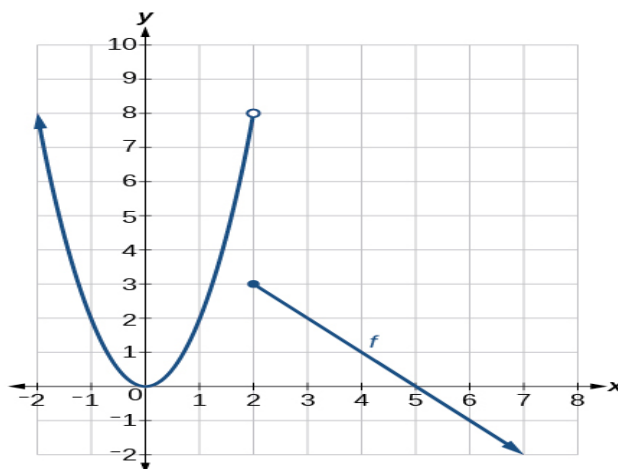
$$\lim_{x \rightarrow 2^+} f(x) = 4$$

Conclusion

$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$, therefore $f(x)$ is discontinuous at $x = 1$

$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$, therefore $f(x)$ is continuous at $x = 2$

2. Given the graph of function f ,



find:

- a. $\lim_{x \rightarrow 2^-} f(x)$ b. $\lim_{x \rightarrow 2^+} f(x)$ c. $\lim_{x \rightarrow 2} f(x)$ d. $f(2)$

Solution

- a. $\lim_{x \rightarrow 2^-} f(x) = 8$ b. $\lim_{x \rightarrow 2^+} f(x) = 3$ c. $\lim_{x \rightarrow 2} f(x)$ does not exist since $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$
 d. $f(2) = 3$

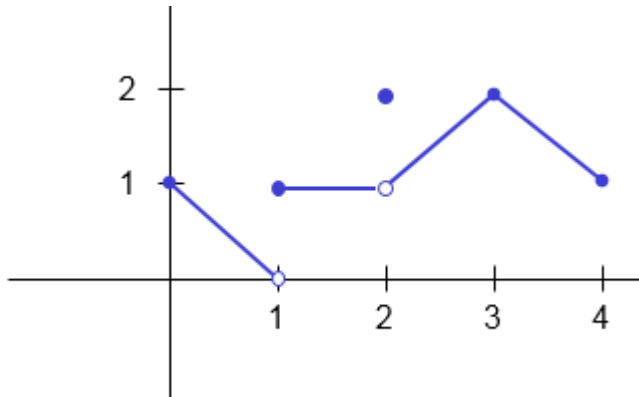
3. Discuss the continuity of the following function $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ \frac{x+1}{x} & \text{if } x \geq 1 \end{cases}$ at $x = 1$

Solution

$\lim_{x \rightarrow 1^-} f(x) = 2$ and $\lim_{x \rightarrow 1^+} f(x) = 2$, therefore $\lim_{x \rightarrow 1} f(x) = 2$, hence, the function is continuous at $x = 1$.

6.9.2 Consolidation activities

1. Observe the graph below, analyze it and interpret it then find out if $\lim_{x \rightarrow 3} f(x)$ exists or not.



Solution:

$$\lim_{x \rightarrow 3^-} f(x) = 2$$

$$\lim_{x \rightarrow 3^+} f(x) = 2$$

$$f(3) = 2$$

Hence, $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x)$ then $f(x)$ exists

2. Consider the function $g(x)$ defined for all real numbers by

$$g(x) = \begin{cases} m(x+3) & \text{if } x < 0 \\ 2 + \sqrt{x} & \text{if } 0 \leq x \leq K \\ 8 - x & \text{if } x > K \end{cases}$$

Where m and K are real positive numbers.

i. Evaluate: $\lim_{x \rightarrow 0^-} g(x)$ and $\lim_{x \rightarrow 0^+} g(x)$.

ii. Find the values of m and K if $g(x)$ is continuous for all real numbers.

Solution

i. $\lim_{x \rightarrow 0^-} g(x) = m(0+3) = 3m$ and $\lim_{x \rightarrow 0^+} g(x) = 2$

ii. But to be continuous at m , $\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^+} g(x) = g(0)$

$$\Rightarrow 3m = 2 \text{ or } m = \frac{2}{3}$$

The function is continuous at k if $\lim_{x \rightarrow K^-} g(x) = \lim_{x \rightarrow K^+} g(x) = g(K)$

$$\lim_{x \rightarrow K^-} g(x) = 2 + \sqrt{k} \text{ and } \lim_{x \rightarrow K^+} g(x) = 8 - K$$

So, $2 + \sqrt{K} = 8 - K$

$$\Rightarrow \sqrt{K} = 6 - k$$

$$(\sqrt{K})^2 = (6 - k)^2 \text{ Squaring both sides.}$$

Hence, the values of K are 4 and 9 for the function to be continuous.

6.9.3 Extended activities

1. Consider the numerical function defined by $f(x) = \frac{x^3}{1-x^2}$

i. Find the domain of definition of the function $f(x)$.

ii. Find the limits at the endpoints of the domain of definition and deduce the equations of all possible asymptotes.

iii. Discuss the symmetry of the function $f(x)$.

Solution

i. **Restrictions on x :** $1 - x^2 \neq 0$

$$(1-x)(1+x) \neq 0 \Rightarrow x \neq 1 \text{ and } x \neq -1$$

$$\text{Dom}f = \mathbb{R} \setminus \{-1, 1\}$$

$$\text{ii. } \begin{cases} \lim_{x \rightarrow -\infty} \frac{x^3}{1-x^2} = +\infty \\ \lim_{x \rightarrow +\infty} \frac{x^3}{1-x^2} = -\infty \end{cases} \quad \text{The horizontal asymptote does not exist.}$$

$$\begin{cases} \lim_{x \rightarrow -1^-} \frac{x^3}{1-x^2} = +\infty \\ \lim_{x \rightarrow -1^+} \frac{x^3}{1-x^2} = -\infty \end{cases} \quad \text{V.A} \equiv x = -1$$

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{x^3}{1-x^2} = +\infty \\ \lim_{x \rightarrow 1^+} \frac{x^3}{1-x^2} = -\infty \end{cases} \quad \text{V.A} \equiv x = 1$$

Let us find the oblique asymptote

$$\text{O.A} \equiv y = ax + b \text{ where } \begin{cases} a \in \mathbb{R}^* \\ b \in \mathbb{R} \end{cases}$$

$$a = \lim_{x \rightarrow \infty} \frac{f(x)}{x} \Rightarrow a = \lim_{x \rightarrow \infty} \frac{x^3}{x-x^3} = -1$$

$$b = \lim_{x \rightarrow \infty} f(x) - ax \Rightarrow b = \lim_{x \rightarrow \infty} \frac{x^3}{1-x^2} + x = 0$$

Therefore, the O.A $\equiv y = -x$

iii. The function is even if $f(x) = f(-x)$

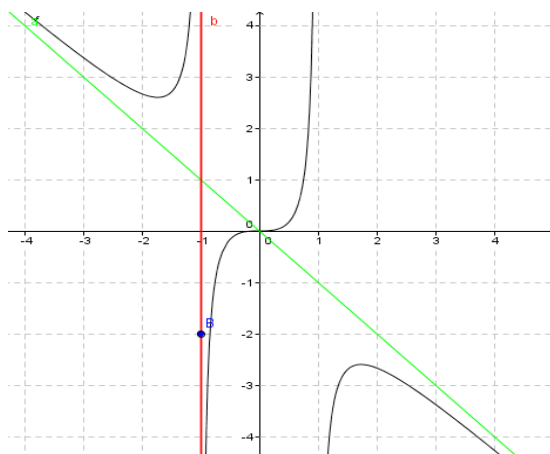
$$\text{But } f(-x) = \frac{(-x)^3}{1-(-x)^2} = \frac{-x^3}{1-x^2}, \text{ hence the function is not even (the graph is not}$$

symmetric about the y-axis). since $f(x) \neq f(-x)$

The function is odd if and only if $f(-x) = -f(x)$

$$\text{But, } -f(x) = \frac{-x^3}{1-x^2} = \frac{-x^3}{1-x^2}, \text{ hence the function is odd since } f(-x) = -f(x), \text{ the}$$

graph of this function is symmetric about the origin. (See graph)



Note: a curve never crosses its vertical asymptote.

2.

Evaluate $\lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x}$

Solution:

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} = \frac{\sqrt{\infty - \infty}}{\infty} \text{ I.F}$$

To evaluate this limit, we try the algebraic manipulations such that the denominator will be cancelled.

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} &= \lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2 \left(1 - \frac{11}{4x} - \frac{3}{4x^2}\right)}}{x} \\ &= \lim_{x \rightarrow +\infty} \frac{\left(\sqrt{4x^2}\right) \sqrt{1 - \frac{11}{4x} - \frac{3}{4x^2}}}{x} \\ &= \lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2}}{x} \lim_{x \rightarrow +\infty} \sqrt{1 - \frac{11}{4x} - \frac{3}{4x^2}} \end{aligned}$$

$$\begin{aligned}
&= \left(\lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2}}{x} \right) \times 1 \\
&= \lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2}}{x} \\
&= \lim_{x \rightarrow +\infty} \frac{2\sqrt{x^2}}{x}
\end{aligned}$$

Recall that $\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

We need to find the domain of the given function: $Domf =]-\infty, -\frac{1}{4}] \cup [3, +\infty[$.

As x tends to $+\infty$, $x \in [3, +\infty[$ and then $\sqrt{x^2} = x$.

Thus,

$$\begin{aligned}
\lim_{x \rightarrow +\infty} \frac{\sqrt{4x^2 - 11x - 3}}{x} &= \lim_{x \rightarrow +\infty} \frac{2\sqrt{x^2}}{x} \\
&= \lim_{x \rightarrow +\infty} \frac{2x}{x} \\
&= 2
\end{aligned}$$

UNIT 7

DIFFERENTIATION OR DERIVATIVE OF NUMERICAL FUNCTIONS

7.1 Key unit competence:

Differentiate a real function and apply derivatives to sketch the graphs and solve problems involving optimization

7.2 Prerequisite

Learners will easily learn this unit, if he/she is able to

- Calculate the limits of numerical functions taught in year 2 (UNIT6),
- Calculate the slope/gradient of graph as taught in year 3(unit
- Identify the six trigonometric functions (year 1,unit 8) and their transformation formulae of sum to product and vice versa (year 2,unit3)
- Determine equations of a straight line. S3(UNIT6) and Year 1,unit 9,
- Be accurate in plotting/graphing and calculations.

7.3 Cross-cutting issues to be addressed

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (Learners encouraged to adhere to rules and standards of calculations and drawing of graphs)

7.4 Guidance on introductory activity

- a) Student- teachers work on the introductory activity to be aware of this unit.
- b) The student- teachers have encountered gradients before. Ask them what they recall about the concept of gradients.

c) Guide the student-teachers , to do the activity 7.1, in pairs, move around in order to help the slower student learners .

d) Guide the student- teacher to present their findings

Answer of introductory activity

a)

we have $x_0 = 1$ and $h = \Delta x = 1$

$$\text{The slope is given by } m_p = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{4 - 2}{2 - 1} = 2$$

b)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 1] - (x^2 + 1)}{h}$$

$$= \frac{x^2 + 2hx + 1 - x^2 - 1}{h}$$

$$= \frac{2hx}{h} = 2x$$

$$f'(x) = 2x$$

$$\text{for } x_0 = 1 \Rightarrow f'(x_0) = f'(1) = 2$$

The slope $m_p = f'(x_0) = 2$

2) Possible answers:

The derivative of a function $f(x)$ with respect to x is denoted by $f'(x)$ or $\frac{d}{dx} f(x)$ and defined as $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ provided that the limit exists.

Or the derivative of a function $y = f(x)$ of a variable x is a measure of the rate at which the value y of the **function** changes with respect to the change of the variable x . It is called the **derivative** of $f(x)$ with respect to x

7.5 List of lessons/sub-heading

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity	To arouse the curiosity of student teachers on the content of unit 7	1
1	Derivative concepts: Definition and graphical interpretation	Define the derivative of a function using the gradient and limit	2
2	Rules of differentiation of elementary functions	Explore rules and use properties of derivatives to differentiate polynomial, rational and irrational functions	2
3	Derivative of trigonometric functions	Explore rules and use properties of derivatives to differentiate the trigonometric functions	2
4	Derivative of inverse trigonometric functions	Explore rules and use properties of derivatives to differentiate the inverse trigonometric functions	2
5	Derivative and the Variation of a function (extrema points, increasing and decreasing interval or turning points)	Determine Variation of a function (Increasing, decreasing interval or turning points)	4
6	Derivative and Concavity of a function's curve (inflexion points)	Determine and interpret the inflection points of a function's curve	2
7	Derivative and the table of variation for a function	Use derivative and the table of variation to Sketch and interpret the function's graph.	4
8	Derivative and limit with indeterminate cases: Hospital's rule	Use first derivative to determine the limit of some rational functions	1

9	Derivative, tangent line equation and Normal line equation	Use first derivative to determine the gradient of the tangent line and normal line to a curve at a point	4
10	Applications of differentiation in Economics and finance	Apply differentiation in economics and other social sciences	4
11	Application of differentiation: rates of change problems, optimization problems	Apply the concepts and techniques of differentiation to model, analyze and solve optimization problems in different situations.	4
12	End unit assessment		2
Total number of periods in this unit.			34

Lesson 1: Derivative concepts, Definition and graphical interpretation

a) Learning objective

Define the derivative of a function using the gradient and limit

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, ...

c) Prerequisites/Revision/Introduction

Learners should have knowledge and skills on how to calculate the gradient(slope) and limit of numerical functions, draw axes of coordinates in Cartesian plane.

c) Learning activities

- Ask learners in small groups to read and discuss on the **activity 7.1** and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling learners.
- Call learners to present the findings, and help them to harmonize the answer.

- By harmonization, lead the student-teachers to discover that the

derivative of a function $f(x)$ with respect to x is denoted by

$f'(x)$ or y' or $\frac{dy}{dx}$ or $\frac{df}{dx}$ or $\frac{d}{dx} f(x)$ and defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ provided that the limit exists.}$$

- Lead them to read through and re-do examples 1 and 2
- Guide them to do **application activity 7.1.1** to master the content.

Answer of activity 7.1.1

a)

Two points are given, $P(x_0, x_1)$ and $Q(y_0, y_1)$

$$m_{\text{sec}} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

b) If $x_1 \rightarrow x_0$, P and Q tends to be on vertical line and the slope tends to infinity.

c)

$$m_{\text{tan}} = \lim_{x \rightarrow x_0} m_{\text{sec}}$$

$$m_{\text{tan}} = \lim_{x \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

d) Let $h = x_1 - x_0$

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(x_1) - f(x_0)}{h}$$

Answer of application activity 7.1.1

1. The slope $m = 12$ m

2. a. $f'(x) = 2x$ b. $f'(x) = 2x + 2$

3. a. The secant line PQ to the curve of $y = f(x)$ has the slope $m_{PQ} = \frac{f(x) - F(a)}{x - a}$

b.

Let $h = x - a$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f'(x) - f(a)}{h}$$

c. $f'(x) = 1$

4.

$$f'(x) = 2x - 8$$

$$f'(a) = 2a - 8$$

Lesson 2: Rules of differentiation of elementary functions

a) Learning objective

Explore rules and use properties of derivatives to differentiate polynomial, rational and irrational functions

b) Teaching resources

Student- teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, ...

c) Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on identification of polynomials, power, and composition, rational and irrational functions.

d) Learning activities

- Ask learners in small groups/individual to read and discuss on the **activity 7.2 in student teacher's book** to calculate the derivatives and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling learners.

- Call learners to present the findings, and help them to harmonize the answer.
- Guide the student-teachers to discover the different rules of derivatives of numerical functions and using different examples.
- In small groups or individually, help the student teacher to work out the **application activity 7.2**

Answer of activity 7.2

$$1. \quad f'(x) = 2x \quad b. \quad h'(x) = 3 \quad c. \quad g'(x) = 2x + 3$$

$$S'(x) = g'(x) + h'(x) = (2x + 3) + 3$$

$$= 2x + 6$$

$$2. \quad f'(x) = \frac{-1}{t^2}$$

Answer of application activity 7.2

$$1. \quad a) [f(g(x))]' = 2x + 5 \quad b) f'[g(x)] = 2x + 5 \quad c) f'[g(x)] \cdot g'(x) = 2x + 5$$

$$2. \quad y' = 3000x^2(x^2 - 1)^{999}$$

$$3. \quad a(t) = 6t$$

$$4) \quad \frac{dy}{dx} = \frac{x + 2}{3y^2}$$

Lesson 3: Differentiation of trigonometric functions

a) Learning objective

Explore rules and use properties of derivatives to differentiate trigonometric functions

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils,...

c) Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on relationship between trigonometric ratios and complementary angles (**in unit 3, year 2**) and the fundamental theorem of trigonometry (**unit 8, year 1**).

d) Learning activities

- Ask learners in small groups/individually to read and discuss on the **activity 7.3** in student teacher's book
- Help the student teachers to recall (**in unit 8, year 1**):
 - the trigonometric identities
 - the six trigonometric ratios and the relationship between them
- Guide them to establish the formulae of the derivatives of inverse trigonometric functions and make sure that everybody is engaged/involved.
- Facilitate working, especially straggling learners.
- Call learners to present the findings, and help them to harmonize the answer.
- After presentation, tutor will help the student-teachers to generalize the derivatives of trigonometric functions and guide them to re-work the provided examples for better understanding.
- In small groups or individually, help the student teacher to work out the **application activity 7.3**

Answers of activity 7.3

1. $y = f(x) = \sin x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2 \sin\left(\frac{x+h-x}{2}\right) \cos\left(\frac{x+h+x}{2}\right)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{2 \sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \lim_{h \rightarrow 0} \cos\left(x + \frac{h}{2}\right)$$

$$f'(x) = 1 \cdot \lim_{h \rightarrow 0} \cos(x+h) (2x^4 + x)' \cos(2x^4 + x)$$

$$f'(x) = \cos x$$

b) $y = \cos x$

We know that $\cos(x) = \sin\left(\frac{\pi}{2} - x\right) \Rightarrow \frac{dy}{dx} = -\sin x$

2. $y = \tan x$

$y = \tan x$ can be written as $y = \frac{\sin x}{\cos x}$

then

$$\frac{dy}{dx} = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2} = \frac{\cos x \cos x + \sin x \sin x}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{(\cos x)^2}$$

3. $\frac{dy}{dx} = \frac{1}{(\cos x)^2}$ or $\frac{dy}{dx} = \sec^2 x$ $y = \cot x$

In the same way with derivative of tangent, $\frac{d \cot x}{dx} = \frac{-1}{(\sin x)^2}$ or $\frac{d \cot x}{dx} = -\csc^2 x$

4. $y = \sec x$

$$y = \sec x = \frac{1}{\cos x} \Rightarrow y' = \frac{\sin x}{\cos^2 x} \text{ or } y' = \sin x \sec^2 x$$

5. $y = \csc x$

$$y = \csc x = \frac{1}{\sin x} \Rightarrow y' = \frac{-\cos x}{\sin^2 x} \text{ or } y' = -\cos x \csc^2 x$$

Answer of application activity 7.3

1. $6x \cos(x^2 + 4) \sin^2(x^2 + 4)$
2. $-6x \sin 3x^2$
3. $\tan x + x(1 + \tan^2 x)$
4. $3[1 + \tan^2(3x + 2)]$
5. $-2x[1 + \cot^2(x^2 - 5)]$
6. $-4 \sin x(1 + \cot^2 4x) + \cos x \cot 4x$
7. $3 \sec(3x + 2) \tan(3x + 2)$
8. $\theta^2 \csc 2\theta(3 - 2\theta \cot 2\theta)$
9. $12 \sec^4 3x \tan 3x$

Lesson 4: Differentiation of inverse trigonometric functions

a. Learning objective

Explore rules and use properties of derivatives to differentiate the inverse trigonometric functions

b. Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils,...

b. Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on relationship between trigonometric ratios inverse trigonometric functions (**in unit 3, year 2**) and the fundamental theorem of trigonometry (**unit 8, year 1**).

c. Learning activities

- Ask learners in small groups/individually to read and discuss on the **activity 7.4** in student teacher's book
- Help the student teachers to recall (**in unit 8, year 1**):
 - The trigonometric identities
 - The six trigonometric ratios and the relationship between them

- Guide them to establish the formulae of the derivatives of inverse trigonometric functions and make sure that everybody is engaged/involved.
- Facilitate working, especially stragglers.
- Call learners to present the findings, and help them to harmonize the answer.
- After presentation, tutor will help the student-teachers to generalize the derivatives of inverse trigonometric functions and guide them to re-work the provided examples
- In small groups or individually, help the student teacher to work out the **application activity 7.4**

Answers of activity 7.4

- $f(x) = \sin^{-1} x$ for $x \in [-1, 1]$ and $x = \sin y$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $y = f(x)$.

$$\begin{aligned}
 (\sin^{-1} x)' &= \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)} \\
 &= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} \quad \text{since } \cos x = \sqrt{1 - \sin^2 x} \\
 &= \frac{1}{\sqrt{1 - x^2}}
 \end{aligned}$$

- $f(x) = \cos^{-1} x$ for $x \in [-1, 1]$ and $x = \cos y$ for $y \in [0, \pi]$ where $y = f(x)$

$$\begin{aligned}
 (\cos^{-1} x)' &= \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = \frac{-1}{\sin(\cos^{-1} x)} \\
 &= \frac{-1}{\sqrt{1 - \cos^2(\cos^{-1} x)}} \quad \text{since } \sin x = \sqrt{1 - \cos^2 x} \\
 &= \frac{-1}{\sqrt{1 - x^2}}
 \end{aligned}$$

- $f(x) = \tan^{-1} x$ for $x \in \mathbb{R}$ and $x = \tan y$ for $y \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ where $y = f(x)$.

$$(\tan^{-1} x)' = \frac{1}{(\tan y)'} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + \tan^2 (\tan^{-1} x)} = \frac{1}{1 + x^2}$$

- $f(x) = \cot^{-1} x$ for $x \in \mathbb{R}$ and $x = \cot y$ for $y \in]0, \pi[$ where $y = f(x)$

$$(\cot^{-1} x)' = \frac{1}{(\cot y)'} = \frac{-1}{1 + \cot^2 y} = \frac{-1}{1 + \cot^2 (\cot^{-1} x)} = \frac{-1}{1 + x^2}$$

- $f(x) = \sec^{-1} x$ for $x \leq -1$ or $x \geq 1$ and $x = \sec y$ for $y \in [0, \pi], y \neq \frac{\pi}{2}$ where $y = f(x)$

$$(\sec^{-1} x)' = \frac{1}{(\sec y)'} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2 - 1}}$$

- $f(x) = \csc^{-1} x$ for $x \leq -1$ or $x \geq 1$ and $x = \csc y$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$ where $y = f(x)$

$$(\csc^{-1} x)' = \frac{1}{(\csc y)'} = \frac{-1}{\csc y \cot y} = \frac{-1}{x\sqrt{x^2 - 1}}$$

Answer of application activity 7.4

1. $\frac{1}{|x|\sqrt{x^2 - 1}}$

2. $\frac{-2x}{\sqrt{1 - x^4}}$

3. $\frac{-1}{\sqrt{2x - x^2}}$

4. $\frac{1}{\sqrt{2x}\sqrt{1 - 2x}}$

$$5. \frac{1}{2\sqrt{x}(1+x)}$$

$$6. \frac{1}{|x|\sqrt{x^2-1}} + \frac{1}{x^2+1}$$

$$7. \frac{-1}{2x\sqrt{x-1}}$$

$$8. \frac{1}{(2x+1)\sqrt{x^2+x}}$$

$$9. \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}}$$

$$10. \frac{1}{x\sqrt{25x^2-1}}$$

$$11. \frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$$

Lesson 5: Derivative and variation of a function

a) Learning objective

Determine Variation of a function

(Increasing, decreasing interval or turning points)

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, ...

c) Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on polynomial functions (year one, unit 11) on sketching the graph of functions (Year 2, unit 5), derivative (lesson 1 & 2, in this unit)

d) Learning activities

- Lead the student-teachers in small groups/individually to read and discuss on the **activity 7.5** in student teacher's book

- Help the student teachers to recall (**in unit 8, year 1**) :
 - the trigonometric identities
- Guide them to find the answers of given activity 7.5 and make sure that everybody is engaged/ involved.
- Facilitate working, especially stragglers learners.
- Call learners to present the findings, and help them to harmonize the answer.
- After presentation, tutor will help the student-teachers to re-work the provided examples.
- In small groups or individually, help the student teacher to work out the **application activity 7.5**

Answers of activity 7.5

a. Yes, it is differentiable in interval $]-\infty, +\infty[$

b. $y' = 3x^2 - 12$

c. Increasing intervals: $]-4, -2]$ and $[2, 4[$

Decreasing interval : $[-2, 2]$

d) Function is Concave up $[0, +\infty[$ and concave down $]-\infty, 0]$

Answer of application activity 7.5

1. $f(x) = 27x - x^3$

$f(x)$ increases on interval $]-\infty, -108]$ and $]54, +\infty[$

$f(x)$ decreases on interval $]-108, 54]$

$f(x)$ has minimum point $M(-3, -108)$

$f(x)$ has maximum point $M(3, 108)$

2. $f(x) = x^4 - 4x^3 + 5$ decreases on intervals $]-\infty, 0] \cup]0, 3[$

And increasing on $[3, +\infty[$

Relative extremum is $(3, -22)$ which a minimum point.

Lesson 6: Second derivative and Concavity of a function's curve

a. Learning objective

Determine and interpret the inflection points and the sense of concavity of a function's curve

b. Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, ...

e) Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on polynomial functions (year one, unit 11) on sketching the graph of functions (Year 2, unit 5), derivative (lesson 1 & 2, in this unit)

f) Learning activities

- Lead the student-teachers in small groups/individually to read and discuss on the **activity 7.6** in student teacher's book
- Guide them to find the answers of given activity 7.5 and make sure that everybody is engaged/involved.
- Facilitate working, especially stragglers.
- Call learners to present the findings, and help them to harmonize the answer.
- Tutor guide the student teachers to do the examples so that they find the second derivative, sign study of the function, try to sketch the graph according to the sense of concavity obtained.
- Tutor guides the student teachers to work out the **application activity 7.6, in pairs or individually** (everyone must be involved)

Answers of activity 7.6

a) The curve of function $y = x^3 - 12x - 5$ opens down on interval $]-4, 0]$ and opens up on $[0, 4[$

b) $y' = 3x^2 - 12 \Rightarrow y'' = (3x^2 - 12)' = 6x$

Sign study of $\frac{d^2y}{dx^2}$

x	$-\infty$	0	$+\infty$
$f''(x)$	- - - -	0 +	+ + +

c) $f''(x) < 0$ and $f(x)$ opens down on interval $]-4, 0]$

$f''(x) > 0$ and $f(x)$ opens up on interval $[0, 4[$

NB: Tutor will show the student teachers that there is a point $(0, -5)$ in which the curve changes the sense of concavity. It is called inflection point.

Answers of application activity 7.6

a)

$f(x) = x^3 - 12x + 1$

$f'(x) = 3x^2 - 12$

maximum point $(-2, 17)$ and minimum point $(2, -15)$

x	$-\infty$	-2	$+2$	$+\infty$
$f'(x)$	+ + +	0	- - - - -	- 0 + + +

$f(x)$ increases on interval $]-\infty, -2]$ and $[2, +\infty[$

$f(x)$ decreases on interval $[-2, 2]$

c)

$$f''(x) = 6x$$

and $f''(x) = 0 \Leftrightarrow x = 0$ inflection point (0,0)

x	$-\infty$									$+\infty$
$f''(x)$		-	-	-	0	+	+	+	+	+

The function $f(x)$ is concave up on interval $[0, +\infty[$ and concave down on interval on $] -\infty, 0]$

Lesson 7: Derivative and the table of variation for a function

a) Learning objective

Determine and interpret the inflection points and the sense of concavity of a function's curve

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, ...

c) Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on polynomial functions (year one, unit 11) on sketching the graph of functions (Year 2, unit 5), derivative (lesson 1 & 2, in this unit)

d) Learning activities

- Lead the student-teachers in small groups/individually to read and discuss on the **activity 7.7** in student teacher's book
- Guide them to find the answers of given activity 7.7 and make sure that everybody is engaged/involved.
- Facilitate working, especially straggling learners.
- Call learners to present the findings, and help them to harmonize the answer.

- Tutor informs the student- teachers that the Variation table or synthetic table summarizes all information about the function: domain of definition, limits at the boundaries of domain, asymptotes to the curve, extrema points, variation of function (increasing or decreasing) , inflection points and the sense of concavity(opens up or down).
- Tutor guides the student teachers to work out the example of studying the function completely, step by Step. In small groups or individually, help the student teacher to work out the **application activity 7.7**

Answer for activity 7.7

a. consider $f(x) = x^3 - 12x + 1$

$$f'(x) = 3x^2 - 12$$

$$f''(x) = 6x$$

b.

$$f'(x) = 3x^2 - 12$$

maximum point $(-2, 17)$

$$f''(x) = 6x \text{ and } f''(x) = 0 \Leftrightarrow x = 0$$

inflection point $(0, 0)$

x	$-\infty$	-2	0	$+2$	$+\infty$										
$f'(x)$	+	+	+	0	-	-	-	-	-	-	0	+	+	+	+
$f''(x)$	-	-	-	-	-	0	+	+	+	+	+	+	+		
$f(x)$			$(-2, 17)$	$(0, 1)$							$(2, -15)$				

C) Analysis \ interpretation

$f(x)$ is defined for all $x \in \mathbb{R}$

$f(x)$ is increasing on interval $]-\infty, -2]$ and $[2, +\infty[$

$f(x)$ is decreasing $[-2, 2]$ and $f'(x) = 0 \Leftrightarrow x = 2$ or $x = -2$ it has

a minimum point $(2, -15)$ and a maximum point $(-2, 17)$

$f(x)$ is concave down $]-\infty, 0]$ concave down $[0, +\infty[$ and changes the concavity at $(0, 0)$

Answer of application activity 7.7

$$1. y = x + \frac{1}{4x} \text{ or } y = \frac{4x^2 + 1}{4x}$$

Condition of existence: $4x \neq 0 \Rightarrow x \neq 0$

$$1) \text{ Dom}f = \mathbb{R} \setminus \{0\} =]-\infty, 0[\cup]0, +\infty[$$

2) Limits at the boundaries of Domf

$$\bullet \lim_{x \rightarrow -\infty} \frac{4x^2 + 1}{4x} = \frac{\infty}{\infty} \quad (IF)$$

$$\text{T.V: } \lim_{x \rightarrow -\infty} \frac{4x^2 + 1}{4x} = \lim_{x \rightarrow -\infty} \frac{4x^2(1 + \frac{1}{4x^2})}{4x} = \lim_{x \rightarrow -\infty} x = -\infty \Rightarrow \text{There is no Horizontal asymptote}$$

$$\bullet \lim_{x \rightarrow 0} \frac{4x^2 + 1}{4x} = \frac{1}{0} \Rightarrow \text{There is a vertical asymptote : } x = 0$$

$$\lim_{x \rightarrow 0^-} \frac{4x^2 + 1}{4x} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \rightarrow 0^+} \frac{4x^2 + 1}{4x} = \frac{1}{0^+} = +\infty$$

$$\bullet \lim_{x \rightarrow +\infty} \frac{4x^2 + 1}{4x} = \frac{\infty}{\infty} \quad (IF)$$

$$\text{T.V: } \lim_{x \rightarrow +\infty} \frac{4x^2 + 1}{4x} = \lim_{x \rightarrow +\infty} \frac{4x^2(1 + \frac{1}{4x^2})}{4x} = \lim_{x \rightarrow +\infty} x = +\infty$$

\Rightarrow There is no Horizontal asymptote

Let find the Oblique asymptote (OA)

$$OA \equiv y = ax + b \text{ with } a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} \text{ and } b = \lim_{x \rightarrow \pm\infty} [f(x) - ax]$$

$$a = \lim_{x \rightarrow \pm\infty} \frac{4x^2 + 1}{4x^2} = \lim_{x \rightarrow \pm\infty} \frac{4x^2}{4x^2} = 1$$

$$b = \lim_{x \rightarrow \pm\infty} \left(\frac{4x^2 + 1}{4x} - x \right) = \lim_{x \rightarrow \pm\infty} \left(\frac{4x^2 + 1 - 4x^2}{4x} \right) = \lim_{x \rightarrow \pm\infty} \left(\frac{1}{4x} \right) = 0$$

$a = 1, b = 0 \Rightarrow$ there is no OA

3) Parity of $f(x)$

- $f(x)$ is even function if $f(x) = f(-x)$

$$f(-x) = \frac{4(-x)^2 + 1}{-4x} = -\frac{4x^2 + 1}{4x} \neq f(x), \text{ so } f(x) = \frac{4x^2 + 1}{4x} \text{ is not even}$$

- $f(x)$ is odd function if $-f(x) = f(-x)$

$$f(-x) = \frac{4(-x)^2 + 1}{-4x} = -\frac{4x^2 + 1}{4x}$$

- $-f(x) = -\frac{4x^2 + 1}{4x}$, so $-f(x) = f(-x)$ and $f(x)$ is odd function

4) Parity of $f(x)$

The given function is not periodic (because it is not trigonometric)

5) First derivative of $f(x)$

$$f'(x) = \left(\frac{4x^2 + 1}{4x} \right)' = \frac{8x \cdot 4x - 4 \cdot (4x^2 + 1)}{16x^2} = \frac{32x^2 - 16x^2 - 4}{16x^2}$$

$$f'(x) = \frac{16x^2 - 4}{16x^2} = \frac{4x^2 - 1}{4x^2}$$

Critical points:

- Condition of existence of $f'(x)$

$$4x^2 \neq 0 \Leftrightarrow x \neq 0$$

$$f'(x) = 0 \Leftrightarrow \frac{4x^2 - 1}{4x^2} = 0 \Leftrightarrow 4x^2 - 1 = 0$$

$$\Rightarrow x = -\frac{1}{2} \text{ or } x = \frac{1}{2}$$

Extrema points: if $x = -\frac{1}{2} \Rightarrow y = -1$, function has a maximum point $(-\frac{1}{2}, -1)$

$x = \frac{1}{2} \Rightarrow y = 1$, function has a minimum point $(\frac{1}{2}, 1)$

Sign table of $f'(x)$

x	$-\infty$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$+\infty$									
$f'(x)$	+	+	+	0	-	-	-	-	-	0	+	+	+	+
$f(x)$	$-\infty$					$+\infty$								

6) Second derivative of $f(x)$

$$f''(x) = \left(\frac{4x^2 - 1}{4x^2} \right)' = \frac{8x \cdot 4x^2 - 8x \cdot (4x^2 - 1)}{16x^4} = \frac{32x^3 - 32x^3 + 8x}{16x^4} = \frac{8x}{16x^4}$$

$$f''(x) = \frac{1}{2x^3}$$

Critical points: • Existence condition of $f''(x)$: $x^3 \neq 0 \Rightarrow x \neq 0$

• $f''(x) = 0 \Leftrightarrow \frac{1}{2x^3} = 0$ It doesn't exist in \mathbb{R}

Sign table of $f''(x)$

x	$-\infty$	0				$+\infty$
$f''(x)$	-	-	-	-	-	+
$f(x)$	$-\infty$					$+\infty$

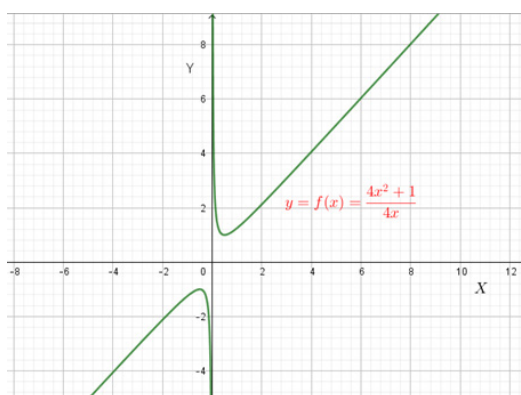
7) Variation table or synthetic table

x	$-\infty$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$+\infty$								
$f'(x)$	+	+	0	-	-	-	-	-	0	+	+	+	
$f''(x)$	-	-	-	-	-	-	-	-	+	+	+	+	+
$f(x)$	$-\infty$												

8) Supplementary points

x	2	-2	-3	3
$f(x)$	$17/8$	$-17/8$	$-37/12$	$37/12$

9) Graph of the function $f(x) = \frac{4x^2 + 1}{4x}$



$$2) y = \frac{1}{x-2}$$

Condition of existence: $x - 2 \neq 0$ i.e $x \neq 2$

$$1) \text{Dom}f = \mathbb{R} \setminus \{2\} =]-\infty, 2[\cup]2, +\infty[$$

2) Limits at the boundaries of Domf

$$\bullet \lim_{x \rightarrow -\infty} \frac{1}{x-2} = \frac{1}{\infty} = 0$$

\Rightarrow there is Horizontal asymptote $y = 0$

$$\lim_{x \rightarrow 2} \frac{1}{x-2} = \frac{1}{0} = \infty$$

$$\bullet \lim_{x \rightarrow 2^-} \frac{1}{x-2} = \frac{1}{0^-} = -\infty \Rightarrow \text{There is a vertical asymptote : } x = 2$$

$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = \frac{1}{0^+} = +\infty$$

$$\bullet \lim_{x \rightarrow +\infty} \frac{1}{x-2} = \frac{1}{\infty} = 0 \Rightarrow \text{There is Horizontal asymptote } y = 0$$

Let find the Oblique asymptote (OA)

$$OA \equiv y = ax + b \text{ with } a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} \text{ and } b = \lim_{x \rightarrow \pm\infty} [f(x) - ax]$$

$$a = \lim_{x \rightarrow \pm\infty} \frac{1}{x^2 - 2x} = \lim_{x \rightarrow \pm\infty} \frac{1}{\infty} = 0$$

$a = 0 \Rightarrow$ there is no OA

3) Parity of $f(x)$

$$\bullet f(x) \text{ is even function iff } f(x) = f(-x)$$

$$f(-x) = \frac{1}{-x-2} = -\frac{1}{x+2} \neq f(x), \text{ so } f(x) = \frac{1}{x-2} \text{ is not even}$$

$$\bullet f(x) \text{ is odd function iff } -f(x) = f(-x)$$

$$f(-x) = \frac{-1}{x+2}$$

- $-f(x) = -\frac{1}{x-2} = \frac{1}{2-x}$, so $-f(x) \neq f(-x)$ and $f(x)$ is not odd function

The given function is neither even nor odd

4) Periodicity of $f(x)$

The given function is not periodic (because it is not trigonometric)

5) First derivative of $f(x)$

$$f'(x) = \left(\frac{1}{x-2} \right)' = \frac{-1 \cdot (1)}{(x-2)^2} = \frac{-1}{(x-2)^2}$$

$$f'(x) = \frac{-1}{(x-2)^2}$$

Critical points:

- Condition of existence of $f'(x)$

$$(x-2)^2 \neq 0 \Leftrightarrow x \neq 2$$

$$f'(x) = 0 \Leftrightarrow \frac{-1}{(x-2)^2} = 0 \text{ which is impossible in } \mathbb{R}$$

There are no extrema points because $f'(x) < 0$ always.

Sign table of $f'(x)$

x	$-\infty$	2	$+\infty$
$f'(x)$	- - - - -	-	- - - - -
$f(x)$	0	$+\infty$	0
	↘	↘	↘
		$-\infty$	

6) Second derivative of $f(x)$



$$f''(x) = \left(\frac{-1}{(x-2)^2} \right)' = - \left[\frac{-2(x-2)}{(x-2)^4} \right] = \frac{2}{(x-2)^3}$$

Critical points:



- Existence condition of $f''(x)$: $(x-2)^3 \neq 0 \Rightarrow x \neq 2$
- $f''(x) = \frac{2}{(x-2)^3}$ can not be zero

The signs of $f''(x) = \frac{2}{(x-2)^3}$ depends on signs of denominator where $x \neq 2$

Sign table of $f''(x)$

x	$-\infty$	2	$+\infty$
$f''(x)$	- - - - -		+ + + + +
$f(x)$	0 		$+\infty$ 

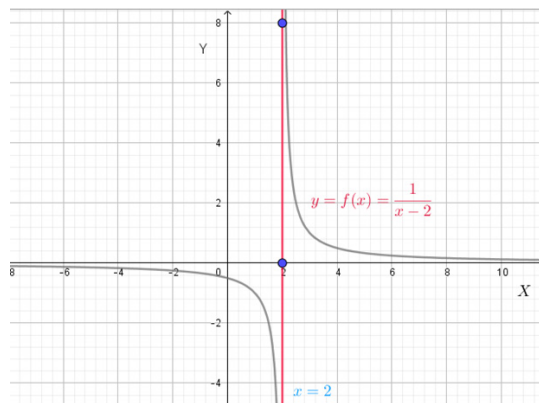
7) Variation table or synthetic table

x	$-\infty$	2	∞
$f'(x)$	- - - - -		- - - - -
$f''(x)$	- - - - -		+ + + + +
$f(x)$	0 		$+\infty$ 

8) Supplementary points

x	-1	0	1	3	4
$f(x)$	-1/3	-1/2	-1	1	1/2

9) Graph of function $f(x) = \frac{1}{x-2}$



Lesson 8: Derivative and limit with indeterminate cases: Hospital's rule

a) Learning objective

Use first derivative to determine the limit of some rational functions

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, calculator, manila paper, markers, pens, pencils, ...

c) Prerequisites/Revision/Introduction

Student -teachers should have knowledge and skills on calculations of limits (**year two,unit6**) and derivatives(**lesson 1&2 year two,unit 7**).

d) Learning activities

- Ask learners in small groups/individually to read and discuss on the **activity 7.8** in student teacher's book
- Help the student teachers to recall the indeterminate forms seen in calculations of limits
- Guides them to apply the Hospital rule and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling learners.
- Call learners to present the findings, and help them to harmonize the answer.

- Tutor will verify if the student- teachers found the same limits by calculating the limits of derivative of numerator over derivative of denominator for above limits. And after presentation of their findings ,Tutor help them to state the Hospital's rule.
- In small groups or individually, help the student teacher to work out the **application activity 7.8**

Answers of activity 7.8

a)

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 5x + 6} = \frac{0}{0} (IF) \qquad \lim_{t \rightarrow 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^2} = \frac{0}{0} (IF)$$

$$T.V : \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 5x + 6} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)(x-2)}$$

$$= \lim_{x \rightarrow 3} \frac{(x+3)}{(x-2)} = \frac{6}{1} = 6$$

b) $\lim_{x \rightarrow \infty} \frac{2x^2 - 3}{5x^2 + 4x - 6} = \frac{\infty}{\infty} (IF)$

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 3}{5x^2 + 4x - 6} = \lim_{x \rightarrow \infty} \frac{x^2 \left(2 - \frac{3}{x^2}\right)}{x^2 \left(5 + \frac{4}{x} - \frac{6}{x^2}\right)}$$

$$= \lim_{x \rightarrow \infty} \frac{\left(2 - \frac{3}{x^2}\right)}{\left(5 + \frac{4}{x} - \frac{6}{x^2}\right)} = \frac{2}{5}$$

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3} = \frac{0}{0} (IF)$$

$$\lim_{x \rightarrow 3} \frac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)}{(x-3)(\sqrt{x+1} + 2)}$$

$$= \lim_{x \rightarrow 3} \frac{1}{(\sqrt{x+1} + 2)}$$

$$= \frac{1}{4}$$

Answer of Application activity 7.8

By using Hospital rule ,

1) $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x} = 6$

$$2) \lim_{x \rightarrow \infty} \frac{2x+7}{3x^2-5} = 0$$

$$3) \lim_{x \rightarrow \infty} \frac{x^2-1}{x^2+3x-4} = 1$$

$$4) \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} = 4$$

$$5) \lim_{x \rightarrow 0} \frac{\arcsin 4x}{\arctan 5x} = \frac{4}{5}$$

Lesson 9: Derivative, tangent line equation and Normal line equation

a) Learning objective

Use first derivative to determine the gradient of the tangent line and normal line to a curve at the point

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, manila paper, markers, pens, pencils, ...

c) Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on sketching the graph of function and derivative of functions **(unit 7, lesson 1&2, year 2)**

d) Learning activities

- Ask learners in small groups/individually to read and discuss on the **activity 7.9** in student teacher's book
- Help the student teachers to recall the gradient by first derivative of function **(unit 7, lesson 1&2, year 2)**
- Guide them to establish the equation tangent and normal line to the curve at the point. and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling learners.
- Call student-teachers to present the findings, and help them to harmonize the answers.
- After presentation, tutor will help the student-teachers to establish the equation of the tangent and normal line to the curve at the point

- Tutor guides them to work the provided example.
- In small groups or individually, help the student teacher to work out the **application activity 7.9**

Answers of activity 7.9

1.

$$\begin{cases} y = -x^3 + 3x \\ y = 3x \end{cases} \Rightarrow 3x = -x^3 + 3x \Rightarrow x = 0$$

$$\text{if } x = 0 \Rightarrow y = 0$$

So the point (0,0) is the intersection point of $y = -x^3 + 3x$ and $y = 3x$

2.

$$f'(x) = -3x^2 + 3$$

$$f'(0) = 3$$

$$3. \text{ Gradient} = f'(0) = 3$$

Answer of application activity 7.9

1.

$$\text{a) } T \equiv y - 2 = -\frac{1}{2}(x + 1) \text{ and } N \equiv y - 2 = 2(x + 1)$$

$$\text{b) } T \equiv y - 5 = 2(x - 1) \text{ and } N \equiv y - 5 = -\frac{1}{2}(x - 1)$$

$$\text{c) } T \equiv y - 8 = 12(x - 2) \text{ and } N \equiv y - 8 = -\frac{1}{12}(x - 2)$$

2.

$$f(x) = x^2 - x$$

$$f'(x) = 2x - 1$$

$$f'(x_0) = 2x_0 - 1 = -3 \Rightarrow x_0 = -1$$

$$y_0 = 2$$

$$\text{So, } T \equiv y - 2 = -3(x + 1) \text{ or } T \equiv y = -3x + 3$$

Lesson 10: Applications of differentiation in Economics and finance

a) Learning objective:

Apply differentiation in economics and other social sciences

b) Teaching resources:

Student-teacher's book and other Reference textbooks to facilitate research, Mathematical set, calculator, Manila paper, graph paper, ruler, markers, pens, pencils, etc.

c) Prerequisites/Revision/Introduction:

Student-teachers will perform well in this unit if they have good background in previous lessons of this unit.

d) Learning activities

- In group discussions, invite student-teachers to conduct research on the internet or in library about the application of differentiation in economics and finance.
- Invite student-teachers to work in group and do the second question of activity 7.10 found in their Mathematics books;
- Move around in the class for facilitating groups where necessary and ask some guiding questions on eventual challenges they may face during their work;
- Invite groups with different working steps to present their work;
- As a tutor, harmonize the findings from presentation of student-teachers and guide them to explore the content and examples given in the student's book where they will be able to solve real life problem of economics and finance by using differentiation.
- After the lesson, guide students-Teacher to do the application activity 5.5 and evaluate whether lesson objectives were achieved.

Answer for activity 7.10

1. Answers vary accordingly, facilitate student-teachers to present their findings and harmonize their answers from their own research.

$$2. MC = \frac{dTC}{dq}$$

$$TC = 6 + 4q^2$$

$$MC = \frac{dTC}{dq} = \frac{d(6 + 4q^2)}{dq} = 8q$$

The function $MC = 8q$

d) Answers of application activity 7.10

1. To find the value of x that maximizes the daily profit, we must first have a

$$P(x) = R(x) - c(x)$$

profit function. Since $P(x) = 28x - (x^3 - 6x^2 + 13x + 15)$

$$P(x) = 28x^3 + 6x^2 + 15x - 15$$

So, the value of x which we are seeking, will be the one that corresponds to the highest point on the $P(x)$ graph for $x \geq 0$. Upon looking at the graph, it would appear that this occurs at about $x = 5$

Lesson 11: Applications of differentiation: rates of change problems, optimization problems

a) Learning objective

Apply the concepts and techniques of differentiation to model, analyse and solve optimization problems in different situations.

b) Teaching resources

Student-teacher's book and other Reference textbooks to facilitate research, calculator, manila paper, markers, pens, pencils, ...

c) Prerequisites/Revision/Introduction

Student teachers should have knowledge and skills on sketching the graph of function and derivative of functions **(unit 7, lesson 1&2, year 2)**

d) Learning activities

- Ask learners in small groups/individually to read and discuss on the **activity 7.11** in student teacher's book
- Help the student teachers to recall the gradient by first derivative of function (**unit 7, lesson 1&2, year 2**)
- Guides them to establish the rate of change of function and make sure that everybody is engaged/ involved.
- Facilitate working, especially straggling learners.
- Call student-teachers to present the findings, and help them to harmonize the answers.
- After presentation, tutor will help the student-teachers to establish the formula of rate of change and deduce the mean value, Rolle's theorem, of the function and guide them to work the provided examples.
- In small groups or individually, help the student teacher to work out the **application activity 7.4**

Answers of activity 7.11

Slope of the secant line connecting the two points of the function is 6 and -3 it means that the secant line is going downhill or decreasing as you look at it from left to right.

Answer of application activity 7.11

1) Specific rate of change = $4 \times 2 + (5 - 2) = 8 + 6 = 14$

2) Specific rate of change = $3 \times 0 + 3(0)(2) + (2)2 = 4$

7.6 Unit summary

1) The derivative of a function with respect to x is denoted by or $\frac{d}{dx}f(x)$ and defined as provided that the limit exists.

2) Derivative of a constant function: $\frac{df}{dx} = \frac{d}{dx}(c) = 0$

3) Derivative of identity function: if $f(x) = x$, $\frac{df}{dx} = \frac{dx}{dx} = 1$

4) with $c \in \mathbb{R}$

$$5) [f^n(x)]' = nf^{n-1}(x)f'(x).$$

$$6) \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$7) \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

$$8) \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$9) \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

10) If f and g are both differentiable and F is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x)) \cdot g'(x)$

11) Let $y = f(v)$ with $v = \Psi(x)$ and derivative of y respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

$$12) [\sin u(x)]' = u'(x) \cos u(x)$$

$$13) (\cos u(x))' = -u'(x) \sin u(x)$$

$$14) (\tan u(x))' = \frac{u'(x)}{\cos^2 u(x)} \text{ or } (\tan u(x))' = u'(x) \sec^2 u(x)$$

$$15) (\cot u(x))' = -\frac{u'(x)}{\csc^2 u(x)} \text{ or } (\cot u(x))' = -u'(x) \csc^2 u(x)$$

$$16) (\sec u(x))' = u'(x) \frac{\sin u(x)}{\cos^2 u(x)} \text{ or } (\sec u(x))' = u'(x) \sin u(x) \sec^2 u(x)$$

$$(\csc u(x))' = -u'(x) \frac{\cos u(x)}{\sin^2 u(x)} \text{ or } (\csc u(x))' = -u'(x) \cos u(x) \csc^2 u(x)$$

$$\text{and } (\tan^{-1} u)' = \frac{u'}{1+u^2} \text{ and } (\cot^{-1} u)' = \frac{-u'}{1+u^2}$$

17) Theorem:

Let f be a function differentiable on an interval $]a, b[$

a) If $f'(x) > 0$ on each point x of $]a, b[$, then f is increasing on $]a, b[$

b) If $f'(x) < 0$ on each point x of $]a, b[$, then f is decreasing on $]a, b[$

c) If $f'(x) = 0$ for all $x \in]a, b[$, then f is constant on this interval, that is $f(x_1) = f(x_2)$ for all $x_1, x_2 \in]a, b[$, or equivalently, there exists a real c such that $f(x) = c$ for all $x \in]a, b[$.

18) a. If f' changes from negative to positive at c , then f has a **local minimum** at c

b. If f' changes from positive to negative at c , then f has a **local maximum** at c

c. If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has **no** local extremum.

19) Theorem:

Let f be a function that is defined and is twice differentiable on an open interval $]a, b[$.

1) If $f''(x) > 0$ for all $x \in]a, b[$, then f is convex on $]a, b[$

2) If $f''(x) < 0$ for all $x \in]a, b[$, then f is concave on $]a, b[$

20) Hospital rule states that, if the limit of $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$ with $g(x) \neq 0$ and x_0 a finite

number or infinity is indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then, it can be calcu-

lated by $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ with $g'(x) \neq 0$. If this result is indeterminate form, the

procedure can be repeated.

21) the equation of the tangent line on the curve at the point is

22) equation of the Normal line on the curve at the point is of the form

$$23) \text{ Marginal Revenue} = \frac{dR}{dq}$$

Let $P(q)$ be the profit of producing and selling q units of the product, that is,

$$P(q) = R(q) - C(q)$$

Thus P is a function of q and it is called the profit function.

24) If (x_1, y_1) and (x_2, y_2) are points on the graph of $f(x)$, then we define $\frac{y_2 - y_1}{x_2 - x_1}$ to be the average rate at which y changes with x over the interval $[x_1, x_2]$.

25) Mean value theorem

Suppose that $f(x)$ is a function that satisfies both of the following.

- $f(x)$ is continuous on the closed interval $[a, b]$.
- $f(x)$ is differentiable on the open interval $]a, b[$.

Then, there is a number c such that $a < c < b$ and $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\text{Or, } f(b) - f(a) = f'(c)(b - a)$$

Note that the Mean Value Theorem doesn't tell us what c is. It only tells us that there is at least one number c that will satisfy the conclusion of the theorem.

Also note that if $f(a) = f(b)$ we can think of Rolle's Theorem as a particular case of the Mean Value Theorem.

7.7 End unit assessment

1. $f'(x) = 1$
2. $y = 3 \sin x - 4 \cos x$
 $y' = 3 \cos x - 4(-\sin x)$
 $= 3 \cos x + 4 \sin x$

3.

$$y = \frac{\cos x}{1 + \sin x}$$

$$y' = \frac{(1 + \sin x)(\cos x)' - (1 + \sin x) \cos x}{(1 + \sin x)'}$$

$$y' = \frac{(1 + \sin x)(-\sin x) - (\cos x) \cos x}{(1 + \sin x)'}, \text{ after simplifying}$$

$$y' = \frac{-1}{1 + \sin x}$$

4. Find an equation of the line tangent to the graph of $y = x^2 + \sin\left(\frac{\pi}{2}x\right)$ at $x = -1$

if $x = -1$ then $y = 0$

$$y' = 2x + \frac{\pi}{2} \cos\left(\frac{\pi}{2}x\right)$$

slope of tangent to the curve at $x = -1$ is $m = -2$

Equation of tangent, $T \equiv y = -2x - 2$

5.

$$\lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{3} + h\right) - \sin \frac{\pi}{3}}{h}$$

$$\text{recall that } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

if $f(x) = \sin x$ then $f'(x) = \cos x$

and letting $x = \frac{\pi}{3}$

$$\text{it follows that } \lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{3} + h\right) - \sin \frac{\pi}{3}}{h} = \lim_{h \rightarrow 0} \frac{f\left(\frac{\pi}{3} + h\right) - f\left(\frac{\pi}{3}\right)}{h}$$

$$= f'\left(\frac{\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}$$

6. a. $p(x) = 19 - \frac{1}{3000}x$

b. ticket prices be set to maximize revenue is 9.50 dollars

7. $f'(x) = 6x^2 - 10x + 4 = 3x^2 - 5x + 2$

$$f''(x) = 12x - 10$$

These points are $(1, 3)$, $(\frac{2}{3}, \frac{82}{27})$.

$$f'(x) = 6x^2 - 10x + 4 = 3x^2 - 5x + 2$$

$$f''(x) = 12x - 10 = 6x - 5$$

The interval of increasing: $]-\infty, 1[\cup]0.667, +\infty[$

The interval of decreasing: $]1, 0.667[$

7.8. Additional information for the tutor

For the educative action of the teacher to be effective (in order to respond to all aspects of the learners' needs), it is worth mentioning that the teacher needs a wide range of skills, attitudes, a rich and deep understanding of the subject matter and the pedagogical processes to develop the understanding that is required from the learner. It is, therefore, imperative for the teacher to not limit himself/herself to the only to the present book, but also to consider getting information from other relevant books, such as those mentioned in the reference.

7.9 Additional activities

7.9.1 Remedial activities

Calculate the derivative of $f(x) = (2x+1)^6$

Solution:

$$f(x) = (2x+1)^6$$

$$f(x) = (2x+1)^6$$

$$f'(x) = 6(2x+1)(2x+1)^5$$

$$f'(x) = 6(2)(2x+1)^5$$

$$f'(x) = 12(2x+1)^5$$

7.9.2 Consolidation activities

- a. Find the derivative of
- b. A firm was assumed to have the total cost function $TC = 18q$ and the total revenue function $TR = 240 + 14q$ calculate the profit-maximizing output at $q = 1$

Solution

The profit function will be $TR - TC$

$$= TR - TC = 240 + 14q - 18q$$

$$= 240 - 4q$$

The profit-maximizing output = $240 - 4 = 236$

7.9.3 Extended activities

A firm faces the demand schedule $p = 200 - 2q$ and the total cost function given by the function $TC = \frac{2}{3}q^3 - 14q^2 + 222q + 50$. Derive expressions for the following functions and find out whether they have maximum or minimum points.

If they do, say what value of q this occurs at and calculate the actual value of the function at this output.

- | | | |
|-------------------|---------------------------|------------------------|
| (a) Marginal cost | (b) Average variable cost | (c) Average fixed cost |
| (d) Total revenue | (e) Marginal revenue | (f) Profit |

UNIT 8

MATRIX OF ORDER 2 AND ORDER 3

8.1 Key unit competence

Solve problem involving the system of linear equations using matrices

8.2 Prerequisite

Learners will easily learn this unit, if he/she is able to

- Present the information in array of rows and columns as they did in statistics (unit 10 year 1 and unit 4, year 2)
- Solve linear equations and quadratic equations (unit 5 year 1)
- Solve system of linear equations using substitution or elimination method
- Manipulate the operations of vectors (Senior2)
- Be accurate in calculations.

8.3 Cross-cutting issues to be addressed:

- Inclusive education (promote education for all while teaching)
- Peace and value Education (respect others view and thoughts during class discussions)
- Gender (equal opportunity of boys and girls in the lesson participation)
- Standardization culture (Learners encouraged to adhere to rules and standards of calculations)

8.4 Guidance on introductory activity

- a) Student- teachers work on the introductory activity to be aware of this unit.
- b) Guide the student-teachers , to do the activity 8.1, in pairs, move around in order to help the slower learners .
- c) Guide the student- teachers to present their findings and help them to harmonize the answers

Answer of introductory activity

a)

Cocks	Rabits	Price
5	4	35,000
3	6	30,000

a) Let x be the cost of one cock and y be the cost of one rabbit, then ,

$$\begin{cases} 5x + 4y = 35,000 \\ 3x + 6y = 30,000 \end{cases}$$

$$\begin{cases} 5x + 4y = 35,000 \times (3) \\ 3x + 6y = 30,000 \times (-5) \end{cases} \Rightarrow \begin{cases} 15x + 12y = 105,000 \\ -15x - 30y = -150,000 \end{cases} \Rightarrow -18y = -45,000 \Rightarrow y = 2,500$$

If we replace y in the first equation we obtain

$$5x + 4(2500) = 35,000 \Rightarrow 5x = 25,000 \Rightarrow x = 5,000$$

Thus the cost of 1 cock is 5,000Frw and the cost of one rabbit is 2,500Frw.

8.5 List of lessons/sub-heading

No	Lesson title	Learning objectives	Number of periods
0	Introductory activity 8	To arouse the curiosity of student teachers on the content of unit 8	1
1	Definition, size and types of matrices	Define and differentiate the types of matrices and identify the equality of two matrices	1
2	Operations on matrix: -Addition and subtraction -Multiplication	Perform operations on matrices of order 2 and 3.	1
3	Operations on matrix: -Multiplication	To perform operations of multiplication on matrices of order 2 and order 3	1

4	Properties of Matrices Multiplication	Perform well the use of properties of multiplication on matrices of order 2 and order 3.	1
5	Transpose of a matrix	Construct a transpose of a given matrix	1
6	Matrix of Linear transformation in 3D	To Construct a matrix representative of transformation of order 2 and order	1
7	Determinant of matrices of order 2 and 3	Calculate the determinants of matrices of order 2 and 3.	1
8	Properties of determinant of order 2 and 3	Apply correctly the properties of the determinant of a matrix of order 2 and order 3	1
9	Inverse of matrix of order 2 and 3	Determine the inverse of a matrix of order 2 or 3.	2
10	Properties of the Inverse Matrix	Apply correctly the properties of the inverse of a matrix of order 2 and order 3.	1
11	Solving System of linear equations using inverse matrix	Use method of inverse matrix to solve a system of two or three linear equations with two or three unknowns.	1
12	Solving System of linear equations using Cramer method	Use Cramer's method to solve a system of two or three linear equations with two or three unknowns.	1

13	Solving System of linear equations using Gaussian method (use of elementary transformations)	Use Gaussian method to solve a system of two or three linear equations with two or three unknowns.	1
14	End unit assessment		2
Total number of periods in this unit.			17

Lesson 1: Definition, size and types of matrices

a) Learning objectives

To define and differentiate types of matrices.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they have a good background on Arithmetic (Unit 1, Year 1) and on Equations and inequalities (unit 2 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and differentiate types of matrices
- After this step, guide students to do the application activity 8.1 and evaluate whether lesson objectives were achieved.

Answer of activity 8.1

$$1) \begin{pmatrix} 20 & 31 \\ 45 & 23 \end{pmatrix}$$

2) The answers vary according to the present situation.

a) Application activity 8.1

Answer of Application activity 8.1

1) (a) Matrix square of order 3 or 3×3

(b) Matrix of one row and two column 1×2

(c) Matrix of one row and one column 1 or 1×1

(d) Matrix of two rows and five column 2×5

(e) Matrix square of order 2 or 2×2

2)

a) $(a \ b \ c)$: row matrix

b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$: Zero matrix

c) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$: column matrix

d) $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$: diagonal matrix

e) $\begin{pmatrix} a & b & c \\ 0 & b & d \\ 0 & 0 & e \end{pmatrix}$: upper triangular matrix

f) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: identity matrix

g) $\begin{pmatrix} a & 0 & 0 \\ c & b & 0 \\ d & 0 & c \end{pmatrix}$: Lower triangular matrix

h) $\begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$: scalar matrix

3)

$$\begin{cases} 3y + 2 = y - 3 \\ 2x + 1 = 5 \end{cases} \Rightarrow \begin{cases} y = -\frac{5}{2} \\ x = 2 \end{cases}$$

Lesson 2: Operations on matrices, Addition and subtraction of matrices

a) Learning objectives

Perform operations on matrices of order 2 and 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they have a good background on Arithmetic (Unit 1, Year 1) and on Equations and inequalities (unit 2 Year 1).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.2.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to define and differentiate types of matrices .

After this step, guide students to do the application activity 8.2.1 and evaluate whether lesson objectives were achieved.

Answer of activity 8.2.1

1)

	Too high	Too Low	No opinion
Male	200	150	45
Female	315	125	65

Then, in matrix form: $\begin{pmatrix} 200 & 150 & 45 \\ 315 & 125 & 65 \end{pmatrix}$

2) The student-teachers understand that they will add the numbers occupying the same position for getting the combined situation in two classes :

$$A+B = \begin{bmatrix} 23 & 2 \\ 20 & 4 \end{bmatrix} + \begin{bmatrix} 25 & 2 \\ 23 & 3 \end{bmatrix} = \begin{bmatrix} 48 & 4 \\ 43 & 7 \end{bmatrix}$$

3) The above situation 1) Helps the student-teachers to work easily the following operations :

Consider the matrices $A = \begin{pmatrix} 2 & -4 & 12 \\ 1 & 0 & -4 \\ 5 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 6 & 4 \\ 1 & 7 & 8 \\ 3 & 21 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 3 & -2 & 0 \end{pmatrix}$

find

$$1. \quad A+3B = \begin{pmatrix} 5 & 14 & 24 \\ 4 & 21 & 20 \\ 14 & 65 & 12 \end{pmatrix}$$

$$2. \quad 2A-B = \begin{pmatrix} 3 & -14 & 20 \\ 1 & -7 & -16 \\ 7 & -17 & 3 \end{pmatrix}$$

$$3. \quad A+(-A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Comment: $-A$ is additive inverse of A

$$4. \quad A+B = \begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix} \quad \text{and} \quad B+A = \begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix}$$

The two results are equal. This implies that the addition of matrices is commutative

$$5. A+(B+C)=\begin{pmatrix} 2 & -4 & 12 \\ 1 & 0 & -4 \\ 5 & 2 & 3 \end{pmatrix}+\begin{pmatrix} 2 & 9 & 3 \\ 1 & 9 & 12 \\ 6 & 19 & 3 \end{pmatrix}=\begin{pmatrix} 4 & 5 & 15 \\ 2 & 9 & 8 \\ 11 & 21 & 6 \end{pmatrix} \text{ and}$$

$$(A+B)+C=\begin{pmatrix} 3 & 2 & 16 \\ 2 & 7 & 4 \\ 8 & 23 & 6 \end{pmatrix}+\begin{pmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 3 & -2 & 0 \end{pmatrix}=\begin{pmatrix} 4 & 5 & 15 \\ 2 & 9 & 8 \\ 11 & 21 & 6 \end{pmatrix}$$

The two results are equal. This implies that the addition of matrices is associative

6. Interchanging the rows and columns of A, B and C we get the following matrices:

$$\begin{pmatrix} 2 & 1 & 5 \\ -4 & 0 & 2 \\ 12 & -4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 3 \\ 6 & 7 & 21 \\ 4 & 8 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 3 & 2 & -2 \\ -1 & 4 & 0 \end{pmatrix}$$

Answer for application activity 8.2.1

$$1) \text{ (a) } \begin{pmatrix} 13 & 4 \\ 6 & 10 \end{pmatrix} + \begin{pmatrix} 21 & 30 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 34 & 34 \\ 15 & 22 \end{pmatrix} \quad \text{(b) } \begin{pmatrix} 26 & 8 \\ 12 & 20 \end{pmatrix} - \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 19 & -2 \\ 9 & 16 \end{pmatrix}$$

2)a)

$$\begin{pmatrix} 1 & -10 & -3 \\ -12 & -2 & -11 \\ 0 & -2 & 1 \end{pmatrix}$$

b)

$$\begin{pmatrix} -25 & 10 & 15 \\ -4 & 6 & -5 \\ -18 & 8 & 25 \end{pmatrix}$$

c)

$$\begin{pmatrix} 15 & -14 & 3 \\ 0 & 0 & -13 \\ 9 & 3 & 4 \end{pmatrix}$$

Lesson 3: Operations on matrices : Multiplication of matrices

a) Learning objectives

To perform operations of multiplication on matrices of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will well perform in this lesson, if they have a good background on Arithmetic (Unit 1, Year 1) , on Equations and inequalities (unit 2 Year 1) , on lesson 1 and lesson 2 of this Unit.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.2.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation: Two matrices A and B can be multiplied together if and only if the number of columns of A is equal to the number of rows of B. $A_{m \times n} \times B_{n \times p} = M_{m \times p}$
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to perform operations of multiplication on matrices of order 2 and order 3.
- After this step, guide students to do the **application activity 8.2.2** and evaluate whether lesson objectives were achieved.

Answer activity 8.2.2

1) We can represent prices of each item as $P = \begin{pmatrix} 40\$ \\ 20\$ \\ 400\$ \end{pmatrix}$.

The total number of each item in the store is $N = (100 \ 200 \ 50)$.

Then, the total revenue due to these sales is given by

$$N \times P = (100 \ 200 \ 50) \begin{pmatrix} 40\$ \\ 20\$ \\ 400\$ \end{pmatrix} = 28000\$$$

2)

$$A \times B = \begin{pmatrix} -2+3-1 & -1+2+6 & 1+3+4 \\ 4+6-5 & 2+4+30 & -2+6+20 \\ 0+9-4 & 0+6+24 & 0+9+15 \end{pmatrix} = \begin{pmatrix} 0 & 7 & 8 \\ 5 & 36 & 24 \\ 5 & 30 & 25 \end{pmatrix}$$

Answers of application activity 8.2.2

$$1. \quad A \times B = \begin{pmatrix} -28 & 36 & 39 \\ 28 & -6 & -5 \\ 56 & 64 & 80 \end{pmatrix}$$

$$2. \quad A \times C = \begin{pmatrix} 47 & 4 & -36 \\ 1 & -9 & 31 \\ 112 & 8 & -28 \end{pmatrix}$$

$$3. \quad B \times C = \begin{pmatrix} 161 & 9 & -21 \\ 276 & -22 & -18 \\ 123 & 7 & -17 \end{pmatrix}$$

Lesson 4.: Properties of Matrices Multiplication

a) Learning objectives

To perform operations of multiplication on matrices of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will well perform in this lesson, if they have a good background on multiplying matrices (lesson 3 of this Unit).

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.2.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation on properties of multiplication of matrices;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to perform operations of multiplication on matrices of order 2 and order 3.
- After this step, guide students to do the application activity 8.2.3 and evaluate whether lesson objectives were achieved.

Answer for activity 8.2.3

$$1. A \times B = \begin{pmatrix} -1 & 3 & -2 \\ -2 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix}, B \times A = \begin{pmatrix} 2 & -4 & -1 \\ 7 & -7 & -2 \\ -5 & 3 & 1 \end{pmatrix}$$

$A \times B \neq B \times A$. Multiplication of matrices is not commutative

$$2. (A \times B)^t = \begin{pmatrix} -1 & -2 & 1 \\ 3 & -1 & 1 \\ -2 & 3 & -2 \end{pmatrix}, B^t \times A^t = \begin{pmatrix} -1 & -2 & 1 \\ 3 & -1 & 1 \\ -2 & 3 & -2 \end{pmatrix}$$

$$(A \times B)^t = B^t \times A^t$$

$$3. A \times (B \times C) = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 1 \\ -2 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 & 1 \\ 4 & -3 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -9 & 4 \\ -6 & 3 & 1 \\ 4 & -3 & 0 \end{pmatrix},$$

$$(A \times B) \times C = \begin{pmatrix} -1 & 3 & -2 \\ -2 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -9 & 4 \\ -6 & 3 & 1 \\ 4 & -3 & 0 \end{pmatrix}$$

$A \times (B \times C) = (A \times B) \times C$. Multiplication of matrices is associative

$$4. A \times (B + C) = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 1 \\ -2 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & -1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & -5 \\ -4 & 2 & 2 \\ 0 & -5 & 3 \end{pmatrix},$$

$$A \times B + A \times C = \begin{pmatrix} -1 & 3 & -2 \\ -2 & -1 & 3 \\ 1 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 4 & -3 \\ -2 & 3 & -1 \\ -1 & -6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 7 & -5 \\ -4 & 2 & 2 \\ 0 & -5 & 3 \end{pmatrix}$$

$A \times (B + C) = A \times B + A \times C$. Multiplication of matrices is distributive over addition

Answers of application activity 8.2.3

1)

$$a) A \times B = \begin{pmatrix} -3 & 0 & 2 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B \times A = \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

$$b) (A \times B) \times C = \begin{pmatrix} -3 & 0 & 2 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 3 \\ -4 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A \times (B \times C) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 & 2 \\ -1 & 2 & -1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 3 \\ -4 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$c) A \times (B+C) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ -1 & 2 & 2 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ -2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A \times B + A \times C = \begin{pmatrix} -3 & 0 & 2 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ -2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$d) tr(A \times B) = tr \begin{pmatrix} -3 & 0 & 2 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -3$$

Lesson 5: Transpose of Matrix

a) Learning objectives

To Construct a transpose of a given matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication) on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.3.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation: Given matrix A, **the transpose of matrix A**, noted A^t , is another matrix where the elements in

the columns and rows have interchanged. In other words, the rows become the columns and the columns become the rows.

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to construct a transpose of a given matrix of order 2 and order 3.
- After this step, guide students to do the application activity 8.3.1 and evaluate whether lesson objectives were achieved.

Answers of activity 8.3.1

$$1. \begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 12 & 3 & -4 \\ 3 & -2 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$2. \begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$$

$$3. \begin{pmatrix} 13 & 6 & 0 \\ 4 & -4 & 2 \\ -1 & -3 & 0 \end{pmatrix}$$

$$4. \begin{pmatrix} 13 & 4 & -1 \\ 6 & -4 & -3 \\ 0 & 2 & 0 \end{pmatrix}$$

5. Matrix obtained in 2 is equal to the matrix obtained in 4

$$6. A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$$

Interchanging the rows and columns of matrix A once we get the new matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 3 & -2 & -2 \\ 1 & 2 & 0 \end{pmatrix}$$

Interchanging the rows and columns of matrix A twice we get the new matrix

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & -2 & 0 \end{pmatrix}$$

The last matrix is equal to matrix A

Answers of application activity 8.3.1

1. Let $A = \begin{pmatrix} x+30 & 2 \\ 4z+x & 3y+4 \end{pmatrix}$ and $B = \begin{pmatrix} 6x+5 & 2 \\ 1 & 1 \end{pmatrix}$

$$A = B \Rightarrow \begin{cases} x+30 = 6x+5 \\ 4z+x = 1 \\ 3y+4 = 1 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = -1 \\ z = -1 \end{cases}$$

a) $A = \begin{pmatrix} 35 & 2 \\ 1 & 1 \end{pmatrix}$

b) $A^t = \begin{pmatrix} 35 & 1 \\ 2 & 1 \end{pmatrix}$

2.

a) $(A+B)^t = \begin{pmatrix} 1 & -3 & 9 \\ 6 & 3 & 0 \\ 3 & 9 & 13 \end{pmatrix}$

b) $3A^t + B = \begin{pmatrix} 1 & 5 & 10 \\ 8 & 9 & -3 \\ 12 & 20 & 29 \end{pmatrix}$

c) $(-3B+4A)^t = \begin{pmatrix} 4 & -19 & 15 \\ -4 & -9 & 14 \\ -2 & -6 & -4 \end{pmatrix}$

d) $M^t = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 0 \\ 1 & 1 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & x^2 \\ 4 & 1 & 0 \\ 1 & x+3 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 0 \\ 1 & 1 & 8 \end{pmatrix}$

$$\begin{cases} x^2 = 4 \\ x + 3 = 1 \end{cases} \Rightarrow x = -2$$

$$3. \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Lesson 6: Matrix of Linear transformation in 3D

a) Learning objectives

To Construct a matrix representative of transformation of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication) on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.3.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation .
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to construct a matrix of a linear transformation. .
- After this step, guide students to do the application activity 8.3.2 and evaluate whether lesson objectives were achieved.

Answers activity 8.3.2

1. $(1,0,2)$

2. $(0,1,0)$

3. $(1,-1,0)$

4. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix}$

Answers of application activity 8.3.2

1. a) $\begin{pmatrix} 3 & 2 & 0 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 4 & -3 & 2 \\ -1 & 0 & 0 \\ -3 & 5 & -1 \end{pmatrix}$

2. a) $\begin{pmatrix} 2 & 1 & -6 \\ 1 & 8 & -15 \\ 11 & -20 & -22 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 9 & 20 \\ -4 & -3 & 4 \\ 10 & 12 & -13 \end{pmatrix}$ c) $\begin{pmatrix} 11 & 4 & -4 \\ 9 & -19 & -10 \\ 11 & -6 & -5 \end{pmatrix}$

Lesson 7 : Determinants of matrices of order two and three

a) Learning objectives

To Calculate the determinant of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication) on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.4.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation:
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to calculate the determinant of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 8.4.1 and evaluate whether lesson objectives were achieved.

Answers of activity 8.4.1

1. a) $1 \times 1 - 2 \times 0 = 1$ b) $-2 \times 6 - 3 \times (-4) = 0$

c) $3 \times 8 - 6 \times 1 = 18$ d) $12 \times 9 - (-2) \times 3 = 114$

2. $(1 \times 6 \times 1) + (3 \times 0 \times 2) + (5 \times 1 \times (-4)) - (2 \times 6 \times 5) - (1 \times 0 \times 1) - (1 \times 3 \times (-4)) = -62$

3. $(10 \times 2 \times 4) + ((-6) \times 5 \times 2) + (0 \times 3 \times 1) - (4 \times 5 \times 0) - (2 \times 3 \times 10) - (1 \times (-6) \times 2) = -70$

Answer of application activity 8.4.1

1) 82 2) 10 3) -19

Lesson 8: Properties of determinant

a) Learning objectives

Apply correctly the properties of the determinant of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.4.2 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation:
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to calculate the determinant of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 8.4.2 and evaluate whether lesson objectives were achieved.

Answers of Activity 8.4.2

1. $|A| = 0, |B| = 0$

2. $|C \cdot D| = -36, |C| \cdot |D| = 6 \times (-6) = -36$

$$|C \cdot D| = |C| \cdot |D|$$

Determinant of product is equal to the product of determinants

3. $tr(C) = 6, |C| = 6$

$$tr(C) = |C|$$

Determinant of a matrix is equal to the trace of that matrix

Answers of Application activity 8.4.2

1) $|A| = 0, |B| = 0, |C| = 14, |D| = -5$

2) $|BC| = |B| \times |C| = 0$

3) $|CD| = |C| \times |D| = -70$

Lesson 9: Inverse of matrices of order two or three

a) Learning objectives

To determine the inverse of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.4.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;

- As a tutor, harmonize the findings from presentation by helping students to realise that calculating matrix inverse of matrix A , is to find matrix A^{-1} such that $A \cdot A^{-1} = A^{-1} \cdot A = I$, where I is identity matrix.
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine the inverse of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 8.4.3 and evaluate whether lesson objectives were achieved.

Answers of Activity 8.4.3

1. $|A| = -1$

2. Cofactor of each element:

$$\begin{aligned} \text{cofactor}(1) &= 3, & \text{cofactor}(1) &= -5, & \text{cofactor}(1) &= 1 \\ \text{cofactor}(2) &= 1, & \text{cofactor}(1) &= -2, & \text{cofactor}(-1) &= 1 \\ \text{cofactor}(3) &= -2, & \text{cofactor}(2) &= 3, & \text{cofactor}(1) &= -1 \end{aligned}$$

Cofactor matrix

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & -2 & 1 \\ -2 & 3 & -1 \end{pmatrix}$$

3. Transpose of cofactor matrix is

$$\begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

4. $\frac{1}{-1} \begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix}$

$$\begin{aligned}
 5. \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} -3 & -1 & 2 \\ 5 & 2 & -3 \\ -1 & -1 & 1 \end{pmatrix} &= \begin{pmatrix} -3+5-1 & -1+2-1 & 2-3+1 \\ -6+5+1 & -2+2-1 & 4-3-1 \\ -9+10+1 & -3+4-1 & 6-6+1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I
 \end{aligned}$$

The product of these two matrices is a unity (identity) matrix I .

Application activity 8.4.3

1. Since determinant is 0, then, there is no inverse

$$2. \begin{pmatrix} \frac{23}{268} & -\frac{29}{268} & \frac{5}{268} \\ -\frac{3}{268} & -\frac{37}{268} & \frac{11}{268} \\ -\frac{9}{268} & \frac{23}{268} & \frac{33}{268} \end{pmatrix}$$

$$3. \begin{pmatrix} \frac{6}{7} & -\frac{45}{14} & \frac{16}{7} \\ -\frac{5}{7} & \frac{24}{7} & -\frac{18}{7} \\ \frac{1}{7} & -\frac{11}{14} & \frac{5}{7} \end{pmatrix}$$

$$4. \begin{pmatrix} -\frac{2}{5} & -\frac{3}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & -1 \\ -\frac{6}{5} & -\frac{29}{5} & 8 \end{pmatrix}$$

Lesson 10: Properties of the Inverse Matrix

a) Learning objectives

Apply correctly the properties of the inverse of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication) on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.4.4 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation by letting students discover that for two invertible matrices A and B , the following are

$$\text{always true: } (A \cdot B)^{-1} = B^{-1} \cdot A^{-1}; (A^{-1})^{-1} = A; (\alpha \cdot A)^{-1} = \alpha^{-1} \cdot A^{-1}; \\ (A^t)^{-1} = (A^{-1})^t$$

- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to determine the inverse of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 8.4.4 and evaluate whether lesson objectives were achieved.

e) Answers of activity 8.4.4

$$1. (AB)^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}, B^{-1}A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ -2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$2. (A^{-1})^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A^{-1})^{-1} = A$$

$$3. (4A)^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \frac{1}{4}A^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}, k \neq 0$$

$$4. (A^t)^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, (A^{-1})^t = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$(A^t)^{-1} = (A^{-1})^t$$

Answers of application activity 8.4.4

$$1. A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix}, B^{-1} \text{ doesn't exist, } \mathbf{AB}^{-1} \text{ doesn't exist,}$$

$$(A^t)^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 3 & -1 \end{pmatrix}, (4B)^{-1} \text{ doesn't exist}$$

$$2) \text{ a. } A^{-1} = \frac{1}{11} \begin{pmatrix} 2 & 5 & -14 \\ 1 & -3 & 4 \\ -1 & 3 & 7 \end{pmatrix}, B^{-1} = \begin{pmatrix} -\frac{5}{12} & \frac{1}{4} & -\frac{13}{12} \\ -\frac{3}{4} & \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

$$\text{b. } (A^{-1})^{-1} = \begin{pmatrix} 3 & 7 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$3. (10A)^{-1} = \frac{1}{110} \begin{pmatrix} 2 & 5 & -14 \\ 1 & -3 & 4 \\ -1 & 3 & 7 \end{pmatrix}$$

$$4. (A^t)^{-1} = \frac{1}{11} \begin{pmatrix} 2 & 1 & -1 \\ 5 & -3 & 3 \\ -14 & 4 & 7 \end{pmatrix}$$

Lesson 11: Solving System of linear equations using inverse matrix

a) Learning objectives

To solve system of linear equations using the inverse of a matrix of order 2 and order 3.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication) on matrices.

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.5.1 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;

- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to solve system of 3 linear equations using the inverse of a matrix of order 2 and order 3.
- After this step, guide students to do the application activity 8.5.1 and evaluate whether lesson objectives were achieved.

Answer for activity 8.5.1

1.

a) Let consider x as the cost for one cock and y the cost of one Rabbit, the equations that illustrate the activity of Kalisa is
$$\begin{cases} 5x + 4y = 35,000 \\ 3x + 6y = 30,000 \end{cases}$$

b) The matrix A indicating the number of cocks and rabbits is $A = \begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$

c) In matrix form the activity of Kalisa is written as
$$\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 35,000 \\ 30,000 \end{pmatrix}$$

d) After discussion student-teacher discover that the cost of one cock is 5000Frw and the cost of one rabbit is 2500Frw.

2.

$$1. \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$2. \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Answers for application activity 8.5.1

1. $S = \{(1, 2, 0)\}$

Lesson 12: Solving System of linear equations using Cramer method

a) Learning objectives

To solve system of of linear equations using Cramer method.

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices, on determinant of matrices and on inverse of matrices

d) Learning activities:

- a) Invite student-teachers to work in groups and do the activity 8.5.2 found in their Mathematics Student books;
- b) Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- c) Invite one member from each group with different working steps to present their work where they must explain the working steps;
- d) As a tutor, harmonize the findings from presentation on solving system of of linear equations using Cramer method;
- e) Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to solve system of of linear equations using Cramer method .
- f) After this step, guide students to do the application activity 8.5.2 and evaluate whether lesson objectives were achieved.

Answers of activity 8.5.2

$$\begin{cases} x+2y-3z=0 \\ 3x+3y-z=5 \\ x-2y+2z=1 \end{cases}$$

$$a) \begin{bmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$$b) \Delta = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix} = [6 - 2 + 18] - [-9 + 2 + 12] = 17$$

$$c) \Delta_x = \begin{vmatrix} 0 & 2 & -3 \\ 5 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix} = 0 - 2 + 30 - (-9 + 0 + 20) = 17$$

$$d) \Delta_y = \begin{vmatrix} 1 & 0 & -3 \\ 3 & 5 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 10 + 0 - 9 - (-15 - 1 + 0) = 17$$

$$e) \Delta_z = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 3 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 3 + 10 + 0 - (0 - 10 + 6) = 17$$

$$f) \begin{cases} x = \frac{\Delta_x}{\Delta} = \frac{17}{17} = 1 \\ y = \frac{\Delta_y}{\Delta} = \frac{17}{17} = 1 \\ z = \frac{\Delta_z}{\Delta} = \frac{17}{17} = 1 \end{cases}$$

$$S = \{(1, 1, 1)\}$$

g) Using other method, we get the same solution

Answers of application activity 8.5.2

1. $S = \{(0,0,0)\}$

2. $S = \{ \}$

3. $S = \{(1,2,0)\}$

Lesson 13: Solving system of linear equations using Gaussian method (elimination of Gauss)

a) Learning objectives

To solve system of of linear equations using Gaussian method (elimination of Gauss).

b) Teaching resources

Learner's book and other Reference books to facilitate research, calculator, manila paper, markers, pens, pencils...

c) Prerequisites/Revision/Introduction

Students will learn easily in this lesson, if they well perform on operation (Addition, subtraction and multiplication)on matrices, on determinant of matrices and on inverse of matrices

d) Learning activities:

- Invite student-teachers to work in groups and do the activity 8.5.3 found in their Mathematics Student books;
- Move around in the class for facilitating students where necessary and give more clarification on eventual challenges they may face during their work; Verify and identify groups with different working steps;
- Invite one member from each group with different working steps to present their work where they must explain the working steps;
- As a tutor, harmonize the findings from presentation;
- Use different probing questions and guide them to explore the content and examples given in the student's book and lead them to discover how to solve system of of linear equations using Gaussian method (elimination of Gauss).

- After this step, guide students to do the application activity 8.5.3 and evaluate whether lesson objectives were achieved.

Answer for activity 8.5.3

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 1 & -4 & -1 \\ 4 & -3 & 1 & 1 \end{array} \right] \begin{array}{l} r_1 \\ r_2 = 2r_1 - r_2 \\ r_3 = 4r_1 - r_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 1 & -9 & -7 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 1 & -9 & -7 \end{array} \right] \begin{array}{l} x \\ y \\ z \end{array} \Rightarrow \begin{cases} x + 2y - 2z = 1 \\ 3y = 3 \\ 1y - 9z = -7 \end{cases}$$

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

Or Complete the equivalent matrices of product of augmented matrix and matrix of unknowns we get:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 1 & -4 & -1 \\ 4 & -3 & 1 & 11 \end{array} \right] \begin{array}{l} r_1 \\ r_2 = 2r_1 - r_2 \\ r_3 = 4r_1 - r_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 11 & -9 & -7 \end{array} \right] \begin{array}{l} r_1 \\ r_2 = \frac{1}{3}r_2 \\ r_3 = 11r_2 - 3r_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 27 & 54 \end{array} \right]$$

$$\text{we have } \begin{cases} 27z = 54 \\ y = 1 \\ x + 2y - 2z = 1 \end{cases} \Rightarrow \begin{cases} z = 2 \\ y = 1 \\ x = 3 \end{cases}$$

Answer for application activity 8.5.3

Solve by using elimination of Gauss

$$1. \begin{cases} 2x - 2y = 5 \\ 4x + 3y = 1 \end{cases} \sim \begin{array}{c|c|c} 2 & -2 & 5 \\ 4 & 3 & 1 \end{array} \begin{array}{l} r_1 = \frac{1}{2}r_1 \\ r_2 = 2r_1 - r_2 \end{array} \sim \begin{array}{c|c|c} 1 & -1 & \frac{5}{2} \\ 0 & -7 & 9 \end{array}$$
$$\Rightarrow \begin{cases} -7y = 9 \\ x - y = \frac{5}{2} \end{cases} \Rightarrow \begin{cases} y = -\frac{9}{7} \\ x = \frac{17}{14} \end{cases}$$
$$S = \left\{ \left(\frac{17}{14}, -\frac{9}{7} \right) \right\}$$

$$2. \begin{cases} a - 3b = -5 \\ 3a - b + 2c = 7 \\ 5a - 2b + 4c = 10 \end{cases}$$

(hint: eliminate 3 for 2nd and 5 for 3rd equations, then maintain zeros and eliminate coefficients of b on 2nd and 3rd equations, finally deduce the solutions $S = \{(4, 3, -1)\}$)

8.6 Unit Summary

1. Square matrix of order three has the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

2. In an upper triangular matrix, the elements located below the leading diagonal are zeros
3. In a lower triangular matrix, the elements above the leading diagonal are zeros.
4. In a diagonal matrix, all the elements above and below the leading diagonal are zeros.

5. A scalar matrix is a diagonal matrix in which the leading diagonal elements are equal.

6. An identity matrix (noted by **I**) is a diagonal matrix in which the leading diagonal elements are equal to 1.

$$7. \text{ If } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \text{ then } \begin{matrix} a_{11} = b_{11}, a_{12} = b_{12}, a_{13} = b_{13} \\ a_{21} = b_{21}, a_{22} = b_{22}, a_{23} = b_{23} \\ a_{31} = b_{31}, a_{32} = b_{32}, a_{33} = b_{33} \end{matrix}$$

$$8. \text{ If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \text{ then}$$

$$A + B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \\ a_{31} - b_{31} & a_{32} - b_{32} & a_{33} - b_{33} \end{pmatrix}$$

$$9. \text{ If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ then } kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$$

$$10. \text{ If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ then } A^t = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$11. \text{ If } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \text{ then}$$

$$\begin{aligned}
 A \cdot B &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \\
 &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}
 \end{aligned}$$

12. The sum of the entries on the leading diagonal of a square matrix, A , is known as the **trace** of that matrix, noted $tr(A)$.

13. Every linear transformation $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ can be identified with a matrix of order three, $[f]_e = (a_{ij})$, whose j^{th} column is $f(e_j)$ where $\{\bar{e}_j\}$, $j=1,2,3$ is the standard basis of \mathbb{R}^3 . The matrix $[f]_e$ is called matrix representation of f relative to the standard basis $\{\bar{e}_j\}$.

14. Let $\{\bar{e}_i\}$, $\{\bar{f}_i\}$ and $\{\bar{g}_i\}$ be bases of E , U and V respectively.

Let $f: E \rightarrow U$ and $g: U \rightarrow V$ be linear mappings. Then $[g \circ f]_{\bar{e}_i}^{\bar{g}_i} = [g]_{\bar{f}_i}^{\bar{g}_i} [f]_{\bar{e}_i}^{\bar{f}_i}$. That is, relative to the appropriate bases, the matrix representation of the composition of two linear mappings is equal to the product of the matrix representations of the individual mappings.

15. For any $f, g \in L(E)$ and any scalar $\alpha \in K$,

16. Consider an arbitrary 3x3 matrix, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. The determinant of

A is defined as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

17. Steps to Calculate the Inverse Matrix

a) Calculate the determinant of A , $|A|$. If the determinant is zero the matrix has no inverse.

b) Find the cofactor matrix which is found by replacing every element in matrix A by its cofactor.

c) Find the adjugate (or classical adjoint) matrix, noted $adj(A)$, which is the transpose of the cofactor matrix.

d) The matrix inverse is equal to the inverse value of its determinant multiplied by the adjugate matrix.

18. Consider the following system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \quad (1)$$

The system (1) can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

and the solution of system (1) is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \text{ provided that } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \text{ exists.}$$

Or we can use Cramer's rule as follows

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}$$

8.7. Additional information for the tutor

This unit is combining both matrix of order 2 and matrix of order 3. It is very necessary to help student-teachers to learn it and finish all given activities in the given time. Tutor may prepare many activities to students to be performed in groups at home and then present them in written form to be marked after. This strategy will help tutor to cover all required topics and concepts in this unit. The tutor will use one example and one application task while teaching and let student-teachers do the remaining tasks themselves in groups and after class.

8.8. Answers for end unit assessment

$$1) \text{ a) } \begin{pmatrix} -7 & -3 & 0 \\ 0 & 4 & -12 \\ 0 & -10 & -2 \end{pmatrix} \quad \text{b) } \begin{pmatrix} -9 & -23 & 6 \\ 0 & -4 & -16 \\ -4 & 2 & -8 \end{pmatrix} \quad \text{c) } \begin{pmatrix} 7 & 8 & 3 \\ 2 & 4 & -10 \\ 2 & -14 & 7 \end{pmatrix}$$

$$\text{d) } \begin{pmatrix} 29 & 28 & 15 \\ 10 & -34 & -21 \\ -4 & 28 & -16 \end{pmatrix} \quad \text{e) } \begin{pmatrix} 38 & 36 & 13 \\ -1 & 12 & -42 \\ 0 & 0 & -18 \end{pmatrix} \quad \text{f) } \begin{pmatrix} 118 & 120 & 37 \\ 19 & 12 & 10 \\ 14 & 0 & 76 \end{pmatrix}$$

$$2) \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 4 & 0 \end{pmatrix}$$

$$3) \text{ a. } [f]_e = \begin{pmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{pmatrix}$$

$$\text{b. Let } \vec{v} = (a, b, c) \in \mathbb{R}^3 \quad [\vec{v}]_e = \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix}$$

$$[f]_e [\vec{v}]_e = \begin{pmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{pmatrix} \begin{pmatrix} c \\ b-c \\ a-b \end{pmatrix} = \begin{pmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{pmatrix}$$

$$\text{But, } f(\vec{v}) = f(a, b, c) = (2b+c, a-4b, 3a), \quad [f(\vec{v})]_e = \begin{pmatrix} 3a \\ -2a-4b \\ -a+6b+c \end{pmatrix}, \text{ verified.}$$

$$4) \quad \text{a) } \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{b) no inverse} \quad \text{c) } \begin{pmatrix} \frac{44}{207} & -\frac{8}{207} & \frac{1}{69} \\ \frac{1}{207} & -\frac{19}{207} & \frac{11}{69} \\ -\frac{13}{207} & \frac{40}{207} & -\frac{5}{69} \end{pmatrix}$$

$$5) \quad X = \begin{pmatrix} 3 & -2 & -2 \\ -5 & 5 & 2 \\ 5 & -3 & 1 \end{pmatrix}$$

$$6) \quad \text{a) } S = \{(0, 0, 0)\} \quad \text{b) } S = \{(1, 1, 1)\} \quad \text{c) } S = \{(3, 0, 1)\}$$

$$7) \quad 0$$

$$8) \quad k \neq -\frac{3}{5}, \quad A^{-1} = \frac{1}{8} \begin{pmatrix} -29 & 17 & 14 \\ -9 & 5 & 6 \\ 16 & -8 & -8 \end{pmatrix}$$

$$9) \quad \text{a) } A^{-1} = \frac{1}{7}(4I - A^2) \quad \text{b) } \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

$$10) A^3 = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6I$$

$$a) x = -3, y = 5, z = 2$$

$$b) x = 2, y = 1, z = 0$$

8.9. Additional activities

8.9.1 remedial activities

$$1. \quad \text{Find , in terms of } t \text{ , the determinant of the matrix } A = \begin{pmatrix} 2-t & 1 & 3 \\ 1 & 1-t & 1 \\ -1 & -1 & -2-t \end{pmatrix}$$

2. If A is a square matrix of order 3 such that $\det A = x$, find the value of

$$a) \det(A^2) \quad b) \det(A^n), n \in \mathbb{Z} \quad c) \det(2A) \quad d) \det(mA), m \in \mathbb{R}$$

8.9.2 consolidation activities

1. Find the value of k for which the following system of equations

$$3x - 2y + 2z = 3$$

$$x + ky - 3z = 0 \quad \text{are consistent}$$

$$4x + y + 2z = 5$$

2. For what value of λ and μ the following system of equations

$$2x + 3y + 5z = 9$$

$$7x + 3y - z = 1$$

$$4x + 3y + \lambda z = \mu$$

will have

a) No solution

b) Unique solution

c) More than one solution

8.9.3 Extended activity

1. Discuss the consistency of the following system of equations for different

$$\text{cases of } \lambda \begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = \lambda \\ \lambda x + y + z = \lambda^2 \end{cases}$$

Answers

1) $-t^3 + t^2 + t - 1$

2) a) x^2 b) x^n c) $2x$ d) mx

3) $k = \frac{1}{4}$

4) a) $\lambda = 5, \mu \neq 9$ b) $\lambda \neq 5$ c) $\lambda = 5, \mu = 9$

5) By using Cramer's rule,

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}$$

Three cases will arise

$$\begin{cases} \text{If } \Delta \neq 0 \text{ there is a unique solution} \\ \text{If } \Delta = 0 \text{ and } \Delta_x \text{ or } \Delta_y \text{ or } \Delta_z = 0 \text{ Infinitely many solutions} \\ \text{If } \Delta = 0 \text{ and } \Delta_x \text{ or } \Delta_y \text{ or } \Delta_z \neq 0 \text{ no solution} \end{cases}$$

Let us write the system in matrix form.
$$\begin{pmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}$$

Let $A = \begin{pmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{pmatrix} \Rightarrow |A| = -\lambda^3 + 3\lambda + 2$ or $\lambda = -1$ and $\lambda = 2$

$$\Delta_x = \begin{vmatrix} 1 & 1 & \lambda \\ \lambda & \lambda & 1 \\ \lambda^2 & 1 & 1 \end{vmatrix} = -\lambda^4 + 2\lambda^2 - 1 \text{ or } \lambda = -1 \text{ and } \lambda = 1$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & \lambda^2 & 1 \end{vmatrix} = \lambda + 1 \text{ or } \lambda = -1$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & \lambda \\ \lambda & 1 & \lambda^2 \end{vmatrix} = \lambda^3 - \lambda^2 - \lambda + 1 \text{ or } \lambda = -1 \text{ and } \lambda = 1$$

Discussion

- **For** $\lambda = -1$, the system admits infinitely many solutions. (see conditions above)
- **For** $\lambda = 1$, the system admits a unique solution. (see conditions above)
- **For** $\lambda = 2$, no solution. (see conditions above)

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